ON AN ESTIMATE FOR SEMI-LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

By Yoshikazu Hirasawa

§ 1. Introduction.

In an excellent paper [3],[1] Nagumo obtained a result on the a priori estimate for derivatives of solutions of the semi-linear elliptic differential equation

$$(1.1)\qquad L(u)\equiv \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}=f\left(x,\, u,\, \frac{\partial u}{\partial x}\right),^{2)}$$

and he utilized his estimate in establishing an existence theorem for solutions of the boundary value problem of the first kind concerning the equation (1. 1). Throughout his paper he imposed the Lipschitz condition on the coefficients $a_{ij}(x)$ and used the fact that the functions $a_{ij}(x)$ are differentiable almost everywhere in obtaining his a priori estimate for derivatives of solutions of the equation (1. 1).

In the present paper, it will be shown that, by imposing the Hölder condition on the coefficients $a_{ij}(x)$, we may have also a result of the same type as obtained by Nagumo in his work.

Though our process of proof is, for the most part, similar to Nagumo's one, a part of it has been suggested by a work of Cordes [2] and especially Lemmas 1 and 2 in § 3 of the present paper are the modifications of Cordes' results.

Our main result will be stated in § 4.

We use the notations $\partial_{x_i}u$ or $\partial_i u$ for $\partial u/\partial x_i$, and $\partial_{x_i}\partial_{x_j}u$ or $\partial_i\partial_j u$ for $\partial^2 u/\partial x_i\partial x_j$. x and $\partial_x u$ denote the n-dimensional real vectors $(x_1, x_2, \cdots, x_n)$ and $(\partial_1 u, \partial_2 u, \cdots, \partial_n u)$ respectively. We use $\partial_{x'_i}u$ or $\partial'_i u$ for $\partial u/\partial x'_i$, and the notations of the same kind for the others too. Especially $\partial_r u$ means $\partial u/\partial r$ for $r\equiv(\sum_{i=1}^{n} x_i^2)^{1/2}$.

We define $|x-x'|$ and $|\partial_x u|$ as follows:

$$|x-x'|=\left\{\sum_{i=1}^{n}(x_i-x'_i)^2\right\}^{1/2} \quad\text{and}\quad |\partial_x u|=\left\{\sum_{i=1}^{n}(\partial_i u)^2\right\}^{1/2}.$$

§ 2. Preliminary remarks.

Let D be a bounded domain in the n-dimensional Euclidean space and let

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1) The numbers in the brackets refer to the list of references at the end of this paper.

2) $\partial u/\partial x$ denotes the n-dimensional real vector $(\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. For $\partial u/\partial x_i$, $\partial u/\partial x$ and etc., we shall introduce, in this section, other notations which will be used in the subsequent sections.

$$L(u)\equiv\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u$$

be a differential operator satisfying the uniform ellipticity condition with two positive constants $\underline{A}$, $\bar{A}$ $(\underline{A}\leqq\bar{A})$:

$$\underline{A}|\xi|^2\leqq\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\leqq\bar{A}|\xi|^2 \tag{2. 1}$$

for any real vector $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ and for any point $x\in D$.

Then, for any point $x^{(0)}\in D$, we can find a linear transformation of variables $x'=Tx$ which transforms the form $\sum_{i,j=1}^{n}a_{ij}(x^{(0)})\partial_i\partial_j u$ into the form $\varDelta' u\equiv\sum_{i=1}^{n}\partial_i'\partial_i' u$, and further we see that this transformation, at the same time, changes the form $\sum_{i,j=1}^{n}a_{ij}(x^{(0)})\partial_i u\partial_j v$ into the form $\sum_{i=1}^{n}\partial_i' u\partial_i' v$.

We call the transformation defined by the above character *the linear transformation associated with the point* $x^{(0)}$, and in the following of the present section, we will give, without detailed proofs, certain properties of this transformation which will be necessary later.

(I) First, we have

$$\frac{1}{\sqrt{\bar{A}}}|x|\leqq|Tx|\leqq\frac{1}{\sqrt{\underline{A}}}|x|,\quad \sqrt{\underline{A}}\,|x'|\leqq|T^{-1}x'|\leqq\sqrt{\bar{A}}\,|x'| \tag{2. 2}$$

and since $|\partial_{x'}u'(x')|=\{\sum_{i,j=1}^{n}a_{ij}(x^{(0)})\partial_i u\partial_j u\}^{1/2}$, we get

$$\sqrt{\underline{A}}\,|\partial_x u(x)|\leqq|\partial_{x'}u'(x')|\leqq\sqrt{\bar{A}}\,|\partial_x u(x)| \tag{2. 3}$$

where we have put $u'(x')\equiv u(T^{-1}x')$ for any function $u(x)$.

(II) Now, let a be a point of D and let Σ be a closed sphere defined by

$$\Sigma=\{x;\ |x-a|\leqq\rho\},$$

ρ being a positive number such that $\Sigma\subset D$. Further let $x^{(0)}$ be an interior point of Σ and let S be a closed sphere defined by

$$S=\{x';\ |x'-Tx^{(0)}|\leqq\rho'\}.$$

Then we have the inequalities

$$\sqrt{\underline{A}}\,\rho'\leqq|T^{-1}x'-x^{(0)}|\leqq\sqrt{\bar{A}}\,\rho'$$

for any $x'\in\dot{S}$, where $\dot{S}$ denotes the boundary of S. Therefore $\dot{\Sigma}$ being the boundary of Σ, the inequality

$$\sqrt{\bar{A}}\rho'<\operatorname{dist}(x^{(0)},\dot{\Sigma}) \tag{2. 4}$$

implies an inclusion relation $S\subset T(\Sigma)$, and in this situation we get

$$\operatorname{dist}(x^{(0)},\dot{\Sigma})-\sqrt{\bar{A}}\rho'\leqq\operatorname{dist}(T^{-1}(S),\dot{\Sigma}). \tag{2. 5}$$

