ON AN EXTENSION OF A THEOREM OF WOLFF

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Let $f(z)$ be an analytic function regular and with positive real part in the half-plane $\Re z > 0$. The main theorem of Julia [1] and Wolff [4] on angular derivative states that there exists a non-negative real constant c for which the limit relation $f(z)/z \to c$ holds uniformly as z tends to ∞ through a Stolz angle $|\arg z| \leqq \alpha < \pi/2$. In a recent paper [3] this result was generalized. Namely, it was shown that the derivative $\mathscr{D}^p f(z)$ of any real (not necessarily integral) order p satisfies the limit relation

$$\lim_{z\to\infty} z^{p-1}\mathscr{D}^p f(z) = \frac{c}{\Gamma(2-p)}$$

valid uniformly as z tends to ∞ through any Stolz angle in $\Re z > 0$. The last limit relation can be written in the form

$$\lim_{z\to\infty} \frac{1}{z^{1-p}}\left(\mathscr{D}^p f(z) - \frac{cz^{1-p}}{\Gamma(2-p)}\right) = 0$$

which implies, in particular,

$$\lim_{z\to\infty} \Im \frac{1}{z^{-1-p}\,|\,z\,|^2}\left(\mathscr{D}^p f(z) - \frac{c^{-p}z^1}{\Gamma(2-p)}\right) = 0.$$

In these relations the approach $z \to \infty$ is restricted to a Stolz angle in order to insure the uniformity. However, it was shown by Wolff [5] that the last relation with $p = -1$ holds uniformly even when z approaches ∞ in an arbitrary (i.e., not necessarily non-tangential) manner through $\Re z > 0$.

In the present paper we shall give an alternative proof of Wolff's last mentioned theorem. It seems more direct than Wolff's original proof which depends on a result previously obtained by himself and de Kok [6]. It will further be shown, by making use of this theorem, that the limit relation under consideration and an analogous one hold for any p with $p \leqq -1$ when z approaches ∞ arbitrarily through $\Re z > 0$.

We begin with a proof of Wolff's theorem.

THEOREM 1 (Wolff). *Let $f(z)$ be an analytic function regular and with positive real part in the half-plane $\Re z > 0$. Put*

$$z = x + iy, \qquad \int_1^z f(z)\,dz = \varphi(x, y) + i\psi(x, y),$$

x, y, φ and ψ being real. Then the limit relation

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$$\lim_{x^2+y^2\to\infty} \frac{\psi(x, y) - cxy}{x^2+y^2} = 0$$

holds uniformly as z tends to ∞ *through* $\Re z > 0$ *in an arbitrary manner, where c denotes the angular derivative of* $f(z)$ *at* ∞.

Proof. Together with $f(z)$, the function $f(z) - cz - i\Im f(1)$ is also regular and with positive real part in $\Re z > 0$, unless it reduces identically to zero in which case the assertion of the theorem follows trivially. The imaginary part of its integral becomes

$$\Im\int_1^z (f(z) - cz - i\Im f(1))\,dz = \psi(x, y) - cxy - \Im f(1)\cdot(x-1).$$

Hence, we may suppose $c = 0$ and $\Im f(1) = 0$ without loss of generality. Then, as shown in a previous paper [2], the function admits an integral representation

$$f(z) = \int_{-\infty}^{\infty} \frac{1 - isz}{z - is}\,d\lambda(s)$$

where $\lambda(s)$ is a real-valued increasing function with finite total variation equal to $\Re f(1)$ $(= f(1))$. —This representation holds obviously also for the degenerate case $f(z) \equiv 0$ by taking $\lambda(s) \equiv \text{const.}$— Integration with respect to z yields

$$\int_1^z f(z)\,dz = \int_{-\infty}^{\infty}\left((1+s^2)\lg\frac{z - is}{1 - is} - is(z-1)\right)d\lambda(s),$$

whence follows

$$\psi(x, y) \equiv \Im\int_1^z f(z)\,dz = \int_{-\infty}^{\infty}\left((1+s^2)\arg\frac{x + i(y-s)}{1 - is} - s(x-1)\right)d\lambda(s).$$

