PICARD'S THEOREM ON SOME RIEMANN SURFACES

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Dedicated to Professor K. Kunugi on his sixtieth birthday

1. Introduction.

In the present paper we shall establish the Picard theorem on some Riemann surfaces with automorphisms. Here we shall adopt a special method based on the Schottky theorem and the most far-reaching method due to Nevanlinna-Selberg. We shall roughly say that a class of meromorphic functions is exceptional if its any member has unreasonably many exceptional values. This nomenclature has no meaning in some cases when we impose the conditions guaranteeing the presence of an essential singularity or some growth conditions. The most important and well-known example of the exceptional class is that of functions of bounded type in $|z|<1$. Anyhow it is important to determine and to study the exceptional class in the various cases.

In order to investigate and to determine the number of Picard's exceptional values and the exceptional class of functions it is necessary to prove the existence of the fundamental functions in some cases. The functions play an essential role in the respective cases.

We shall make free use of the notations in [4], [6] and [7]. Any quantities in [7] and in [4] are distinguished from those in [6] by the subscripts A and P, respectively. In a way we shall give some remarks on the general value distribution theory, especially on the general defect relation.

Let W be a Riemann surface admitting a conformal transformation group G_n onto itself, which is a free abelian group with n generators $T_1, \cdots, T_n$. Further we assume that W has only one ideal boundary point defined by $\gamma=\lim_{m\to\pm\infty}T_j{}^m p, j=1, \cdots, n$, when $n\geqq 2$ and just two defined by $\gamma_1=\lim_{m\to+\infty}T^m p$ and $\gamma_2=\lim_{m\to+\infty}T^{-m}p$ when $n=1$ and that W is an unramified abelian covering surface of a closed Riemann surface. This class of surfaces is denoted by $\mathfrak{G}_n$. Then $W\in O_G$ if $W\in\mathfrak{G}_n$, $n=1, 2$, and $W\notin O_G$ but $W\in O_{AD}$ if $W\in\mathfrak{G}_n$, $n\geqq 3$ [3].

We shall adopt an exhaustion $\{W_a\}$ of W whose member W_a is the interior of a set defined by

$$\sum_{\sum_{j=1}^n |m_j|\leqq a} \prod_{j=1}^n T_j{}^{m_j} R^*,$$

Received August 17, 1963.

where R^* is a fundamental domain for G. When $W \in \mathfrak{G}_n, n \geqq 2$, all components of ∂W_a can be joined by a suitable number of analytic curves $\{\gamma_a\}$ lying in $W_a - \overline{W}_{a-2}$. Resulting connected collection of curves is denoted by L_a which can be covered by a set of small discs whose number does not exceed Ca^{n-1} for a suitable fixed number C. In the case $n=1$ we need some modifications in the above.

If W is a Riemann surface conformally equivalent to a surface with a finite spherical area, then we say $W \in P_{MD}$. If it is not the case, then we say $W \in O_{MD}$.

2. Existence of the fundamental functions.

We shall prove the following

THEOREM 1. *If there exists a non-constant bounded regular function $f(p)$ on an end Ω of $W \in \mathfrak{G}_2$, then there exists a function $F_0(p)$ in W satisfying the following conditions*:
(i) *$F_0(p)$ maps W conformally onto a d_0-sheeted covering surface $\Phi(W, F_0)$ spread over the punctured sphere $|w|<\infty$, where d_0 is the minimum local degree of γ in Heins' sense* [2],
(ii) *$F_0(p)$ satisfies two functional equations $F_0(T_j p)=F_0(p)+t_j$, $j=1,2$, for two generators T_1, T_2 of G_2 and for two non-zero constants t_1, t_2 satisfying* $\mathrm{Im}(t_1/t_2)\neq 0$.

