A GEOMETRIC CONDITION FOR SMOOTHABILITY OF COMBINATORIAL MANIFOLDS

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§1. Introduction.

Let us commence with the terminology. For a complex Y, $|Y|$ will denote the polyhedron covered by Y and Y' will stand for the first barycentric subdivision of Y. We say that a subcomplex X of Y is *complete* if the intersection of a (closed) simplex of Y and $|X|$ is either empty or a simplex of X. A combinatorial manifold is a polyhedron with a distinguished class of simplicial subdivisions which are formal manifolds, [5, p. 825]. For a combinatorial manifold P, the boundary of P is written ∂P and the interior $P-\partial P$ is written Int P, and a closed combinatorial manifold will be a compact combinatorial manifold without boundary. Let X be a subcomplex of Y where $|Y|$ is a combinatorial manifold. (Note that X' is a complete subcomplex of Y'.) Then $N(X, Y)$ denotes the *star neighborhood* of X in Y, that is, the polyhedron consists of simplices of Y, which contain simplices of X.

It is well known that $\partial N(K', L')$ (that is, the boundary of the star neighborhood of the first barycentric subdivision of K in the first barycentric subdivision of L) is a closed combinatorial $(m-1)$-manifold if the polyhedron $|L|$ is a combinatorial m-manifold without boundary and K is a finite complete subcomplex of L; [4, p. 293].

For convenience, we say that a polyhedron Q is imbedded *piecewise linearly* in euclidean space R if there are (linear) simplicial subdivisions X and L of Q and R respectively such that X is a subcomplex of L, where it may be assumed without loss of generality that X is a complete subcomplex of L.

Now let us explain the condition for smoothability.

Definition 1. Let M be a closed combinatorial n-manifold imbedded piecewise linearly in euclidean $(n+r)$-space R, $r \geqq 1$. We say that M is in *smoothable position* in R if the following is satisfied. Let K_0 and L_0 be simplicial subdivisions of M and R respectively, where K_0 is a complete subcomplex of L_0. Then there exist piecewise linear homeomorphisms ρ_i: $M_i \rightarrow \partial N(K_i', L_i')$ for each $0 \leqq i \leqq r-1$, where $M_0 = M$ and for $1 \leqq i \leqq r$, $M_i = \rho_{i-1}(M_{i-1})$ and where K_i and L_i are simplicial subdivisions of M_i and $\partial N(K_{i-1}', L_{i-1}')$ respectively such that K_i is a complete subcomplex of L_i. In the text, however, W_i stands for $\partial N(K_{i-1}', L_{i-1}')$ and L_i will be the subcomplex of L_{i-1}' covering W_i for each $1 \leqq i \leqq r$.

Note that M_i is a closed combinatorial n-manifold, which is combinatorially equivalent to M, and W_i is a closed combinatorial $(n+r-i)$-manifold, for each $1 \leqq i \leqq r$, satisfying $M_i \subset W_i$ and $W_1 \supset W_2 \supset \cdots \supset W_r$.

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The result of this paper is the following:

THEOREM A. *If a closed combinatorial n-manifold M is in smoothable position in $(n+r)$-space R, $r \geqq 1$, then M is smoothable.*[1]

Theorem A has the following implications. After Whitehead [5, p. 827] a combinatorial *n-manifold M* is a *π-manifold* if a regular neighborhood $U(M, R)$ of M in $(n+r)$-space R is combinatorially equivalent to the product of M and a combinatorial r-cell C^r, written $U(M, R) \equiv M \times C^r$, for large values of r. Now suppose that a combinatorial π-manifold M be closed. Then, since the star neighborhood $N(K_0', L_0')$ is a regular neighborhood, $N(K_0', L_0') \equiv M \times C^r$ and there is a piecewise linear homeomorphism ρ_0: $M_0 \to \partial N(K_0', L_0') \equiv M \times S^{r-1}$ defined by taking $\rho_0(x) = (x, x_0)$ for $x \in M_0$ where $S^{r-1} = \partial C^r$ is a combinatorial $(r-1)$-sphere and x_0 is a fixed point of S^{r-1}. Let C^{r-1} be a combinatorial $(r-1)$-cell of S^{r-1} containing x_0 in the interior. It is clear that $M \times C^{r-1}$ is a regular neighborhood of $M \times \{x_0\}$ $(=M_1)$ in $M \times S^{r-1} (=W_1)$. Then $N(K_1', L_1') \equiv M \times C^{r-1}$ by [4, Theorem 23], and there is a piecewise linear homeomorphism ρ_1: $M_1 \to \partial N(K_1', L_1') \equiv M \times S^{r-2}$ defined by taking $\rho_1(y) = (y, y_0)$ for $y \in M_1$ where $S^{r-2} = \partial C^{r-1}$ and y_0 is a fixed point of S^{r-2}, and so on. Hence M is in smoothable position in R, and therefore, by Theorem A,