Now, in view of the main theorem on angular derivative, we have $f(x) = o(x)$ as $x \to +\infty$ and hence

$$\psi(x, 0) = o(x^2) \qquad \text{as} \quad x \to +\infty.$$

For any real y, we have

$$\begin{aligned}\psi(x, y) - \psi(x, 0) &= \int_{-\infty}^{\infty} (1+s^2)\arg\frac{x + i(y-s)}{x - is}\,d\lambda(s)\\ &= \int_{-\infty}^{\infty}(1+s^2)\arctan\frac{xy}{x^2 - ys + s^2}\,d\lambda(s),\end{aligned}$$

where the arctan denotes the branch attaining the value 0 for $y = 0$ and continued continuously; in particular, its range is contained in the interval $(-\pi, \pi)$. The integrand of the last integral may be estimated as follows. For $|s| > 2|y|$ we have

$$\left|(1+s^2)\arctan\frac{xy}{x^2 - ys + s^2}\right| \leqq (1+s^2)\frac{x|y|}{x^2 + s^2/2}$$

$$\leqq \left(\frac{1}{x^2 + 2y^2} + \frac{s^2}{x^2 + s^2/2}\right) x|y| = O(x^2 + y^2),$$

while for $|s| \leqq 2|y|$ we have

$$\left|(1+s^2)\arctan\frac{xy}{x^2-ys+s^2}\right| \leqq (1+s^2)\pi = O(x^2+y^2).$$

Hence we have an estimation

$$\frac{1+s^2}{x^2+y^2}\arctan\frac{xy}{x^2-ys+s^2}=O(1) \qquad \text{as} \quad x^2+y^2\to\infty,$$

valid uniformly for $-\infty < s < \infty$. On the other hand, the right member of the last relation can be replaced by $o(1)$ for $|s| \leqq S$ with any fixed S. Hence we get

$$\frac{\psi(x,y)-\psi(x,0)}{x^2+y^2} = o(1) + \int_{|s|>S}\frac{1+s^2}{x^2+y^2}\arctan\frac{xy}{x^2-ys+s^2}\,d\lambda(s).$$

Since, as shown above, the integrand of the last integral is estimated by $O(1)$ uniformly and $\lambda(s)$ is of bounded variation, the value of this integral becomes arbitrarily near zero for S large enough. Consequently, we obtain

$$\frac{\psi(x,y)}{x^2+y^2} = \frac{\psi(x,0)}{x^2+y^2}o(1) = o(1) \qquad \text{as} \quad x^2+y^2\to\infty.$$

This is the result desired.

Now, the theorem of Wolff just re-proved can be extended as in the manner preannounced above.

THEOREM 2. *Let $f(z)$ be an analytic function regular and with positive real part in the half-plane $\Re z > 0$. For any real positive number q, let its (fractional) integral of order q be denoted by*

$$\mathcal{D}^{-q}f(z) = \frac{1}{\Gamma(q)}\int_1^z (z-\zeta)^{q-1}f(\zeta)\,d\zeta$$

where the branch of $(z-\zeta)^{q-1} \equiv \exp((q-1)\lg(z-\zeta))$ is determined by taking the principal value of logarithm and the integration is supposed to be taken along the rectilinear segment connecting 1 with z. Then the limit relation

$$\lim_{z\to\infty}\Im\frac{1}{z^{q-1}|z|^2}\left(\mathcal{D}^{-q}f(z) - \frac{cz^{q+1}}{\Gamma(q+2)}\right) = 0$$

holds uniformly, provided $q \geqq 1$, as z tends to ∞ through $\Re z > 0$ in an arbitrary manner, where c denotes the angular derivative of $f(z)$ at ∞.

Proof. The particular case $q=1$ of this theorem is nothing but the Wolff's theorem 1 discussed above. Remembering $\mathcal{D}^{-q}z = z^{q+1}/\Gamma(q+2)$, we may again suppose $c=0$. For any $q>1$, we have

$$\mathcal{D}^{-q}f(z) = \mathcal{D}^{-(q-1)}(\mathcal{D}^{-1}f(z)) = \frac{1}{\Gamma(q-1)}\int_1^z (z-\zeta)^{q-2}\mathcal{D}^{-1}f(\zeta)\,d\zeta$$

which may be written, by putting $\zeta = 1 + t(z-1)$ with $0 \leqq t \leqq 1$, in the form

$$\mathcal{D}^{-q}f(z)=\frac{(z-1)^{q-1}}{\Gamma(q-1)}\int_0^1(1-t)^{q-2}\mathcal{D}^{-1}f(1+t(z-1))\,dt.$$