Proof. By Heins' composition theorem 5.1 in [2] there exists a subend Ω^* of Ω on which there exists a non-constant bounded regular function $f_1(p)$ such that $f_1(p)$ has the minimum local degree d_0 at γ and

$$f(p)=\varphi\circ f_1(p)$$

for some bounded regular function φ in the unit disc $|w|<1$ and $|f_1(p)|\leqq 1$ in Ω^* and $|f_1(p)|=1$ on $\partial\Omega^*$ and $f_1(\gamma)=0$. Then there hold two functional equations

$$f_1(T_j p)=\varphi_j\circ f_1(p), \quad j=1,2,$$

where φ_j, $j=1,2$, are two bounded regular functions univalent in $|w|<1$ and $\varphi_j(0)=0$, $|\varphi_j(w)|\leqq \max_{\Omega^*}|f_1(T_j p)|$. The above composition theorem shows that by any non-constant regular bounded function $f(p)$ in Ω^* d_0 points $p_1, \cdots, p_{d_0}$ lie over the same point $\varphi(w)$, if these points lie in Ω^* and lie over a point w by f_1, that is, $w=f_1(p_1)=\cdots=f_1(p_{d_0})$. Further, then, for each j $T_j p_1, \cdots, T_j p_{d_0}$ lie over the same point $\varphi_j(w)$ by f_1.

Now we shall define an identification map P in Ω^* in such a manner that $Pp_1=\cdots=Pp_{d_0}$ when $f_1(p_1)=\cdots=f_1(p_{d_0})$. Next we shall define the map P in any general part of W in such a manner that $Pp_1=\cdots=Pp_{d_0}$ if $T_j{}^m p_1, \cdots, T_j{}^m p_{d_0}$ lie in Ω^* and these points are identified by P. In this definition the map P is invariant in any choice of m and j. Indeed, we can construct a finite chain of equivalent points of the following form $\Pi T_k{}^{m_k} p_l$ which joins $T_j{}^m p_l$ to $T_j{}^n p_l$ in Ω^*. If $PT_j{}^m p_1=\cdots$

$=PT_j^m p_{d_0}$, then we have

$$f_1(T_j^m p_1)=\cdots=f_1(T_j^m p_{d_0})$$

and hence

$$f_1(\Pi T_k^{m_k} p_1)=\cdots=f_1(\Pi T_k^{m_k} p_{d_0})$$

and finally

$$f_1(T_j^n p_1)=\cdots=f_1(T_j^n p_{d_0}).$$

Invariance of P for j is quite similarly established.

We shall construct an abstract surface $\mathfrak{W}$ in such a manner that $p_1, \cdots, p_{d_0}$ are identified and a new point $\mathfrak{p}$ is defined when $Pp_1=\cdots=Pp_{d_0}$. In this $\mathfrak{W}$ we introduce a notion of neighborhood $\mathfrak{N}(\mathfrak{p})$ of $\mathfrak{p}$ by

$$\mathfrak{N}(\mathfrak{p})=\bigcap_{j=1}^{d_0} PN_j(p_j),$$

where $N_j(p_j)$ is a neighborhood of P_j on W. Local parameter at $\mathfrak{p}$ is introduced as follows: $w-w_0$ if the inverse image P_1 of $\mathfrak{p}$ belongs to Ω^* and $f_1(p_1)=w_0$, and the same $w-w_0$ if $T_j^m P^{-1}\mathfrak{p}\in\Omega^*$ and $PT_j^m P^{-1}\mathfrak{p}=w_0$ for some m and j and for any choice of $P^{-1}\mathfrak{p}$. This choice of the local parameter has meaning. Indeed, if we choose $T_j^n P^{-1}\mathfrak{p}\in\Omega^*$ and $PT_j^n P^{-1}\mathfrak{p}=z_0$, then there is a relation $w=g(z)$ with a suitable univalent regular function g around z_0 satisfying $g(z_0)=w_0$. Therefore we have $w-w_0=g(z)-g(z_0)=\sum_1^\infty a_n(z-z_0)^n$, $a_1\neq 0$. Similarly we can prove the invariance of the local parameter for j and for a choice of $P^{-1}\mathfrak{p}$.