(III) Let $x'=Tx$ be the linear transformation associated with a point $x^{(0)}$ belonging to D, and suppose that by $x'=Tx$, the form $\sum_{i,j=1}^{n}a_{ij}(x)\partial_i\partial_j u$ is changed into the form

$$L'(u')\equiv \sum_{i,j=1}^{n} b_{ij}(x')\partial_i'\partial_j'u'(x'),$$

then we have

$$b_{ij}(x^{(0)\prime})=\delta_{ij}$$

and

$$\left\{\sum_{i,j=1}^{n}(b_{ij}(x')-b_{ij}(x^{(0)\prime}))^2\right\}^{1/2}\leqq\frac{n}{\underline{A}}\left\{\sum_{i,j=1}^{n}(a_{ij}(x)-a_{ij}(x^{(0)}))^2\right\}^{1/2},$$

where $x'=Tx$ and $x^{(0)\prime}=Tx^{(0)}$.

Supposing further that the coefficients $a_{ij}(x)$ satisfy the Hölder condition with a positive constant H and a positive exponent α $(0<\alpha\leqq 1)$:

$$\left\{\sum_{i,j=1}^{n}(a_{ij}(x^{(1)})-a_{ij}(x^{(2)}))^2\right\}^{1/2}\leqq H|x^{(1)}-x^{(2)}|^{\alpha}, \tag{2. 6}$$

we obtain

$$\left\{\sum_{i,j=1}^{n}(b_{ij}(x')-b_{ij}(x^{(0)\prime}))^2\right\}^{1/2}\leqq\frac{n\bar{A}^{\alpha/2}}{\underline{A}}H|x'-x^{(0)\prime}|^{\alpha}. \tag{2. 7}$$

§ 3. Lemmas.

$C^k[|x|\leqq r_0]$ denotes the family of all functions which are k times continuously differentiable in the domain $\{x;\ |x|\leqq r_0\}$, where r_0 is a positive real number.

LEMMA 1. *Let $u(x)$ be a function belonging to $C^2[|x|\leqq r_0]$. Then we have, for any positive number α such that $0<\alpha\leqq 1$,*

$$\begin{aligned}&\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2\,dx\right\}^{1/2}-\left\{\int_{r\leqq r_0}(\Delta u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2dx\right\}^{1/2}\\&\leqq c r_0^2\left\{\int_{r\leqq r_0}\sum_{i=1}^{n}(\partial_i u)^2 r^{\alpha-n}dx\right\}^{1/2},\end{aligned} \tag{3. 1}$$

where $r=|x|$, and c is a positive constant depending only on α and n, but not on r_0.

Proof. Suppose, for the first time, that $u(x)$ belongs to $C^3[|x|\leqq r_0]$, and multiply the identity

$$(\Delta u)^2-\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2=\sum_{i,j=1}^{n}\partial_i\partial_j\{(\partial_i u)(\partial_j u)\}-\Delta\Big(\sum_{i=1}^{n}(\partial_i u)^2\Big)$$

by the factor $r^{2+\alpha-n}(r_0^2-r^2)^2$, and integrate the result over the sphere $r\equiv|x|\leqq r_0$, then we obtain by integration by parts

$$J\equiv\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2dx-\int_{r\leqq r_0}(\Delta u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2dx$$

(3. 2)

$$=\int_{r\leqq r_0}\left[\sum_{i=1}^{n}(\partial_i u)^2\Delta\{r^{2+\alpha-n}(r_0^2-r^2)^2\}-\sum_{i,j=1}^{n}(\partial_i u)(\partial_j u)\partial_i\partial_j\{r^{2+\alpha-n}(r_0^2-r^2)^2\}\right]dx.$$

Furthermore it may be shown by the limiting process that the above relation (3. 2) holds for any $u(x)\in C^2[|x|\leqq r_0]$.

Since

$$\Delta\{r^{2+\alpha-n}(r_0^2-r^2)^2\}$$
$$=\alpha(2+\alpha-n)r^{\alpha-n}(r_0^2-r^2)^2+4(n-2(2+\alpha))r^{2+\alpha-n}(r_0^2-r^2)+8r^{4+\alpha-n},$$

and

$$\partial_i\partial_j\{r^{2+\alpha-n}(r_0^2-r^2)^2\}$$
$$=(2+\alpha-n)\{r^2+(\alpha-n)x_ix_j\}r^{-2+\alpha-n}(r_0^2-r^2)^2$$
$$-4\{2(2+\alpha-n)x_ix_j+r^2\delta_{ij}\}r^{\alpha-n}(r_0^2-r^2)+8x_ix_jr^{2+\alpha-n},$$

the integrand in the right-hand side of the relation (3. 2) is equal to

$$I_0\equiv(n-2-\alpha)\left\{(1-\alpha)\sum_{i=1}^{n}(\partial_i u)^2-(n-\alpha)(\partial_r u)^2\right\}r^{\alpha-n}(r_0^2-r^2)^2$$
$$+4(n-3-2\alpha)\sum_{i=1}^{n}(\partial_i u)^2 r^{2+\alpha-n}(r_0^2-r^2)$$
$$-8(n-2-\alpha)(\partial_r u)^2 r^{2+\alpha-n}(r_0^2-r^2)+8\left\{\sum_{i=1}^{n}(\partial_i u)^2-(\partial_r u)^2\right\}r^{4+\alpha-n}.$$