In view of theorem 1, we get

$$\Im\mathcal{D}^{-1}f(1+t(z-1))=o(|z|^2)$$

valid uniformly as $z\to\infty$. Consequently, we obtain the estimation

$$\Im\frac{\mathcal{D}^{-q}f(z)}{(z-1)^{q+1}}=o(|z|^2).$$

Since $(z-1)^{q+1}/z^{q+1}\to 1$ as $z\to\infty$, this is evidently equivalent to our assertion under the assumption $c=0$.

As mentioned above, the limit relation in theorem 2 can be replaced by more precise one without $\Im$-sign provided $z=x+iy$ tends to ∞ in satisfying the condition $y=O(x)$. Hence it is an essential matter only when this condition does not satisfied. In this connection it may be of some interest to state the following analogous theorem.

THEOREM 3. *Under the same assumptions as in theorem* 2, *the limit relation*

$$\lim_{z\to\infty}\Im\frac{1}{i^{q-1}|z|^{q+1}}\left(\mathcal{D}^{-q}f(z)-\frac{cz^{q+1}}{\Gamma(q+2)}\right)=0$$

holds also uniformly in the same sense as above.

Proof. Wolff's theorem 1 is also a particular case $q=1$ of this theorem. We suppose here again $c=0$ for the sake of brevity. For any $q>1$, we again write

$$\mathcal{D}^{-q}f(z)=\frac{1}{\Gamma(q-1)}\int_1^z(z-\zeta)^{q-2}\mathcal{D}^{1-}f(\zeta)\,d\zeta.$$

Here, for any fixed z, the path of integration may be deformed continuously within $\Re z>0$. Hence, by putting $z=x+iy$, we take the path consisting of a horizontal segment from 1 to x and a vertical segment from x to $x+iy$. Thus we get

$$\mathcal{D}^{-q}f(x+iy)=\frac{1}{\Gamma(q-1)}\int_1^x(x+iy-\xi)^{q-2}\mathcal{D}^{-1}f(\xi)\,d\xi$$
$$+\frac{i^{q-1}}{\Gamma(q-1)}\int_0^y(y-\eta)^{q-2}\mathcal{D}^{-1}f(x+i\eta)\,d\eta.$$

Since $c=0$, we have $\mathcal{D}^{-1}f(\xi)=o(\xi^2)$ as $\xi\to+\infty$ and hence

$$\int_1^x(x+iy-\xi)^{q-2}\mathcal{D}^{-1}f(\xi)\,d\xi=o(|z|^{q+1})$$

valid uniformly as $z\to\infty$. On the other hand, in view of theorem 1, we get $\Im\mathcal{D}^{-1}f(x+i\eta)=o(x^2+\eta^2)$ and hence

$$\mathfrak{J}\int_0^y (y-\eta)^{q-2}\mathcal{D}^{-1}f(x+i\eta)\,d\eta = o(|z|^{q+1})$$

valid also uniformly as $z\to\infty$. Consequently, we obtain the estimation

$$\mathfrak{J}i^{-(q-1)}\mathcal{D}^{-q}f(z) = o(|z|^{q+1})$$

which is evidently equivalent to our assertion with $c=0$.

REFERENCES

[1] JULIA, G., Extensions nouvelles d'un lemme de Schwarz. Acta Math. **42** (1920), 349–355.

[2] KOMATU, Y., On angular derivative. Kōdai Math. Sem. Rep. **13** (1961), 167–179.

[3] KOMATU, Y., On fractional angular derivative. Kōdai Math. Sem. Rep. **13** (1961), 249–254.

[4] WOLFF, J., Sur une généralisation d'un théorème de Schwarz. C. R. Paris **183** (1926), 500–502.

[5] WOLFF, J., Sur l'intégrale d'une fonction holomorphe à partie réelle positive. C. R. Paris **196** (1933), 1949–1950.

[6] WOLFF, J., AND F. DE KOK, Les fonctions holomorphes à partie réelle positive et l'intégrale de Stieltjes. Bull. Soc. Math. France **60** (1932), 221–227.

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