Thus $\mathfrak{W}$ is a Riemann surface on which W lies. Then the identification map P is a standard projection map from W onto $\mathfrak{W}$. This $\mathfrak{W}$ has only one ideal boundary component and admits a free abelian group of rank 2 as a group of automorphisms. Let $Pp_1=\cdots=Pp_{d_0}=\mathfrak{p}$ and $PT_jp_1=\cdots=PT_jp_{d_0}=\mathfrak{p}_1$, then the neighborhoods around $p_1, \cdots, p_{d_0}$ correspond conformally to the neighborhoods around $T_jp_1, \cdots, T_jp_{d_0}$ by T_j sheet by sheet and hence PT_jP^{-1} is conformal. This induced conformal map is denoted by $\mathfrak{T}_j$. Then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ generate a free abelian group $\mathfrak{G}$ of automorphisms on $\mathfrak{W}$.

By the method of construction we can prove that $\mathfrak{W}$ is simply connected. By the uniformization map Ψ $\mathfrak{W}$ can be mapped conformally onto the punctured sphere $\mathfrak{W}^*$: $\{|w^*|<\infty\}$. Then the group $\mathfrak{G}^*$ corresponding to $\mathfrak{G}$ can be represented by

$$\mathfrak{T}_1^*:\ w^*+t_1,\quad \mathfrak{T}_2^*:\ w^*+t_2.$$

Let Ω_0 be an end of W such that the image $\Psi P(\Omega_0)$ of Ω_0 coincides with $|w^*|>1$. Let $F_0(p)$ be ΨPp, then the function $1/F_0(p)$ has the minimum local degree d_0 at γ by its definition and satisfies $|F_0(p)|>1$ in Ω_0, $|F_0(p)|=1$ on $\partial\Omega_0$ and $F_0(\gamma)=\infty$. F_0 can be defined in the whole W. By the definitions of F_0, Ψ, P, $\mathfrak{T}_j$, $\mathfrak{T}_j^*$ and t_j we have

$$\begin{aligned} F_0(T_j p) &= \Psi P T_j p = \Psi P T_j P^{-1} P\ p = \Psi \mathfrak{T}_j P p \\ &= \Psi \mathfrak{T}_j \Psi^{-1} \Psi \mathrm{P} p = \mathfrak{T}_j{}^* \Psi P p \\ &= \Psi P p + t_j = F_0(p) + t_j. \end{aligned}$$

The final fact in (ii) is evident from the group structure and the finiteness of d_0. This is the desired result.

Under the condition $W \in P_{MD}$, Mizumoto announced theorem 1 in [10].

COROLLARY 1. *Let W belong to the class $\mathfrak{G}_2$. A perfect condition for $W \in P_{MD}$ is the existence of a non-constant bounded regular function on an end.*

THEOREM 2. *Let W belong to $\mathfrak{G}_1$. If there exists a non-constant bounded regular function $f(p)$ on an end Ω, then there is a regular function $F_0(p)$ in W for which $F_0(Tp) = tF_0(p)$ remains true for some constant t $(\neq 0, 1)$, Further $F_0(p)$ is bounded in some end and $F_0(\gamma_1) = 0$ and has d_0 sheets over all points with two exceptions $w = 0$ and ∞, where d_0 is the minimum local degree at γ_1.*

Under some suitable modifications we can prove the above theorem 2 similarly which was already formulated in [5].

COROLLARY 2. *Let $W \in \mathfrak{G}_1$. A perfect condition for $W \in P_{MD}$ is the existence of a non-constant bounded regular function on an end.*

M. Tsuji stated the following facts in [9] p. 486: (We follow his notations.)

THEOREM X. 47. (i) *If $r \geqq 3$, then $\Phi \in O_{MD}{}^*$.*
(ii) *If $r = 2$, then there are two cases:*

(a) $G_0 = m_1\omega_1 + m_2\omega_2$, (b) $G_0 = m_1\omega_1 + m_2\omega_1{}^*$.

