ON BASIC CURVES IN SPACES WITH NORMAL GENERAL CONNECTIONS

By Tominosuke Ōtsuki

In a previous paper [8], the author investigated the developments of curves in spaces with normal general connections into pseudo-affine spaces which are generalized from the affine spaces. In this paper, he will show that the developments of curves called basic in such spaces have special properties which are important for the study of curvatures of these spaces.

§1. Basic curves and their developments.

Making use of the notations in [8], [10] and [11], let $\mathfrak{X}$ be an n-dimensional differentiable manifold with a normal general connection Γ which is written in terms of local coordinates u^i as

$$\Gamma = \partial u_j \otimes (P_i^j d^2 u^i + \Gamma_{ih}^j du^i \otimes du^h) \tag{1.1}$$

and $\operatorname{rank}(P_i^j) = m$.

A development of a curve C: $u^i = u^i(t)$ in $\mathfrak{X}$ is the curve $\bar{C}$: $x^\lambda = x^\lambda(t)$ in R^n given by the solution of the system of equations:

$$\begin{cases} \dfrac{dx^\lambda}{dt} = Y_i^\lambda \dfrac{du^i}{dt}, \\ P_j^i \dfrac{DX_\lambda^j}{dt} = P_j^i\left(P_k^j \dfrac{dX_\lambda^k}{dt} + \Gamma_{ih}^j X_\lambda^i \dfrac{du^h}{dt}\right) = 0, \end{cases} \tag{1.2}$$

where $X_\lambda^i Y_i^\mu = \delta_\lambda^\mu$.[1]

Let Q_i^j, A_i^j and N_i^j be the local components of the tensors Q, A and N in [10], respectively. Let V_λ^j and U_i^λ be the components of the contravariant vectors V_λ and the covariant vectors U^λ in [10], such that

$$A_i^j V_\alpha^i = V_\alpha^j, \qquad A_i^j V_B^i = 0, \qquad \alpha = 1, 2, \cdots, m,$$
$$A_i^j U_j^\alpha = U_i^\alpha, \qquad A_i^j U_j^B = 0, \qquad B = m+1, \cdots, n\,[2]$$

and $V_\lambda^i U_i^\mu = \delta_\lambda^\mu$.

Putting $X_\lambda^i = V_\mu^i \xi_\lambda^\mu$, $Y_\mu^\lambda = \eta_\mu^\lambda U_i^\mu$, the second part of (1.2) is equivalent to

$$\frac{d\xi_\lambda^\mu}{dt} + K_\nu^\mu \xi_\lambda^\nu = 0, \tag{1.3}$$

Received April 12, 1962.

1) See [8], §3.

2) The indices run as follows: $\alpha, \beta, \gamma, \cdots = 1, 2, \cdots, m$; $B, C, D, \cdots = m+1, \cdots, n$.

where

$$K_\lambda^\alpha = U_k^\alpha Q_j^k \frac{DV_\lambda^j}{dt}$$

and K_λ^B are auxiliary functions of t. Therefore, the development of C depends on $(n-m)n$ arbitrary functions K_λ^B.[3)]

Now, by definition, a curve C: $u^i = u^i(t)$ in $\mathfrak{X}$ is called *basic*, if its tangent vectors are A-invariant, that is

$$A_i^j \frac{du^i}{dt} = \frac{du^j}{dt}.$$

THEOREM 1.1. *Any basic curve in $\mathfrak{X}$ can be developed into the m-dimensional subspace R^m: $x^{m+1} = \cdots = x^n = 0$ of R^n, if we put $K_\alpha^B = 0$ in* (1.3). *Furthermore, the development does not depend on the remaining $(n-m)^2$ auxiliary functions K_C^B.*

Proof. Let C: $u^i = u^i(t)$ be a basic curve. Putting $K_\alpha^B = 0$ in (1.3), we have

$$(1.4)\qquad \begin{cases} \dfrac{d\xi_\beta^\alpha}{dt} + K_\gamma^\alpha \xi_\beta^\gamma + K_B^\alpha \xi_\beta^B = 0, & \dfrac{d\xi_B^\alpha}{dt} + K_\gamma^\alpha \xi_B^\gamma + K_C^\alpha \xi_B^C = 0, \\ \dfrac{d\xi_\beta^B}{dt} + K_C^B \xi_\beta^C = 0, & \dfrac{d\xi_C^B}{dt} + K_D^B \xi_C^D = 0. \end{cases}$$

Hence we may put $\xi_\alpha^B = 0$ and $\eta_\alpha^B = 0$. Accordingly we have

$$\frac{dx^B}{dt} = Y_i^B \frac{du^i}{dt} = \eta_\lambda^B U_i^\lambda \frac{du^i}{dt} = \eta_\lambda^B U_j^\lambda A_i^j \frac{du^i}{dt}$$
$$= \eta_\alpha^B U_i^\alpha \frac{du^i}{dt} = 0,$$

which shows that the development of C may be regarded as lying in R^m.