Moreover, by virtue of the inequality $\sum_{i=1}^{n}(\partial_i u)^2\geqq(\partial_r u)^2$, we get

$$I_0\leqq|(n-2-\alpha)(2-\alpha)|\sum_{i=1}^{n}(\partial_i u)^2 r^{\alpha-n}(r_0^2-r^2)^2$$
$$+4|n-3-2\alpha|\sum_{i=1}^{n}(\partial_i u)^2 r^{2+\alpha-n}(r_0^2-r^2)+8\sum_{i=1}^{n}(\partial_i u)^2 r^{4+\alpha-n}$$
$$=\{|(n-2-\alpha)(2-\alpha)|(r_0^2-r^2)^2+4|n-3-2\alpha|r^2(r_0^2-r^2)+8r^4\}\sum_{i=1}^{n}(\partial_i u)^2 r^{\alpha-n}$$
$$\leqq c' r_0^4\sum_{i=1}^{n}(\partial_i u)^2 r^{\alpha-n},$$

where we have put

$$c'=r_0^{-4}\underset{0\leqq r\leqq r_0}{\mathrm{Max}}\{|(n-2-\alpha)(2-\alpha)|(r_0^2-r^2)^2+4|n-3-2\alpha|r^2(r_0^2-r^2)+8r^4\},$$

and hence c' is a positive constant depending only on α and n, but not on r_0.

We have therefore

$$J=\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2\,dx-\int_{r\leqq r_0}(\Delta u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2\,dx$$

$$\leqq c' r_0^4 \int_{r\leqq r_0} \sum_{i=1}^{n} (\partial_i u)^2 r^{\alpha-n} dx,$$

from which the inequality (3. 1) follows with $c=\sqrt{c'}$.

LEMMA 2. *Let $a_{ij}(x)$ be functions defined in the sphere $r\equiv|x|\leqq r_0$ and satisfying the condition*

$$\left\{\sum_{i,j=1}^{n} (a_{ij}(x)-\delta_{ij})^2\right\}^{1/2} \leqq \varepsilon < 1$$

for any point x of the sphere $|x|\leqq r_0$ and for a positive constant ε not depending on x.

Then we have, for any $u(x)\in C^2[|x|\leqq r_0]$,

$$\left\{\int_{r\leqq r_0} \sum_{i,j=1}^{n} (\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

(3. 3)

$$\leqq \frac{1}{1-\varepsilon}\left\{\int_{r\leqq r_0}\left(\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u\right)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2} + \frac{c}{1-\varepsilon} r_0^2 \left\{\int_{r\leqq r_0} \sum_{i=1}^{n} (\partial_i u)^2 r^{\alpha-n} dx\right\}^{1/2}.$$

where c is the same constant as in Lemma 1.

Proof. Put $L(u)\equiv\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u$, then, by virtue of the relation

$$L(u)=\Delta u+\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\partial_i\partial_j u,$$

we obtain

$$\left\{\int_{r\leqq r_0} (L(u))^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

$$\geqq \left\{\int_{r\leqq r_0} (\Delta u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

$$-\left\{\int_{r\leqq r_0}\left(\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})\partial_i\partial_j u\right)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

$$\geqq \left\{\int_{r\leqq r_0} (\Delta u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

$$-\left\{\int_{r\leqq r_0}\left(\sum_{i,j=1}^{n}(a_{ij}(x)-\delta_{ij})^2\right)\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

$$\geqq \left\{\int_{r\leqq r_0} (\Delta u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}$$

$$-\varepsilon\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0^2-r^2)^2 dx\right\}^{1/2}.$$

Hence, combining the above inequality with the result of Lemma 1, we get

$$(1-\varepsilon)\left\{\int_{r\leqq r_0}\sum_{i,j=1}^{n}(\partial_i\partial_j u)^2 r^{2+\alpha-n}(r_0{}^2-r^2)^2 dx\right\}^{1/2}$$

$$\leqq\left\{\int_{r\leqq r_0}(L(u))^2 r^{2+\alpha-n}(r_0{}^2-r^2)^2 dx\right\}^{1/2}+cr_0{}^2\left\{\int_{r\leqq r_0}\sum_{i=1}^{n}(\partial_i u)^2 r^{\alpha-n} dx\right\}^{1/2},$$

from which the inequality (3. 3) follows.

REMARK. We will here find an upper bound of the constant c given in Lemmas 1 and 2.

Put

$$\varphi(t)=|(n-2-\alpha)(2-\alpha)|(1-t)^2+4|n-3-2\alpha|t(1-t)+8t^2,$$

then, by the definition we have

$$c^2=c'\doteqdot \operatorname*{Max}_{0\leqq t\leqq 1}\varphi(t).$$

Since

$$|n-2-\alpha|\leqq n-2+\alpha\leqq n-1, \qquad |n-3-2\alpha|\leqq n-1+2\alpha\leqq n+1$$

for $n=2, 3, \cdots$, we obtain

$$\varphi(t)\leqq 4\{(n-1)(1-t)^2+(n+1)t(1-t)+2t^2\}$$
$$=4\{(n-1)-(n-3)t\}\leqq 4\{(n-1)+t\},$$

and hence we get

$$c^2=c'\leqq 4n.$$

Thus we can take $c=2\sqrt{n}$ as the constant c given in Lemmas 1 and 2.