In case (a), $\Phi \in O_{MD}{}^*$.
In case (b), $\Phi \in P_{MD}{}^*$, *when and only when there exists an abelian integral of the first kind on F, whose periods $\Omega_i, \Omega_i{}^*$ on $C_i, C_i{}^*$ $(i = 1, \cdots, p)$ satisfy the same linear relations as $\omega_i, \omega_i{}^*$.*

Unfortunately there was an erroneous point which was pointed out by L. Sario. (See Math. Rev. **19** (1958), pp. 1043–1044.) In Tsuji's proof of (ii), case (a) there was the same erroneous point. It should be emphasized that this case can be corrected by our theorem 1. Further it is still conjectured, based on various phenomena, that Tsuji's main result, that is, $W \in O_{MD}$, remains true when $W \in \mathfrak{G}_n$, $n \geqq 3$.

Let W belong to $\mathfrak{G}_2 \cap P_{MD}$. Then there exists the fundamental function $F_0(p)$ on W satisfying the conditions in theorem 1. Then the parallelogram R_1 with four vertices 0, $t_1, t_1 + t_2, t_2$ determines a fundamental domain R^* of G_2 in such a manner that R^* is a part of W over the R_1 by the projection map $F_0(p)$. Over the periphery ∂R_1 of R_1 there are at most d_0 closed curves in W, which surrounds R^*. Identifying the corresponding segments of ∂R^*, R^* becomes a closed Riemann surface R $(\equiv W \bmod G_2)$. Let L be a connected image of a part of ∂R^* by the proces $R^* \to R$. Then L is divided into two non-void closed curve classes (M) and (N) called *joint* in later. Each member in (M) intersects with some one in (N)

and vice versa. M and N are not homologous in R. They have the forms $m\alpha$ and $n\beta$ with some integers m and n, respectively. Here α, β are generators of G_2. If Tsuji's case (ii), (a) is the case, then W is obtained from R with two cuts along two non-conjugate non-dividing independent cycles. Thus $W \notin P_{MD}$.

THEOREM 3. *If $W \in \mathfrak{G}_2 \cap P_{MD}$, then the fundamental domain R^* of G_2 is obtained from the closed Riemann surface W* mod *G_2 by cutting along a number of joint systems of closed curves. Especially Tsuji's conclusion for* (ii), (a) *remains true.*

For Tsuji's case (ii), (b), the existence of the fundamental function $F_0(p)$ on W and hence that of an abelian differential dF_0 of the first kind on R play an essential role. Thus to estabilsh a perfect condition for $W \in P_{MD}$ in terms of the Riemann matrix attached to the closed Riemann surface W mod G_2 is now possible.

3. Schottky's method.

Let $w(z)$ be single-valued and regular in the unit disc in which $w(z) \neq 0$, 1, then $w(z)$ satisfies an inequality

$$\overset{+}{\log}|w(z)| \leqq A(1+\overset{+}{\log}|w(0)|)$$

in $|z|<3/4$ with an absolute constant A. See [1], p. 293.

THEOREM 4. *If $f(p)$ is a regular function in $W-\overline{W}_{a_0}$ having two Picard's exceptional values* 0 *and* 1, *then $f(p)$ satisfies either $\overset{+}{\log}|f(p)| \leqq \exp(Ca^{n-1})$ or $\overset{+}{\log}(1/|f(p)|) \leqq \exp(Ca^{n-1})$ in a domain $W_a-\overline{W}_{a_1}$ for a suitable constant C. The same conclusion holds for two functions $f(p)-1$ and $f(p)/(f(p)-1)$.*

Proof. We may assume that $f(p) \neq 0, 1$ for a suitable subend $W-\overline{W}_{a_1}$. In the first place we shall consider a case such that there exists a sequence of $\{\partial W_a\}$ on which there exist two points p_a, q_a such that $|f(p_a)|<\varepsilon_n$, $|f(q_a)|>1/\varepsilon_n$ for a sequence ε_n satisfying $\varepsilon_n \to 0$ when $n \to \infty$. Then there is a point r_a on any L_a such that $|f(r_a)|=1$. Then by the above sharpend Schottky theorem we have

$$\log|f(p)| \leqq \exp(C'a^{n-1}) \text{ and } \log(1/|f(p)|) \leqq \exp(C'a^{n-1})$$

on $\{\partial W_a\}$ and hence on $W_a-\overline{W}_{a_1}$. In the second place, if either $|f(p)|$ or $1/|f(p)|$ is bounded on $\{\partial W_a\}$, then the same holds for $W_a-\overline{W}_{a_1}$. Thus we have the desired result, which may be considered as a sort of Picard's theorem.