Making use of these relations, we have

$$(1.5)\qquad \frac{dx^\alpha}{dt} = \eta_\beta^\alpha U_i^\beta \frac{du^i}{dt}, \qquad \frac{d\eta_\beta^\alpha}{dt} - \eta_\gamma^\alpha K_\beta^\gamma = 0,$$

which shows that the developments of C does not depend on the choice of auxiliary functions K_C^B.

THEOREM 1.2. *According to the method in Theorem* 1.1, *the developments of a basic curve with respect to the normal general connections Γ and $A\Gamma A$ are identical with each other.*

Proof. Between the covariant differential operators D and $\tilde{D}$ of Γ and

3) See Theorem 3.1 in [10].

$A\Gamma A$, there exists the relation

$$\tilde{D} = \iota_A \cdot D \cdot A$$

by Theorem 2.1 in [11]. Hence we have

$$\tilde{K}^\alpha_\beta = U^\alpha_k Q^k_j \frac{\tilde{D} V^j_\beta}{dt} = U^\alpha_k Q^k_l A^l_j \frac{D}{dt}(A^j_h V^h_\beta)$$
$$= U^\alpha_k Q^k_j \frac{D V^j_\beta}{dt} = K^\alpha_\beta,$$

making use of $P^j_i = A^j_l P^l_h A^h_i$ and $Q^j_i = A^j_l Q^l_h A^h_i$. This shows that the theorem holds good. q.e.d.

Lastly we consider the development of A-invariant contravariant vectors and covariant vectors defined along basic curves in $\mathfrak{X}$.

By definition, the developments[4] of a contravariant vector V^i and a covariant vector W_i defined along a curve in $\mathfrak{X}$ are the contravariant vector $\bar{V}$ and the covariant vector $\bar{W}$ defined along the development $\bar{C}$ of C by

$$\bar{V}^\lambda = Y^\lambda_i V^i \quad \text{and} \quad \bar{W}_\lambda = X^i_\lambda W_i.$$

LEMMA 1.1. *The development of an A-invariant contravariant vector defined along a basic curve lies in R^m and does not depend on the choice of auxiliary functions in* (1.3).

Proof. Using the above notations, since we have

$$\bar{V}^\lambda = Y^\lambda_i V^i = \eta^\lambda_\mu U^\mu_j A^j_i V^i = \eta^\lambda_\alpha U^\alpha_i V^i,$$

we get $\bar{V}^B = 0$. By means of Theorem 1.1, this lemma holds good. q.e.d.

LEMMA 1.2. *The induced covariant vector in R^m from the development of an A-invariant covariant vector defined along a basic curve does not depend on the choice of auxiliary functions in* (1.3).

Proof. Using the above notations, we have

$$\bar{W}_\lambda = W_i X^i_\lambda = W_j A^j_i V^i_\mu \xi^\mu_\lambda = W_i V^i_\alpha \xi^\alpha_\lambda,$$

which shows that $\bar{W}_\alpha$ does not depend on the choice of auxiliary functions in (1.3).

§2. Basic geodesics.

THEOREM 2.1. *If a normal general connection Γ satisfies*

$$N^j_k \Gamma^k_{lm} A^l_i A^m_h = 0, \tag{2.1}$$

4) See [8], §5.

then there exists a basic geodesic through a given point in $\mathfrak{X}$ and with a given A-invariant tangent vector of $\mathfrak{X}$ at the point as its tangent vector.

Proof. The conditions that a curve C: $u^i = u^i(t)$ in $\mathfrak{X}$ is basic and a geodesic are

$$A_i^j \frac{du^i}{dt} = \frac{du^j}{dt} \tag{2.2}$$

and

$$\frac{D}{dt}\frac{du^j}{dt} = \psi P_i^j \frac{du^i}{dt}, \tag{2.3}$$

where ψ is a function of the parameter t of C. (2.3) can be written as

$$P_i^j\left(\frac{d^2u^i}{dt} - \psi\frac{du^i}{dt}\right) + \Gamma^j_{ih}\frac{du^i}{dt}\frac{du^h}{dt} = 0,$$

from which we get

$$A_i^j\left(\frac{d^2u^i}{dt^2} - \psi\frac{du^i}{dt}\right) + {}'\Gamma^j_{ih}\frac{du^i}{dt}\frac{du^h}{dt} = 0, \tag{2.4}$$

where ${}'\Gamma^j_{ih} = Q_l^j \Gamma^l_{ih}$.