Now, for the proof of the main theorem in the next section, we will use also the following lemma due to Akō [1]:

LEMMA 3. *Let $G(x, \xi)$ be Green's function of Laplace's equation $\Delta u=0$ with respect to a sphere $|x|=r_0$, and let $f(x)$ be a bounded function in $|x|\leqq r_0$ ($|f(x)|\leqq M$). Then we have the estimate*

$$\left|\omega_n{}^{-1}\int_{|\xi|\leqq r_0}\partial_x G(0, \xi)f(\xi)\,d\xi\right|\leqq c_n M r_0,$$

where ω_n denotes the surface area of the n-dimensional unit sphere, that is, $\omega_n=2\pi^{n/2}/\Gamma(n/2)$, and

$$c_n=\frac{2}{n+1}B\left(\frac{1}{2}, \frac{n+1}{2}\right), \quad B\left(\frac{1}{2}, \frac{n+1}{2}\right)=\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{n+2}{2}\right).$$

For the proof of this lemma, see Lemma 3 in a paper of Akō [1], p. 384.

§4. Main result.

In the present section, we consider the semi-linear differential equation of elliptic type

(4. 1) $$\sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u=f(x, u, \partial_x u)$$

and give a result about the a priori estimate for derivatives of solutions of this equation.

Let D be a bounded domain with the boundary $\dot{D}$ in the n-dimensional Euclidean space and let d be the diameter of D.

The differential operator

$$L(u)\equiv \sum_{i,j=1}^{n} a_{ij}(x)\partial_i\partial_j u$$

is supposed to satisfy the ellipticity condition with two positive constants $\underline{A}$ and $\overline{A}$ ($\underline{A}\leqq\overline{A}$):

(4. 2) $$\underline{A}\,|\xi|^2\leqq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j\leqq\overline{A}|\xi|^2$$

for any real vector $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ and for any point $x\in D$, and further the coefficients $a_{ij}(x)$ are supposed to fulfil the Hölder condition with a positive constant H and a positive exponent α $(0<\alpha\leqq1)$:

(4. 3) $$\left\{\sum_{i,j=1}^{n} (a_{ij}(x^{(1)})-a_{ij}(x^{(2)}))^2\right\}^{1/2}\leqq H|x^{(1)}-x^{(2)}|^{\alpha}$$

for any $x^{(1)}$, $x^{(2)}\in D$.

The function $f(x, u, p)$ is defined in a $(2n+1)$-dimensional domain

$$\mathfrak{D}=\{(x, u, p);\ x\in D,\ |u|\leqq M,\ |p|<+\infty\},$$

where M is a positive quantity and p denotes the n-dimensional real vector $(p_1, p_2, \cdots, p_n)$. Furthermore we suppose that the function $f(x, u, p)$ satisfies the condition of the growth order with respect to p:

(4. 4) $$|f(x, u, p)|\leqq B|p|^2+\Gamma$$

for any $(x, u, p)\in\mathfrak{D}$, where B and Γ are positive constants.

THEOREM. *Suppose that the above-mentioned hypotheses are fulfilled. Let $u(x)$ be a solution of the equation* (4. 1) *such that $|u(x)|\leqq M$, and let N be the oscillation of $u(x)$ in the domain D.*

If the condition

(4. 5) $$8k(n)\frac{BN}{\underline{A}}<1 \qquad \left(k(n)=\frac{2}{n+1}B\left(\frac{1}{2}, \frac{n+1}{2}\right)^{-2}\right)^{3)}$$

is fulfilled, then we have

(4. 6) $$|\partial_x u(a)|<C^{(1)}\rho(a)^{-1}\operatorname*{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\}+C^{(2)}\rho(a)^{4)}$$

3) As seen in Lemma 3, $B(1/2, 1/(n+1))$ is the Beta-function, which can be expressed by the Gamma-function as follows: $B(1/2, 1/(n+1))=\Gamma(1/2)\Gamma((n+1)/2)/\Gamma((n+2)/2)$.

4) The notation $\operatorname{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\}$ denotes the oscillation of $u(x)$ in the sphere $|x-a|\leqq\rho(a)$ possessing the point a as the center.

for any point $a\in D$, *where* $\rho(a)=\operatorname{dist}(a, \dot{D})$, *and* $C^{(1)}$ *and* $C^{(2)}$ *are positive constants depending only on* $\underline{A}$, $\bar{A}$, B, Γ, N, n, d, H *and* α.

Proof. Let a be a point of D and let Σ_κ be a closed sphere defined by

$$\Sigma_\kappa=\{x;\ |x-a|\leqq\kappa\rho(a)\} \qquad (0<\kappa<1),$$

and put

$$\mu_\kappa=\operatorname*{Max}_{x\in\Sigma_\kappa}\{|\partial_x u(x)|\rho_\kappa(x)\},$$

where $\rho_\kappa(x)=\operatorname{dist}(x, \dot{\Sigma}_\kappa)$ and $\dot{\Sigma}_\kappa$ denotes the boundary of Σ_κ. Then we can find a point $x^{(0)}$ in the interior of Σ_κ such that

$$|\partial_x u(x^{(0)})|\rho_\kappa(x^{(0)})=\mu_\kappa.$$

Now, let $x'=Tx$ be the linear transformation associated with the point $x^{(0)}$, that is, suppose that by $x'=Tx$ we have

$$\sum_{i,j=1}^n a_{ij}(x^{(0)})\partial_i\partial_j u(x)=\sum_{i=1}^n \partial_i'\partial_i' u'(x'),$$

where $u'(x')\equiv u(T^{-1}x')$, and further put

$$f(x, u, \partial_x u)\equiv f'(x', u', \partial_{x'}u'),$$

$$\sum_{i,j=1}^n a_{ij}(x)\partial_i\partial_j u(x)\equiv\sum_{i,j=1}^n b_{ij}(x')\partial_i'\partial_j' u'(x').$$