We shall study more precisely the case $n=1$. Let $f(p)$ be a meromorphic function on an end Ω of W. If $\overline{f(\Omega_0)}$ is not the extended plane for some subend Ω_0 of Ω, then we say $f \in AeB(\Omega)$. If there is no non-constant meromorphic function $f(p)$ in any end Ω in which $f(p)$ has at least $n+1$ Picard's exceptional values, then we say that W is of strong n-Picard type. If any non-constant meromorphic function $f(p)$ in Ω which admits at least $n+1$ Picard's exceptional values belongs to the class $AeB(\Omega)$ for some end Ω, then we say that W is of n-Picard type. Ordinary big Picard theorem shows that the finite plane is of 2-Picard type. Heins [2] gave an example of Riemann surface of strong 2-Picard type.

We shall prove the following

THEOREM 5. *Let W belong to the class $\mathfrak{S}_1$. If $W \in P_{MD}$, then W is of 2-Picard type. If $W \in O_{MD}$, then W is of strong 2-Picard type.*

Proof. Let R^* be a fundamental domain for G. Let $f(p)$ be a non-constant meromorphic function in an end Ω which takes a value infinitely often but takes three values $0, 1$ and ∞ only finitely often, then we may assume that $f(p)$ does not take three values $0, 1$ and ∞ in Ω and further $\overline{f(\Omega)}$ is the extended plane by Heins' theorem 4.2 in [2]. Then by a similar argument as in theorem 4 we arrive at a contradiction.

Let $f(p)$ be a non-constant meromorphic function in Ω which takes every value finitely often, then by Heins' theorems 4.1 and 4.2 in [2] we may assume that $\overline{f(\Omega_0)}$ is not the extended plane for some subend Ω_0. Thus $f \in AeB(\Omega)$. Further we may assume that $f(p)$ is bounded and regular in a subend Ω_0 of Ω. Then by theorem 2 $W \in P_{MD}$.

If $W \in P_{MD}$, there is a rational function φ for any meromorphic function $f(p)$ with a finite spherical area such that $f(p) = \varphi \circ F_0(p)$ by the fundamental function $F_0(p)$. $F_0(p)$ excludes evidently three different values in any end and $F_0(p)$ is regular in the whole W. $F_0(p)$ maps W conformally onto a finitely-sheeted covering surface $\Phi(W, F_0)$ spread over the punctured sphere $0 < |w| < \infty$. Thus W is of 2-Picard type. Thus we have the desired result.

In [5] we established a perfect condition by using the periodicity moduli of the abelian differentials of the first kind of the closed surface R ($\equiv W \bmod G_1$) in order that $W \in P_{MD}$. By this theorem we can roughly say that almost all W in $\mathfrak{S}_1$ belong to the class O_{MD}, when the genus of R is not less than 2, and hence to the strong 2-Picard type. By theorem 5 the exceptional class is completely determined.

4. Nevanlinna-Selberg's method.

In the case $W \in \mathfrak{S}_2$ we cannot obtain a perfect result on the number P of Picard's exceptional values by the Schottky method. When $W \in \mathfrak{S}_2 \cap P_{MD}$, we can obtain a somewhat precise information for P by the far-reaching result due to Nevanlinna and to Selberg [4], [7].

Let $f(p)$ be single-valued and meromorphic on W. Then $F(w)$, defined by $f \circ F_0^{-1}(w)$, is at most d_0-valued in the punctured disc $r_0 \leqq |w| < \infty$. Then we have

$$P \leqq 2d_0$$

in any cases with the exception of at most d_0-valued algebraic functions over the disc $r_0 \leqq |w|$.