Conversely, from (2.4) we get

$$P_i^j\left(\frac{d^2u^i}{dt^2} - \psi\frac{du^i}{dt}\right) + A_l^j \Gamma^l_{ih}\frac{du^i}{dt}\frac{du^h}{dt} = 0.$$

Making use of (2.1) and (2.2), we have

$$N_l^j \Gamma^l_{ih}\frac{du^i}{dt}\frac{du^h}{dt} = 0.$$

Since $A_i^j + N_i^j = \delta_i^j$, summing up the two equations we get (2.3). Accordingly, the conditions (2.2) and (2.3) are equivalent to (2.2) and (2.4) under the condition (2.1).

Putting

$$\frac{du^j}{dt} = V_\alpha^j v^\alpha$$

by (2.2) and substituting these into (2.4), we have

$$A_i^j\left(V_\alpha^i\frac{dv^\alpha}{dt} + \frac{\partial V_\alpha^i}{\partial u^h}V_\beta^h v^\alpha v^\beta - \psi V_\alpha^i v^\alpha\right) + {}'\Gamma^j_{ih}V_\alpha^i V_\beta^h v^\alpha v^\beta = 0,$$

that is

$$V_\alpha^j\left\{\left(\frac{dv^\alpha}{dt} - \psi v^\alpha\right) + U_i^\alpha\left(\frac{\partial V_\beta^i}{\partial u^h}V_\gamma^h + {}'\Gamma^i_{kh}V_\beta^k V_\gamma^h\right)v^\beta v^\gamma\right\} = 0.$$

Thus, the equations (2.2) and (2.4) can be replaced with

$$\frac{du^j}{dt} = V^j_\alpha v^\alpha, \tag{2.5}$$

$$\frac{dv^\alpha}{dt} + U^\alpha_i\left(\frac{\partial V^i_\beta}{\partial u^h} V^h_\gamma + {}'\Gamma^i_{kh} V^k_\beta V^h_\gamma\right) v^\beta v^\gamma = \psi v^\alpha. \tag{2.6}$$

These equations regarding u^j and v^α as unknown functions have always a solution satisfying any given initial condition such as

$$u^j(t_0) = u^j_0 \quad \text{and} \quad v^\alpha(t_0) = v^\alpha_0.$$

Hence, the theorem is proved.

Remark. A geometrical meaning of the condition (2.1) is given as follows: the general connections $N\Gamma$ and $N\Gamma A$ are written as[5)]

$$N\Gamma = \partial u_j \otimes N^j_l \Gamma^l_{ih} du^i \otimes du^h$$

and

$$N\Gamma A = \partial u_j \otimes N^j_l \Gamma^l_{kh} A^k_i du^i \otimes du^h.$$

Hence, (2.1) is equivalent to the condition that any contravariant vector defined along any basic curve in $\mathfrak{X}$ with respect to the normal general connection Γ is covariantly constant with respect to the general connection $N\Gamma A$ which is a tensor of type (1, 2).

§3. The parallel displacement of vectors along basic curves.

THEOREM 3.1. *If a normal general connection Γ satisfies* (2.1), *then along a basic curve C: $u^i = u^i(t)$ the conditions*

$$A^j_i V^i = V^j \tag{3.1}$$

and

$$\frac{DV^j}{dt} = 0 \tag{3.2}$$

are compatible, that is, the $V^j(t)$ satisfying these equations are uniquely determined by their initial conditions $V^j(t_0) = V^j_0$.

Proof. We first show that for a basic curve with respect to a normal general connection Γ satisfying (2.1) we have

$$\frac{DX^j_\alpha}{dt} = 0, \tag{3.3}$$

5) See [11], §1, (1.14).