THEOREM B. *Closed combinatorial π-manifolds are smoothable.*[2]

By [3, p. 619], Theorem 2 obtained by the second named author [2, p. 214] is a consequence of the result. That is,

COROLLARY. *If a closed combinatorial n-manifold M is imbedded piecewise linearly in $(n+1)$-space R, then M is smoothable.*

(Note that the Schoenflies conjecture is not required.)

To see the way to get Theorem A we have to have more terminology. For convenience, eliminating the restriction on the dimensions of sets questioned we alter the definition of [1, p. 52] as follows:

DEFINITION 2. Consider a set X in euclidean space R. An i-plane (an i-dimensional hyperplane) will be called *transverse to X* if it makes angles bounded away from zero with the secant lines of X. Let x be a point of X. An i-plane will be called *transverse to X at x* if it is transverse to some neighborhood of x on X. A set of i-planes is called a *transverse i-plane field* over X if for each point x of X there corresponds continuously an i-plane of the set, which is transverse to X at x.

1) Theorem A was announced at the meeting on Differential Topology, Kyoto, October 1961.

2) The authors were informed when this paper was completed that Theorem B was proved by J. Milnor, Microbundles and differentiable structures, Princeton University, 1961 (mimeographed).

Suppose that a closed combinatorial n-manifold M is in smoothable position in R. Using Definition 1, M_r is combinatorially equivalent to M. Therefore Theorem A follows from Theorem C below in accordance with Theorem 2. 4 of Cairns; see [1, p. 53] or [6, p. 159].

THEOREM C. *Let a closed combinatorial n-manifold M be in smoothable position in euclidean $(n+r)$-space R, $r\geqq 1$. Then M_r admits a transverse r-plane field over M_r.*

In fact Theorem C is a generalization of Theorem 1 of [2, p. 214] and the establishment of Theorem C is the purpose of the paper. The method used here is so simple that the Schoenflies conjecture needed in the previous paper is no longer required in this paper.

§ 2. Lemmas.

For distinct points x and y of euclidean space R, $\overleftrightarrow{xy}$ denotes the line through x and y. Let X and Y be subsets of R, the join of X and Y will be denoted by $X*Y$. The parallelism between planes and lines will be denoted by the symbol // between them.

Using Definition 2, Remark 1 of [2, p. 205] may be restated as follows:

LEMMA 1. *Let Q be a polyhedron imbedded piecewise linearly in euclidean space R and let K be a simplicial subdivision of Q. If an i-plane P^i through a point x of Q is not transverse to Q at x, then there are points s, t of* St(x, K) *(that is, the star of x in K) arbitrarily near x such that $\overleftrightarrow{st}$ // P^i.*

Let a closed combinatorial n-manifold M be in smoothable position in $(n+r)$-space R, $r\geqq 1$. Then by Definition 1, there exist piecewise linear homeomorphisms ρ_i: $M_i\to W_{i+1}$ where i ranges $0\leqq i\leqq r-1$. Recall that $|K_i|=M_i$, $|L_{i+1}|=W_{i+1}=\partial N(K_i', L_i')$ and L_{i+1} is a subcomplex of L_i'.

It is well known that for each point x of W_{i+1} any simplex α of L_i containing x has the dimension greater than 0, and the intersection $\alpha\cap M_i$ is a non-empty proper subset of α; see [4, p. 294]. Since K_i is a complete subcomplex of L_i, the intersection $\alpha\cap M_i$ is a simplex, say β, of K_i, which is a non-empty proper face of α. By γ we denote the non-empty proper face of α which is the face opposite β in α.