When $W\in\mathfrak{S}_1\cap P_{MD}$, we can say by Schottky's method that $P\leqq 2$. However by Nevanlinna-Selberg's method we cannot yet say that $P\leqq 2$. For the method we have $P\leqq 2d_0$. It is not known that $P\leqq 2d_0$ is the best possible inequality in the case $W\in\mathfrak{S}_2\cap P_{MD}$. It is conjectured that $P\leqq 2$.

5. Remarks on the value distribution theory.

We do not aim at to establish any general value distribution theory, but we intend to study an interesting phenomenon arising in the theory. The so-called defect relation and the number of Picard's exceptional values are our problem. We shall show the following fact: In a special type of Riemann surfaces we cannot derive any effective conclusions for any non-trivial functions of slow growth from the general defect relation, however the special theory gives some effective conclusions.

Let $W\in\mathfrak{S}_2\cap P_{MD}$. By the fundamental function $F_0(p)$ we can determine a conformal metric

$$ds\equiv\lambda(z)|dz|=\frac{1}{2\pi d_0}|\mathrm{grad}_z u||dz|=\frac{1}{2\pi d_0}\ \frac{|dF_0|}{|F_0|}$$

with $u(p)+iv(p)=\log F_0(p)$. This metric satisfies the following conditions: $\lambda(z)$ is non-negative and continuous with no points of accumulation of its zeros and further

$$\int_{L_\sigma} ds=1,$$

where L_σ is the level line $u=\sigma$. Let W_k be the domain satisfying $u<k$. Then the distance $d(p, W_k)$ between p and W_k tends to ∞ when p tends to γ. Further $\log\lambda$ is harmonic except for logarithmic poles. Thus we can apply Sario's formulation of the second fundamental theorem [6]: Let f be a meromorphic function on W. Then

$$\sum_1^q m(\sigma, a_j)<2T(\sigma)-N_1(\sigma)+N(\sigma, 1/\lambda)+O(\sigma+\log T(\sigma)) \tag{1}$$

with some exceptional intervals Δ with

$$\int_\Delta e^{\alpha\sigma}d\sigma<\infty$$

for $\alpha\geqq 0$.

Now we shall examine the various quantities in the above general second fundamental theorem more precisely.

Evidently $dF_0(p)$ is an abelian differential of the first kind on the closed

Riemann surface $R(\equiv W \bmod G_2)$, and hence the sum of orders of its zeros in R is equal to $2g-2$, where g is the genus of R. We may assume that $g>1$ and $d_0>1$. Thus by definition we have asymptotically

$$n(\sigma, 1/\lambda)=(2g-2)\frac{\pi e^{2\sigma}}{A_0}+o(e^{2\sigma}),$$

where A_0 is the area of a fundamental parallelogram. Therefore, if

$$(2) \qquad \varlimsup_{\sigma\to\infty}\frac{e^{2\sigma}}{T(\sigma)}=+\infty, \qquad \varlimsup_{\sigma\to\infty}\frac{T(\sigma)}{\sigma}=+\infty,$$

then there holds

$$\varlimsup_{\sigma\to\infty}\frac{N(\sigma, 1/\lambda)}{T(\sigma)}=\infty.$$

Let $e(\sigma)$ be the Euler characteristic of W_σ, then we have $n(\sigma, 1/\lambda)=e(\sigma)-d_0$. Therefore the integrated Euler characteristic of W_σ satisfies

$$E(\sigma)=\int_0^\sigma e(\sigma)d\sigma=N(\sigma, 1/\lambda)+O(\sigma).$$

This shows that

$$\eta=\varlimsup_{\sigma\to\infty}\frac{E(\sigma)}{T(\sigma)}=\varlimsup_{\sigma\to\infty}\frac{N(\sigma, 1/\lambda)}{T(\sigma)}=\infty.$$

Under the condition (2) the general defect relation

$$\sum_{(a)}\delta(a)\leqq 2+\eta$$

gives no effective conclusions for the number of defect values and the number P.