Then we obtain

$$\begin{aligned}\Delta' u'(x')&\equiv\sum_{i=1}^n \partial_i'\partial_i' u'(x')\\ &=f'(x', u', \partial_{x'}u')+\sum_{i,j=1}^n(\delta_{ij}-b_{ij}(x'))\partial_i'\partial_j' u'(x').\end{aligned}$$

Let $S_{(\lambda)}$ and $S_{(2\lambda)}$ be two concentric closed spheres with radii $\lambda\rho_\kappa(x^{(0)})$ and $2\lambda\rho_\kappa(x^{(0)})$, and having the point $x^{(0)\prime}=Tx^{(0)}$ as the common center, where λ is a positive constant. If we take λ so small that $2\lambda\sqrt{\bar{A}}\,\rho_\kappa(x^{(0)})<\rho_\kappa(x^{(0)})$, that is,

$$\lambda<\frac{1}{2\sqrt{\bar{A}}}, \tag{4.7}$$

then, by (2.4) and (2.5) in §2, we have

$$S_{(\lambda)}\subset S_{(2\lambda)}\subset T(\Sigma_\kappa).$$

Now, let $G(x', \xi)$ be Green's function of the equation $\Delta' u'=0$ with respect to the sphere $S_{(\lambda)}$. Then we obtain for any $x'\in S_{(\lambda)}$

$$\begin{aligned}u'(x')=h'(x')&-\omega_n^{-1}\int_{S_{(\lambda)}} G(x', \xi)f'(\xi, u'(\xi), \partial_\xi u'(\xi))d\xi\\ &-\omega_n^{-1}\int_{S_{(\lambda)}} G(x', \xi)\sum_{i,j=1}^n(\delta_{ij}-b_{ij}(\xi))\partial_i'\partial_j' u'(\xi)d\xi\end{aligned}$$

where ω_n denotes the surface area of the n-dimensional unit sphere, and $h'(x')$ is the harmonic function defined in $S_{(\lambda)}$ and taking the boundary values $u'(x')$ on the boundary $\dot{S}_{(\lambda)}$ of the sphere $S_{(\lambda)}$.

We get therefore

$$|\partial_{x'}u'(x^{(0)'})|\leqq(\mathrm{I})+(\mathrm{II})+(\mathrm{III}), \tag{4. 8}$$

where

$$(\mathrm{I})=|\partial_{x'}h'(x^{(0)'})|,$$

$$(\mathrm{II})=\left|\omega_n{}^{-1}\int_{S_{(\lambda)}}\partial_{x'}G(x^{(0)'},\ \xi)f'(\xi,\ u'(\xi),\ \partial_\xi u'(\xi))\,d\xi\right|,$$

$$(\mathrm{III})=\left|\omega_n{}^{-1}\int_{S_{(\lambda)}}\partial_{x'}G(x^{(0)'},\ \xi)\sum_{i,j=1}^{n}(\delta_{ij}-b_{ij}(\xi))\partial_i'\partial_j'u'(\xi)d\xi\right|.$$

We see firstly

$$(\mathrm{I})\leqq\frac{1}{\lambda\rho_\kappa(x^{(0)})}B\left(\frac{1}{2},\ \frac{n+1}{2}\right)^{-1}\operatorname*{Osc}_{S_{(\lambda)}}\{u'(x')\}. \tag{4. 9}$$

On the other hand, by (2. 5) in §2, we have

$$|\partial_x u(x)|(1-2\lambda\sqrt{A})\rho_\kappa(x^{(0)})\leqq|\partial_x u(x)|\rho_\kappa(x)\leqq\mu_\kappa$$

for any $x\in T^{-1}(S_{(2\lambda)})\subset\Sigma_\kappa$, and hence by taking λ so small that $(1-2\lambda\sqrt{A})^{-1}\leqq\sqrt{2}$, that is,

$$\lambda\leqq\frac{\sqrt{2}-1}{2\sqrt{2}\sqrt{A}}, \tag{4. 10}$$

we obtain

$$|\partial_x u(x)|\leqq\sqrt{2}\,\rho(x^{(0)})^{-1}\mu_\kappa \tag{4. 11}$$

for any $x\in T^{-1}(S_{(2\lambda)})\subset\Sigma_\kappa$.

Therefore it follows from the condition (4. 4), that

$$|f'(\xi,\ u'(\xi),\ \partial_\xi u'(\xi))|=|f(x,\ u(x),\ \partial_x u(x))|\leqq 2B\rho(x^{(0)})^{-2}\mu_\kappa^2+\Gamma \tag{4. 12}$$

in the sphere $S_{(2\lambda)}$, and thus, by virtue of Lemma 3 in §3, we get

$$(\mathrm{II})\leqq\frac{2}{n+1}B\left(\frac{1}{2},\ \frac{n+1}{2}\right)^{-1}\{2B\rho_\kappa(x^{(0)})^{-1}\mu_\kappa^2+\Gamma\rho_\kappa(x^{(0)})\}\lambda. \tag{4. 13}$$

In the next place we estimate the third term (III) of the right-hand side of the inequality (4. 8).