because

$$\begin{aligned}\frac{DX^j_\alpha}{dt} &= \frac{D}{dt}(V^j_\beta \xi^\beta_\alpha) = \frac{DV^j_\beta}{dt}\xi^\beta_\alpha + P^j_i V^i_\beta \frac{d\xi^\beta_\alpha}{dt} \\ &= \left(\frac{DV^j_\beta}{dt} - P^j_i V^i_\gamma K^\gamma_\beta\right)\xi^\beta_\alpha \\ &= \left(\frac{DV^j_\beta}{dt} - P^j_i V^i_\gamma U^\gamma_k Q^k_h \frac{DV^h_\beta}{dt}\right)\xi^\beta_\alpha \\ &= \left(\frac{DV^j_\beta}{dt} - A^j_k \frac{DV^k_\beta}{dt}\right)\xi^\beta_\alpha = N^j_k \frac{DV^k_\beta}{dt}\xi^\beta_\alpha \\ &= N^j_k\left(P^k_i \frac{dV^i_\beta}{dt} + \Gamma^k_{ih} V^i_\beta \frac{du^h}{dt}\right)\xi^\beta_\alpha \\ &= (N^j_k \Gamma^k_{lm} A^l_i A^m_h) V^i_\beta \frac{du^h}{dt}\xi^\beta_\alpha = 0.\end{aligned}$$

Now, by means of (3.1) we may put $V^j = X^j_\alpha v^\alpha$. Hence we have

$$\frac{DV^j}{dt} = \frac{DX^j_\alpha}{dt} v^\alpha + P^j_i X^i_\alpha \frac{dv^\alpha}{dt} = P^j_i X^i_\alpha \frac{dv^\alpha}{dt}.$$

Accordingly, (3.2) is satisfied, if and only if $v^\alpha = \text{constant}$.

THEOREM 3.2. *If a normal general connection Γ satisfies the conditions*

$$A^j_k \Lambda^k_{lm} N^l_i A^m_h = 0, \tag{3.4}$$

then along a basic curve C: $u^i = u^i(t)$ the conditions

$$W_j A^j_i = W_i \tag{3.5}$$

and

$$\frac{DW_i}{dt} = 0 \tag{3.6}$$

are compatible, that is, the $W_i(t)$ satisfying these equations are uniquely determined by their initial conditions $W_i(t_0) = W^0_i$.

Proof. By means of (3.5), we may put $W_i = U^\alpha_i w_\alpha$. Hence we have

$$\begin{aligned}\frac{DW_i}{dt} &= \frac{DU^\alpha_i}{dt} w_\alpha + U^\alpha_j P^j_i \frac{dw_\alpha}{dt} \\ &= \left(\frac{DU^\alpha_j}{dt} V^j_\beta w_\alpha + W^\alpha_\beta \frac{dw_\alpha}{dt}\right) U^\beta_i + \left(\frac{DU^\alpha_j}{dt} V^j_B w_\alpha\right) U^B_i,\end{aligned}$$

where W^α_β are defined by $U^\alpha_j P^j_i = W^\alpha_\beta U^\beta_i$. Accordingly, in order that (3.6) is satisfied, it is necessary and sufficient that

(3.7)
$$W^\alpha_\beta \frac{dw_\alpha}{dt} + \frac{DU^\alpha_j}{dt} V^j_\beta w_\alpha = 0$$

and

(3.8)
$$\frac{DU^\alpha_j}{dt} V^j_B w_\alpha = 0.$$

Since $|W^\alpha_\beta| \neq 0$, (3.7) uniquely determines $w_\alpha(t)$ by the initial conditions $w_\alpha(t_0) = w^0_\alpha$. By means of (3.4), we have

$$\frac{DU^\alpha_i}{dt} V^i_B = \left(\frac{dU^\alpha_j}{dt} P^j_i - U^\alpha_j \Lambda^j_{ih} \frac{du^h}{dt} \right) V^i_B$$
$$= - U^\alpha_j (A^j_k \Lambda^k_{lm} N^l_i A^m_h) V^i_B \frac{du^h}{dt} = 0.$$

Hence (3.8) is identically satisfied along the basic curve. q.e.d.

REMARK. A geometrical meaning of the condition (3.4) is given as follows: the general connections ΓN and $A\Gamma N$ are written as

$$\Gamma N = \partial u_j \otimes \Lambda^j_{lh} N^l_i du^i \otimes du^h.$$

and

$$A\Gamma N = \partial u_j \otimes A^j_k \Lambda^k_{lh} N^l_i du^i \otimes du^h.$$

Hence, (3.4) is equivalent to the condition that any covariant vector defined along any basic curve in $\mathfrak{X}$ with respect to the normal general connection Γ is covariantly constant with respect to the general connection $A\Gamma N$ which is a tensor of type (1, 2).

§4. Basic vectors and the basic covariant differentiation.