When $T(\sigma)/\sigma\leqq M<\infty$, the spherical area of the image $f(W)$ of W by f is of finite value and hence the covering surface $\Phi(W,f)$ is finitely-sheeted [2] and further $f(p)=F\circ F_0(p)$ with a suitable rational function $F(w)$ of w by theorem 1 This class of functions was already called exceptional.

We shall now prove that there is surely a family of non-exceptional meromorphic functions satisfying (2) and a concrete representation of any functions satisfying (2) can be established. Let $f(p)$ be such a function on W. Let $F(w)$ be a function defined by $f\circ F_0^{-1}(w)$, then $F(w)$ is at most d_0-valued function on $|w|<\infty$ Any fundamental symmetric polynomials of all branches of $F(w)$ are single-valued and meromorphic in $|w|<\infty$. Thus F satisfies an algebraic equation

$$F^{d_0}+A_1F^{d_0-1}+\cdots+A_{d_0}=0$$

with the single-valued meromorphic coefficients A_j in $|w|<\infty$. By the definitions of $T(\sigma)$ and $T_A(\sigma, F)$ we have

$$T(\sigma)=N(\sigma,f)+m(\sigma,f)+O(\sigma)$$
$$=N_A(\sigma, F)+m_A(\sigma, F)+O(\sigma)=T_A(\sigma, F)+O(\sigma).$$

Here σ corresponds to $\log r$, $w=re^{i\theta}$. Therefore any coefficients satisfy

$$T_A(\sigma, A_j)=o(e^{2\sigma}).$$

Let the jth and the kth sheets of $\Phi(W, F_0)$ have at least one common branch point in a fundamental parallelogram and hence in any fundamental parallelogram. Let G be the difference of two branches F_j and F_k of F, then by Selberg's theory on algebroid functions

$$T_A(\sigma, G)\leqq 2T_A(\sigma, F)+O(1)$$
$$\leqq\frac{2}{d_0}\sum_{j=1}^{d_0} T_A(\sigma, A_j)+O(1)=o(e^{2\sigma})$$

and

$$T_A(\sigma, G)=T_A(\sigma, 1/G)+O(\sigma)$$
$$\geqq N_A(\sigma, 1/G)+O(\sigma)\geqq\pi\frac{e^{2\sigma}}{A_0}.$$

This is a contradiction, unless $G\equiv$const. and hence $G\equiv 0$. Therefore $F_j\equiv F_k$. Repeating this process we can say that each branch coincides with any other and hence F must satisfy the single-valuedness in $|w|<\infty$. Therefore $f(p)$ can be represented in the form

$$F\circ F_0(p)$$

with a single-valued meromorphic function $F(w)$ in $|w|<\infty$ satisfying $T_P(\sigma, F)=o(e^{2\sigma})$. Thus we have the desired result.

Let $f(p)$ be any meromorphic function on W with the representation

$$f(p)=F\circ F_0(p)$$

by a suitable single-valued meromorphic function F in $|w|<\infty$. Then we have

$$f_z=F_w\cdot F_{0z}$$

and

$$n_1(\sigma)=n(\sigma, 1/f_z)+2n(\sigma,f)-n(\sigma,f_z).$$

Thus we have

$$n_1(\sigma)=n(\sigma, 1/F_{0z})+d_0(n_P(\sigma, 1/F_w)+2n_P(\sigma, F)-n_P(\sigma, F_w))$$
$$=n(\sigma, 1/\lambda)+d_0 n_{1P}(\sigma).$$

If (2) holds, then Sario's ϑ is in general also of infinite value:

$$\vartheta=\varlimsup_{\sigma\to\infty} \frac{N_1(\sigma)}{T(\sigma)}=\infty.$$

Fortunately the causes of the above insignificance, that is, $\eta=\infty$, and $\vartheta=\infty$, are lying in both sides of (1) with the same term, and hence we can modify (1) to a more effective form. In general we have

$$(1_P) \qquad \sum m_P(\sigma, a_j)\leqq 2T_P(\sigma, F)-N_{1P}(\sigma)+O(\sigma+\log T_P(\sigma, F))$$

with some exceptional intervals. This is nothing but the Nevanlinna second fundamental theorem in $|w|<\infty$. Then we can draw the legitimate conclusions: the defect relation, the ramification relation and so on.