By making use of the estimate

$$|\partial_{x'}G(x^{(0)'},\ \xi)|\leqq|\xi-x^{(0)'}|^{1-n}, \tag{4. 14}$$

and by the inequality (2. 7), we have

$$(\mathrm{III})\leqq\omega_n{}^{-1}\int_{S_{(\lambda)}}|\partial_{x'}G(x^{(0)'},\ \xi)|\left\{\sum_{i,j=1}^{n}(\delta_{ij}-b_{ij}(\xi))^2\right\}^{1/2}\left\{\sum_{i,j=1}^{n}(\partial_i'\partial_j'u'(\xi))^2\right\}^{1/2}d\xi$$

$$\leqq \frac{n\bar{A}^{\alpha/2}}{\omega_n \underline{A}} H \int_{r\leqq r_1} \left\{ \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 \right\}^{1/2} r^{1+\alpha-n} d\xi,$$

where $r=|\xi-x^{(0)\prime}|$ and $r_1=\lambda\rho_\kappa(x^{(0)})$. Furthermore we see

$$J_1 \equiv \int_{r\leqq r_1} \left\{ \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 \right\}^{1/2} r^{1+\alpha-n} d\xi$$

$$\leqq \left\{ \int_{r\leqq r_1} r^{\alpha-n} d\xi \right\}^{1/2} \left\{ \int_{r\leqq r_1} \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 r^{2+\alpha-n} d\xi \right\}^{1/2}$$

$$= \left\{ \frac{\omega_n r_1^{\alpha}}{\alpha} \right\}^{1/2} \left\{ \int_{r\leqq r_1} \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 r^{2+\alpha-n} d\xi \right\}^{1/2}$$

and hence, by putting $r_0=2r_1=2\lambda\rho_\kappa(x^{(0)})$, we get

$$J_1 \leqq \frac{1}{(r_0^2-r_1^2)} \left\{ \frac{\omega_n r_1^{\alpha}}{\alpha} \right\}^{1/2} \left\{ \int_{r\leqq r_0} \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 r^{2+\alpha-n} (r_0^2-r^2)^2 d\xi \right\}^{1/2}$$

$$= \frac{4}{3r_0^2} \left\{ \frac{\omega_n r_1^{\alpha}}{\alpha} \right\}^{1/2} \left\{ \int_{r\leqq r_0} \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 r^{2+\alpha-n} (r_0^2-r^2)^2 d\xi \right\}^{1/2}.$$

We have therefore

$$\text{(III)} \leqq K'Hr_0^{-2+\alpha/2} \left\{ \int_{r\leqq r_0} \sum_{i,j=1}^{n} (\partial_i'\partial_j' u'(\xi))^2 r^{2+\alpha-n} (r_0^2-r^2)^2 d\xi \right\}^{1/2},$$

where

$$K' = \frac{4}{3} \left(\frac{n^2 \bar{A}^{\alpha}}{2^{\alpha} \alpha \omega_n \underline{A}^2} \right)^{1/2}.$$

Now, by (2. 7) of § 2, and by $2\rho_\kappa(x^{(0)}) \leqq 2\rho(a) \leqq d$, we obtain

$$\left\{ \sum_{i,j=1}^{n} (b_{ij}(x')-\delta_{ij})^2 \right\}^{1/2} \leqq H_1|x'-x^{(0)\prime}|^{\alpha}$$

$$\leqq H_1(2\lambda\rho_\kappa(x^{(0)}))^{\alpha} \leqq H_1 d^{\alpha} \lambda^{\alpha}$$

for any $x' \in S_{(2\lambda)}$, where $H_1 = n\bar{A}^{\alpha/2}H/\underline{A}$. Hence, if we take λ so small that $H_1 d^{\alpha}\lambda^{\alpha} \leqq 1/2$, that is,

(4. 15) $$\lambda \leqq \frac{d}{\sqrt{\bar{A}}} \left(\frac{\underline{A}}{2nH} \right)^{1/\alpha},$$

then we can choose 1/2 as ε in Lemma 2 of § 3.

Consequently we have by Lemma 2 of § 3,

$$\text{(III)} \leqq K'Hr_0^{-2+\alpha/2} \left[2\left\{ \int_{r\leqq r_0} \left(\sum_{i,j=1}^{n} b_{ij}(\xi)\partial_i'\partial_j' u'(\xi) \right)^2 r^{2+\alpha-n} (r_0^2-r^2)^2 d\xi \right\}^{1/2} \right.$$

$$\left. + 4\sqrt{n}\, r_0^2 \left\{ \int_{r\leqq r_0} \sum_{i=1}^{n} (\partial_i' u'(\xi))^2 r^{\alpha-n} d\xi \right\}^{1/2} \right].$$

Furthermore, since it follows from (4. 11) and (4. 12), that

$$\left\{\int_{r\leqq r_0}\left(\sum_{i,j=1}^{n} b_{ij}(\xi)\partial'_i\partial'_j u'(\xi)\right)^2 r^{2+\alpha-n}(r_0{}^2-r^2)^2 d\xi\right\}^{1/2}$$

$$\leqq r_0{}^2\left\{\int_{r\leqq r_0} |f'(\xi, u'(\xi), \partial_\xi u'(\xi))|^2 r^{2+\alpha-n} d\xi\right\}^{1/2}$$

$$\leqq\sqrt{\frac{\omega_n}{2+\alpha}}(2B\rho_\kappa(x^{(0)})^{-2}\mu_\kappa{}^2+\Gamma)r_0{}^{3+\alpha/2}$$

and

$$\left\{\int_{r\leqq r_0}\sum_{i=1}^{n}(\partial'_i u'(\xi))^2 r^{\alpha-n} d\xi\right\}^{1/2}\leqq\sqrt{\frac{2\omega_n}{\alpha}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa r_0{}^{\alpha/2},$$

we obtain

$$\text{(III)}\leqq KHr_0{}^\alpha\left\{(2B\rho_\kappa(x^{(0)})^{-2}\mu_\kappa{}^2+\Gamma)r_0+4\sqrt{\frac{n}{\alpha}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa\right\}, \tag{4. 16}$$

where

$$K=\sqrt{2\omega_n}K'=\frac{4}{3}\left(\frac{2^{1-\alpha}n^2\bar{A}^\alpha}{\alpha\underline{A}^2}\right)^{1/2}. \tag{4. 17}$$