For a normal general connection Γ, we denote its basic covariant differential operator by $\bar{D}$.[6] Let $'\Gamma = Q\Gamma$ and $''\Gamma = \Gamma Q$ be its contravariant part and its convariant part respectively.

THEOREM 4.1. *If a normal general connection Γ satisfies* (2.1), *then along a basic curve the following two conditions are equivalent to each other:*

(α')
$$A^j_i V^i = V^j, \qquad \frac{DV^j}{dt} = 0$$

and

(β')
$$A^j_i V^i = V^j, \qquad \frac{\bar{D}V^j}{dt} = 0,$$

where V^i are components of a contravariant vector.

6) See [10], §4.

Proof. Regarding the basic covariant differential operator $\bar{D}$, we have the formula

$$\iota_A \cdot D = \iota_P \cdot \bar{D},^{7)}$$

hence

$$P_i^j \frac{\bar{D}V^i}{dt} = A_i^j \frac{DV^i}{dt}.$$

By virtue of Theorem (4.1) in [10], we have

$$\frac{\bar{D}V^j}{dt} = A_i^j \frac{\bar{D}V^i}{dt} = Q_k^j P_i^k \frac{\bar{D}V^i}{dt} = Q_k^j A_i^k \frac{DV^i}{dt} = Q_i^j \frac{DV^i}{dt}.$$

Assuming now $A_i^j V^i = V^j$ and (2.1), we have

$$N_i^j \frac{DV^i}{dt} = N_l^j \Gamma_{kh}^l V^k \frac{du^h}{dt} = (N_k^j \Gamma_{lm}^k A_i^l A_h^m) V^i \frac{du^h}{dt} = 0.$$

These equations easily show that the above conditions (α') and (β') are equivalent to each other.

THEOREM 4.2. *If a normal general connection Γ satisfies* (3.4), *then along a basic curve the following two conditions are equivalent to each other*:

$$(\alpha'') \qquad A_i^j W_j = W_i, \qquad \frac{DW_i}{dt} = 0$$

and

$$(\beta'') \qquad A_i^j W_j = W_i, \qquad \frac{\bar{D}W_i}{dt} = 0,$$

where W_i are the components of a covariant vector.

Proof. We have analogously the equations

$$P_i^j \frac{\bar{D}W_j}{dt} = A_i^j \frac{DW_j}{dt}$$

and

$$\frac{\bar{D}W_i}{dt} = A_i^j \frac{\bar{D}W_j}{dt} = Q_i^k P_k^j \frac{\bar{D}W_j}{dt} = Q_i^j \frac{DW_j}{dt}.$$

Assuming now $A_i^j W_j = W_i$ and (3.4), we have

7) See Theorem 3.2 in [10].

$$N_i^j \frac{DW_j}{dt} = N_i^j \left(\frac{dW_k}{dt} P_j^k - W_k \Lambda_{jh}^k \frac{du^h}{dt} \right)$$
$$= - W_j (A_k^j \Lambda_{lm}^k N_i^l A_h^m) \frac{du^h}{dt} = 0.$$

These equations easily show that the above condition (α'') and (β'') are equivalent to each other.

REFERENCES

[1] CHERN, S. S., Lecture note on differential geometry. Chicago Univ. (1950).

[2] EHRESMANN, G., Les connexions infinitésimales dans un espace fibré différentiable. Colloque de Topologie (Espaces fibrés) (1950), 29–55.

[3] EHRESMANN, G., Les prolongements d'une variété différentiables I, Calcul des jets, prolongement principal. C. R. Paris 233 (1951), 598–600.

[4] ŌTSUKI, T., Geometries of connections. Kyōritsu Shuppan Co. (1957). (Japanese)

[5] ŌTSUKI, T., On tangent bundles of order 2 and affine connections. Proc. Japan Acad. 34 (1958), 325–330.

[6] ŌTSUKI, T., Tangent bundles of order 2 and general connections. Math. J. Okayama Univ. 8 (1958), 143–179.

[7] ŌTSUKI, T., On general connections, I. Math. J. Okayama Univ. **9** (1960), 99–164.

[8] ŌTSUKI, T., On general connections, II. Math. J. Okayama Univ. **10** (1961), 113–124.

[9] ŌTSUKI, T., On metric general connections. Proc. Japan Acad. 37 (1961), 183–188.

[10] ŌTSUKI, T., On normal general connections. Kōdai Math. Sem. Rep. **13** (1961), 152–166.

[11] ŌTSUKI, T., General connections $A\Gamma A$ and the parallelism of Levi-Civita. Kōdai Math. Sem. Rep. **13** (1962), 40–52.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.