Let $v_0, v_1, \cdots, v_q$ be the vertices of α, $\alpha=v_0*v_1*\cdots*v_q$. Then it may be assumed that $\beta=v_0*\cdots*v_e$ and $\gamma=v_{e+1}*\cdots*v_q$. By $(a_0, a_1, \cdots, a_q)$ we denote the barycentric coordinates of x with respect to α. Then the points $y(x)$ and $z(x)$ of α are defined by x such that the barycentric coordinates of $y(x)$ and $z(x)$ with respect to α are

$$\left(\frac{a_0}{\sum_{i=0}^{e} a_i}, \cdots, \frac{a_e}{\sum_{i=0}^{e} a_i}, 0, \cdots, 0\right) \quad \text{and} \quad \left(0, \cdots, 0, \frac{a_{e+1}}{\sum_{i=e+1}^{q} a_i}, \cdots, \frac{a_q}{\sum_{i=e+1}^{q} a_i}\right)$$

respectively. By the definition we immediately see that $y(x)$ and $z(x)$ are contained in the simplices β and γ respectively, and x is contained in the interior of the join $y(x)*z(x)$. There may be another simplex α_1 of L_i containing the point x of W_{i+1}. Then, using α_1 instead of α, we have the points $y_1(x)$ and $z_1(x)$ for x. However, it is trivial to check that $y_1(x)$ and $z_1(x)$ are $y(x)$ and $z(x)$ previously determined by α respectively. Therefore the points $y(x)$ and $z(x)$ are well defined for each point x of W_{i+1}.

Since the points $y(x)$ and $z(x)$ vary continuously if x ranges over $\alpha\cap W_{i+1}$, we deduce the following:

LEMMA 2. *The set of the lines* $\overleftrightarrow{y(x)z(x)}$, *where* x *ranges over* W_{i+1}, *is a continuous line field over* W_{i+1}.

LEMMA 3. *Let* x *be a point of* W_{i+1} *and let* s *be a point of* $\mathrm{St}(x, L_{i+1})$. *Then the intersection* $\mathrm{Int}(s*z(x))\cap W_{i+1}$ *is empty.*

Proof. Let $e_0*\cdots*e_{q-1}$ be a $(q-1)$-simplex of L_{i+1} containing x and s where $q=n+r-i$ and where e_j is the barycenter of the $(j+1)$-simplex σ^{j+1} of L_i such that $\sigma^1\subset\sigma^2\subset\cdots\subset\sigma^q=\alpha$; and furthermore, one of the vertices of σ^1 is contained in β and the other is contained in γ (for α, β and γ see above); see [4, p. 294].

Let y_j be the barycenter of the simplex $\sigma^{j+1}\cap M_i$, and let z_j be the barycenter of the face opposite $\sigma^{j+1}\cap M_i$ in σ^{j+1}. It is easily verified that the points $y(x)$ and $z(x)$ are in $y_0*\cdots*y_{q-1}$ $(\subset\beta)$ and $z_0*\cdots*z_{q-1}$ $(\subset\gamma)$ respectively, where these joins may be singular. Since y_j, z_j and e_j are collinear, $(e_0*\cdots*e_{q-1})*(z_0*\cdots*z_{q-1})$ is contained in $(y_0*\cdots*y_{q-1})*(z_0*\cdots*z_{q-1})$. Since s is contained in $e_0*\cdots*e_{q-1}$, $W_{i+1}\cap(s*z(x))$ is contained in $W_{i+1}\cap(y_0*\cdots*y_{q-1}*z_0*\cdots*z_{q-1})$. From [4, p. 294], it is immediately seen that $W_{i+1}\cap(y_0*\cdots*y_{q-1}*z_0*\cdots*z_{q-1})$ is contained in the cell δ dual to σ^1 in α. Using the barycentric coordinate system with respect to α, it is calculated that the intersection of the dual cell δ and the join $s*z(x)$ is the point s, and then $W_{i+1}\cap(s*z(x))=s$. This completes the proof.

LEMMA 4. *The set of lines* $\overleftrightarrow{y(x)z(x)}$ *obtained in Lemma* 2 *is a transverse line field over* W_{i+1} *for each* $0\leqq i\leqq r-1$.