On the other hand, by (2. 3) of §2, we get

$$|\partial_{x'}u'(x^{(0)\prime})|\geqq\sqrt{\underline{A}}\,|\partial_x u(x^{(0)})|=\sqrt{\underline{A}}\,\rho_\kappa(x^{(0)})^{-1}\mu_\kappa. \tag{4. 18}$$

Thus by virtue of (4. 8), (4. 9), (4. 13), (4. 16) and (4. 18), we have

$$\sqrt{\underline{A}}\,\rho_\kappa(x^{(0)})^{-1}\mu_\kappa$$

$$\leqq\frac{1}{\lambda\rho_\kappa(x^{(0)})}B\left(\frac{1}{2},\frac{n+1}{2}\right)^{-1}\underset{S(\lambda)}{\mathrm{Osc}}\{u'(x')\}$$

$$+\frac{2}{n+1}B\left(\frac{1}{2},\frac{n+1}{2}\right)^{-1}(2B\rho_\kappa(x^{(0)})^{-1}\mu_\kappa{}^2+\Gamma\rho_\kappa(x^{(0)}))\lambda$$

$$+KHr_0{}^\alpha\left\{(2B\rho_\kappa(x^{(0)})^{-2}\mu_\kappa{}^2+\Gamma)r_0+4\sqrt{\frac{n}{\alpha}}\rho_\kappa(x^{(0)})^{-1}\mu_\kappa\right\}$$

and hence we obtain

$$\lambda C_0\mu_\kappa{}^2-(1-\lambda^\alpha C_1)\mu_\kappa+\frac{1}{\lambda}C_2\geqq0,$$

where

$$C_0=\frac{2B}{\sqrt{\underline{A}}}\left\{\frac{2}{n+1}B\left(\frac{1}{2},\frac{1}{n+1}\right)^{-1}+2KHd^\alpha\lambda^\alpha\right\},\qquad C_1=4\sqrt{\frac{n}{\alpha\underline{A}}}KHd^\alpha,$$

$$C_2=\frac{1}{\sqrt{\underline{A}}}\left\{B\left(\frac{1}{2},\frac{1}{n+1}\right)^{-1}\underset{|x-a|\leqq\rho(a)}{\mathrm{Osc}}\{u(x)\}+\lambda^2\rho(a)^2\left(\frac{2}{n+1}B\left(\frac{1}{2},\frac{1}{n+1}\right)^{-1}+2KHd^\alpha\lambda^\alpha\right)\Gamma\right\}.$$

Furthermore we see by (4. 7), (4. 10) and (4. 15), that it is sufficient for us to take λ as follows:

$$\lambda \leqq \operatorname{Min}\left\{\frac{1}{8\sqrt{\bar{A}}},\ \frac{d}{\sqrt{\bar{A}}}\left(\frac{\underline{A}}{2nH}\right)^{1/\alpha}\right\}. \tag{4. 19}$$

While, since

$$C_0C_2 \leqq \frac{2B}{\underline{A}}\left\{B\left(\frac{1}{2},\ \frac{1}{n+1}\right)^{-1}N+\lambda^2\rho(a)^2\left(\frac{2}{n+1}B\left(\frac{1}{2},\ \frac{1}{n+1}\right)^{-1}+2KHd^\alpha\lambda^\alpha\right)\Gamma\right\}$$
$$\cdot\left\{\frac{2}{n+1}B\left(\frac{1}{2},\ \frac{1}{n+1}\right)^{-1}+2KHd^\alpha\lambda^\alpha\right\},$$

the condition (4. 5) implies that we can take λ so small that

$$\lambda^\alpha C_1<\frac{1}{2} \quad\text{and}\quad (1-\lambda^\alpha C_1)^2>4(\lambda C_0)(\lambda^{-1}C_2). \tag{4. 20}$$

If we choose λ in this manner, the equation in X:

$$\lambda C_0X^2-(1-\lambda^\alpha C_1)X+\lambda^{-1}C_2=0$$

has two distinct real positive roots R_1 and R_2 $(R_1<R_2)$, and it holds that

$$\mu_\kappa \leqq R_1 \qquad\text{or}\qquad R_2 \leqq \mu_\kappa;$$

however μ_κ depends continuously on κ and $\lim_{\kappa\to+0}\mu_\kappa=0$.

We see therefore $\mu_\kappa \leqq R_1$, and by letting κ tend to unity, we have

$$|\partial_x u(a)| \leqq R_1\rho(a)^{-1}.$$

Since

$$R_1 \leqq \frac{4C_0C_2}{2\lambda C_0(1-\lambda^\alpha C_1)} \leqq \frac{4C_2}{\lambda},$$

we get

$$|\partial_x u(a)| \leqq C^{(1)}\rho(a)^{-1}\operatorname*{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\}+C^{(2)}\rho(a),$$

where

$$C^{(1)}=\frac{4}{\lambda\sqrt{\underline{A}}}B\left(\frac{1}{2},\ \frac{n+1}{2}\right)^{-1},\quad C^{(2)}=\frac{\lambda}{\sqrt{\underline{A}}}\left\{\frac{2}{n+1}B\left(\frac{1}{2},\ \frac{n+1}{2}\right)^{-1}+2KHd^\alpha\lambda^\alpha\right\}\Gamma,$$

which both depend only on $\underline{A}$, $\bar{A}$, B, Γ, N, n, d, H and α.

§ 5. Corollaries.