Further we can conclude the following fact: Let $f(p)$ be a non-exceptional meromorphic function on W satisfying

$$\varlimsup_{\sigma\to\infty}\frac{\log T(\sigma)}{\sigma}=\rho<2,$$

then the number P of Picard's exceptional values is at most one, when $\rho\neq 1$. Indeed, by the condition on the order of $f(p)$ we can apply the rigidity of projection map stated already and we have the representation $f(p)=F\circ F_0(p)$ with a single-valued meromorphic function F in $|w|<\infty$ which has the same order. Then by the classical theory we have the desired result.

Let $f(p)$ be represented by $F\circ F_0(p)$ with a d_0-valued algebroid function F in the whole plane. Then (1) is reduced to the second fundamental theorem on algebroid functions formulated by Selberg [7]. Indeed, we have

$$N_A(\sigma;\ \mathfrak{X})=(g-1)\frac{\pi e^{2\sigma}}{A_0}+o(e^{2\sigma})=N\left(\sigma, \frac{1}{\lambda}\right)+O(\sigma),$$

$$T_A(\sigma;\ F)=T(\sigma)+O(\sigma),$$

$$N_A(\sigma;\ \mathfrak{Z}_F)-N_A(\sigma;\ \mathfrak{X})=N_{1A}(\sigma;\ F),$$

$$N_1(\sigma)=N\left(\sigma, \frac{1}{\lambda}\right)+N_{1A}(\sigma;\ F)$$

and the ramification theorem

$$N_A(\sigma;\ \mathfrak{X})\leqq(2d_0-2)T_A(\sigma;\ F)+O(1).$$

By these relations we have

$$(1_A) \qquad \sum_{1}^{q} m_A(\sigma;\ a_j) < 2T_A(\sigma;\ F) - N_A(\sigma;\ \mathfrak{Z}_F) + N_A(\sigma;\ \mathfrak{X}) + O(\sigma + \log T_A(\sigma;\ F)).$$

This is nothing but the second fundamental theorem on algebroid functions.

We can summarize our earlier results in the following

THEOREM 6. *With the exception of arbitrary exceptional functions the number P of Picard's exceptional values of a meromorphic function $f(p)$ on $W \in \mathfrak{G}_2 \cap P_{MD}$ satisfies the following inequalities:*

$$P \leqq \begin{cases} 2 & \text{if} \quad \rho > 2, \\ 2d_0 & \text{if} \quad \rho = 2, \\ 2 & \text{if} \quad \rho = 1, \\ 1 & \text{if} \quad \rho < 2, \ \rho \neq 1, \end{cases}$$

where ρ is the order of f. Any exceptional function is representable as a composite function $F \circ F_0(p)$ with a suitable rational function F.

As a byproduct we have the following somewhat curious inequality: For any d_0-valued algebroid function $F(w)$ whose proper existence domain is $\Phi(W, F_0)$ there holds

$$\varliminf_{\sigma \to \infty} \frac{T(\sigma)}{e^{2\sigma}} \geqq \frac{(g-1)\pi}{(2d_0 - 2)A_0}.$$

In some places we have stated several unsolved problems. Further we shall state an important problem, which seems very difficult to settle.

It is not known what class plays the exceptional role in the cases $W \in \mathfrak{G}_n$, $n \geqq 3$ and $W \in \mathfrak{G}_2 \cap O_{MD}$.

We can construct a Riemann surface on which $\eta = +\infty$ holds for any meromorphic functions f of finite order or of finite hyperorder. Then no effective conclusion from the general defect relation can also be drawn. However the presence of the ramification theorem and the rigidity property of projection map for any functions of lower order help us out of some senselessness. Then the Nevanlinna theory and the Selberg theory play an essential role and give somewhat significant conclusions. It is not known whether there is another type of Riemann surfaces for which $\eta = +\infty$ without any inevitability for the function excepting some trivial functions. The unit disc does not belong to this type.

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