Proof. Suppose that $\overleftrightarrow{y(x)z(x)}$ is not transverse to W_{i+1} at x. Then there are points s, t of $\mathrm{St}(x, L_{i+1})$, such that $\overleftrightarrow{st}\,//\,\overleftrightarrow{y(x)z(x)}$ by Lemma 1. Since $\overleftrightarrow{st}\,//\,\overleftrightarrow{y(x)z(x)}$, the points x, $y(x)$, $z(x)$, s and t are in a plane (or a line). Since $\overleftrightarrow{st}\,//\,\overleftrightarrow{y(x)z(x)}$ and the segment $y(x)*z(x)$ contains x in the interior, it is seen that the intersection $x*t\cap\mathrm{Int}(s*z(x))$ is not empty, where s, t may be replaced by t, s respectively if necessary. Since $t\in\mathrm{St}(x, L_{i+1})$ and $|L_{i+1}|=W_{i+1}$, $x*t$ is contained in W_{i+1}. Therefore the intersection $\mathrm{Int}(s*z(x))\cap W_{i+1}$ is not empty. This contradicts Lemma 3. Therefore the line $\overleftrightarrow{y(x)z(x)}$ is transverse to W_{i+1} at x. Lemma 2 now completes the proof of Lemma 4.

§ 3. Proof of Theorem C.

The line $\overleftrightarrow{y(x)z(x)}$ and the segment $y(x)*z(x)$ and a simplex α of L_{i-1} containing x obtained in § 2 for each x of W_i will be written $l_i(x)$, $s_i(x)$ and α_i respectively where i ranges over $1 \leqq i \leqq r$ rather than $0 \leqq i \leqq r-1$. In particular, let x be a point of M_r, x is a point of all W_i $(i=1, 2, \cdots, r)$. So there exist lines $l_i(x)$ for all $i=1, \cdots, r$. First we shall prove that the lines $l_1(x), \cdots, l_r(x)$ are linearly independent in R. Since L_i is a subcomplex of L_{i-1}', there is a simplex α_i of L_{i-1} containing x for each simplex α_{i+1} of L_i containing x such that $\alpha_{i+1} \subset \alpha_i$. And since the segment $s_i(x)$ is contained in α_i, the join $s_r(x)*\cdots*s_{j+1}(x)$ is contained in $\mathrm{St}(x, L_j)$ for each $1 \leqq j \leqq r-1$. Since, by Lemma 4, $l_j(x)$ is transverse to $\mathrm{St}(x, L_j)$, the line $l_j(x)$ is linearly independent of the plane which is spanned by $l_r(x), l_{r-1}(x), \cdots, l_{j+1}(x)$ for each $1 \leqq j \leqq r-1$. Hence the lines $l_1(x), \cdots, l_r(x)$ are linearly independent in R.

Let $P^i(x)$ be the i-plane spanned by $l_1(x), \cdots, l_i(x)$. Next we shall show that the set of r-planes $P^r(x)$, where x ranges over M_r, is the required field. Since each $l_i(x)$ varies continuously if x ranges over W_i, the set of r-planes $P^r(x)$ is a continuous field over M_r. On the other hand, by Lemma 4, $P^1(x)$ is transverse to $\mathrm{St}(x, L_1)$. Then, using induction on i, Theorem C will follow if it be shown that $P^i(x)$ is transverse to $\mathrm{St}(x, L_i)$ provided that $P^{i-1}(x)$ is transverse to $\mathrm{St}(x, L_{i-1})$. Since $P^i(x)$ is spanned by $l_i(x)$ and $P^{i-1}(x)$, any line on $P^i(x)$ is parallel to a line $\overleftrightarrow{yz}$ where $y \in s_i(x)$ and $z \in P^{i-1}(x)$. Suppose that $P^i(x)$ is not transverse to $\mathrm{St}(x, L_i)$. Then there are points s and t of $\mathrm{St}(x, L_i)$ such that $\overleftrightarrow{st} \,//\, \overleftrightarrow{yz}$ where $y \in s_i(x)$ and $z \in P^{i-1}(x)$, by Lemma 1. We may choose s, t, y, z as vertices of a parallelogram (it may be degenerate). Let z_1, x_1 be the midpoints of the segments $z*t$, $x*t$ respectively. Then $\overleftrightarrow{z_1x_1} \,//\, \overleftrightarrow{zx}$ and z_1 is the midpoint of the segment $y*s$. Let α_i be a simplex of L_{i-1} containing $x*s$. Then, since $s_i(x) \subset \alpha_i$, $s*s_i(x) \subset \alpha_i \subset \mathrm{St}(x, L_{i-1})$. Since $x_1 \in x*t \subset \mathrm{St}(x, L_{i-1})$ and $z_1 \in y*s \subset s_i(x)*s \subset \mathrm{St}(x, L_{i-1})$, $\overleftrightarrow{zx} \,//\, \overleftrightarrow{z_1x_1}$ implies that $P^{i-1}(x)$ is not transverse to $\mathrm{St}(x, L_{i-1})$.

This contradiction completes the proof.

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