COROLLARY 1. *If we suppose that the condition*

$$|f(x,u,p)| \leqq B'|p|+\Gamma \tag{5. 1}$$

is fulfilled instead of the condition (4. 4) *of Theorem of* § 4, *then the condition* (4. 5) *may be omitted and we obtain*

(5. 2) $$|\partial_x u(a)| \leqq C^{(1)}\rho(a)^{-1} \operatorname*{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\} + C^{(2)}\rho(a)$$

for any point $a \in D$, *where* $C^{(1)}$ *and* $C^{(2)}$ *are positive constants depending only* on $\underline{A}$, $\bar{A}$, B', Γ, n, d, H *and* α.

Proof. In this case, we have, instead of (4. 13) and (4. 16),

(5. 3) $$\text{(II)} \leqq (2B'\mu_\kappa + \Gamma\rho_\kappa(x^{(0)}))\lambda,$$

(5. 4) $$\text{(III)} \leqq KHr_0{}^\alpha \left\{(2B'\rho_\kappa(x^{(0)})^{-1}\mu_\kappa + \Gamma)r_0 + 4\sqrt{\frac{n}{\alpha}}\,\rho_\kappa(x^{(0)})^{-1}\mu_\kappa\right\},$$

but the estimate (5. 3) can be obtained only by (4. 12) and (4. 14) without the use of Lemma 3.

Hence it follows from (4. 8), (4. 9), (4. 18), (5. 3) and (5. 4), that

$$\begin{aligned} &\sqrt{\underline{A}}\,\rho_\kappa(x^{(0)})^{-1}\mu_\kappa \\ &\leqq \frac{1}{\lambda\rho_\kappa(x^{(0)})} B\left(\frac{1}{2}, \frac{n+1}{2}\right)^{-1} \operatorname*{Osc}_{S(\lambda)}\{u'(x')\} + (2B'\mu_\kappa + \Gamma\rho_\kappa(x^{(0)}))\lambda \\ &\quad + KHr_0{}^\alpha\left\{(2B'\rho_\kappa(x^{(0)})^{-1}\mu_\kappa + \Gamma)r_0 + 4\sqrt{\frac{n}{\alpha}}\,\rho_\kappa(x^{(0)})^{-1}\mu_\kappa\right\}, \end{aligned}$$

and further we have

(5. 5) $$(1-\lambda^\alpha C_1)\mu_\kappa \leqq \frac{1}{\lambda} C_2,$$

where

$$C_1 = \frac{B'd}{\sqrt{\underline{A}}}\lambda^{1-\alpha} + \frac{KHd^\alpha}{\sqrt{\underline{A}}}\left(2B'd\lambda + 4\sqrt{\frac{n}{\alpha}}\right),$$

$$C_2 = \frac{1}{\sqrt{\underline{A}}}\left\{B\left(\frac{1}{2}, \frac{n+1}{2}\right)^{-1} \operatorname*{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\} + \lambda^2\rho(a)^2(1+2KHd^\alpha\lambda^\alpha)\Gamma\right\}.$$

Therefore, taking λ so small that

(5. 6) $$\lambda \leqq \operatorname{Min}\left\{\frac{1}{8\sqrt{\bar{A}}}, \frac{d}{\sqrt{\bar{A}}}\left(\frac{\underline{A}}{2nH}\right)^{1/\alpha}\right\} \text{ and } \lambda^\alpha C_1 \leqq \frac{1}{2},$$

we get

(5. 7) $$\mu_\kappa \leqq \frac{2}{\lambda} C_2,$$

from which it follows along the same lines as in the proof of Theorem of § 4, that

$$|\partial_x u(a)| \leqq C^{(1)}\rho(a)^{-1} \operatorname*{Osc}_{|x-a|\leqq\rho(a)}\{u(x)\} + C^{(2)}\rho(a).$$

COROLLARY 2. *If we suppose that the condition*

(5. 8) $$|f(x, u, p)| \leqq \Gamma$$

is fulfilled instead of the condition (4. 4) *of Theorem of* § 4, *then the condition* (4. 5) *may be omitted and we obtain*

$$|\partial_x u(a)| \leqq K^{(1)} \rho(a)^{-1} \operatorname*{Osc}_{|x-a| \leqq \rho(a)} \{u(x)\} + K^{(2)} \rho(a) \tag{5. 9}$$

for any point $a \in D$, *where* $K^{(1)}$ *and* $K^{(2)}$ *are positive constants depending only on* $\underline{A}$, $\bar{A}$, Γ, n, d, H *and* α.

Proof. In this case, we have in the inequality (5. 5) in the proof of Corollary 1,

$$C_1 = \frac{4KHd^\alpha \sqrt{n}}{\sqrt{\alpha \underline{A}}}$$

and the constant C_2 is the same as in Corollary 1.

Therefore, taking λ so small that

$$\lambda \leqq \operatorname{Min}\left\{\frac{1}{8\sqrt{\bar{A}}},\ \frac{d}{\sqrt{\bar{A}}}\left(\frac{\underline{A}}{2nH}\right)^{1/\alpha},\ \frac{1}{d}\left(\frac{\sqrt{\alpha \underline{A}}}{8KH\sqrt{n}}\right)^{1/\alpha}\right\}, \tag{5. 10}$$

we obtain the inequality (5. 7), from which the estimate (5. 9) follows.

Exactly writing, we get

$$K^{(1)} = \frac{2}{\lambda \sqrt{\underline{A}}} B\left(\frac{1}{2}, \frac{n+1}{2}\right)^{-1} \text{ and } K^{(2)} = 2\lambda(1 + 2KHd^\alpha \lambda^\alpha)\Gamma. \tag{5. 11}$$

ADDENDUM. In the preparation of this paper, the condition (4. 5) in the main theorem of § 4 has been improved by a suggestion of Dr. K. Akō and the author wishes to thank him cordially for his valuable suggestion.

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DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.