ON RECURRENT MARKOV PROCESSES

By Tadashi Ueno

Introduction.

There are intimate relations between the behavior of Brownian motions and classical theory of potentials. Such relations can be naturally generalized to a wider class of Markov processes, as Hunt [5] studied systematically. Here, we try to study a generalization of classical Green potentials for a class of recurrent Markov processes. The recurrence assumption much simplifies the situation, where the system of hitting measures for subsets of the state space and a system of measures determined by mean sojourn time on subsets play the essential rôle. They determine the system of Green capacities, equilibrium distributions, Green kernels and the invariant measure of the process, uniquely up to a constant factor. It seems to be important that some of these quantities depend only on the system of hitting measures. By making use of these quantities we have a representation of the generator of the process. Following is the outline of this paper, where rigorous definitions, assumptions, and justifications of some quantities are referred to §1.

Let $\{X(t, w), t \geqq 0, w \in W\}$ be a Markov process taking values of a topological space R, where the path functions are right continuous and have left limits and W is the probability space. $P_x(\cdot)$ is the probability of the event $\cdot$ under the condition that the path starts at x. The *hitting time* $\sigma_K(w)$ for an open or closed subset K of R is given by

$$\begin{aligned} \sigma_K(w) &= \inf\{t \geqq 0 \mid X(t, w) \in K\}, && \text{if such } t \text{ exists,} \\ &= \infty, && \text{otherwise.} \end{aligned}$$

The *hitting measure* $h^K(x, \cdot)$ on K is defined by

$$h^K(x, \cdot) = P_x(X(\sigma_K(w), w) \in \cdot).$$

The *Green measure* $G^{R-K}(x, \cdot)$ for the set $R-K$ is given by

$$G^{R-K}(x, E) = E_x\left(\int_0^{\sigma_K(w)} \chi_E(X(t, w))\, dt\right), \qquad E \subset R,$$

where $\chi_E(\ )$ is the characteristic function of E and $E_x(F(w))$ is the expectation of $F(x)$ with respect to $P_x(\cdot)$. The fundamental assumption is that of

Recurrence. The process hits any set A having an inner point with probability one, starting at any point $x \in R$, i.e.

$$P_x(X(t, w) \in A \quad \textit{for some} \quad 0 \leqq t < \infty) = 1, \quad x \in R.$$

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Let $\mathfrak{F}$ be the family of all $\{K, L\}$, where K and L are mutually disjoint closed sets in R with inner points and satisfy the conditions in §2. There is a unique pair of measures $\mu_L{}^K$ and $\mu_K{}^L$ on K and L respectively with total mass 1, satisfying

$$\mu_L{}^K(\cdot)=\int_L \mu_K{}^L(dx)h^K(x,\ \cdot),$$

$$\mu_K{}^L(\cdot)=\int_K \mu_L{}^K(dx)h^L(x,\ \cdot),$$

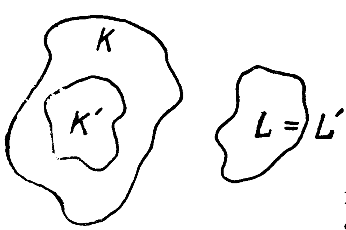
called *equilibrium distributions* on K and L with respect to $R-L$ and $R-K$ respectively. When $X(t, w)$ is the 2-dim. Brownian motion and K is contained in one component of $R-L$, $\mu_L{}^K$ coincides with the equilibrium distribution on K with respect to the classical Green function of the component.

Define a measure $m_{K,L}$ on R by

$$m_{K,L}(\cdot)=\int_K \mu_L{}^K(dx)G^{R-L}(x,\ \cdot)+\int_L \mu_K{}^L(dx)G^{R-K}(x,\ \cdot).$$

This is the *invariant measure* for the process $X(t, w)$, as Maruyama and Tanaka [2] proved. It is also proved that $m_{K,L}$ *depends on the choice of* $\{K, L\}$ *only up to a constant factor.* The proof of the latter assertion[1] leads naturally to the definition of Green capacities.

To study a relation between two pairs of measures $\{\mu_L{}^K, \mu_K{}^L\}$ and $\{\mu_{L'}{}^{K'}, \mu_{K'}{}^{L'}\}$, consider a special case $K'\subset K$ and $L=L'$. Then, we have



$$C_{(K,L)}(K',\ L')\cdot\mu_{L'}{}^{K'}(E)=\int_K \mu_L{}^K(dx)h^{K'\cup L}(x,\ E),\qquad E\subset K',$$

where

$$C_{(K,L)}(K',\ L')=\int_K \mu_L{}^K(dx)h^{K'\cup L}(x,\ K')$$

is the normalizing constant. $C_{(K,L)}(K', L')$ in this special case can be extended uniquely to a positive valued function defined on $\mathfrak{F}\times\mathfrak{F}$ satisfying

$$C_{(K_1,L_1)}(K_2,\ L_2)\cdot C_{(K_2,L_2)}(K_3,\ L_3)=C_{(K_1,L_1)}(K_3,\ L_3),$$

$$C_{(K_1,L_1)}(K_2,\ L_2)=C_{(K_2,L_2)}(K_1,\ L_1)^{-1}.$$

We note that this function depends only on the system of hitting measures and that it is characterized by

$$m_{K,L}(\cdot)=C_{(K,L)}(K',\ L')m_{K',L'}(\cdot).$$

Fix any $\{K_0, L_0\}\in\mathfrak{F}$ and put $C(K, L)=C_{(K_0,L_0)}(K, L)$ and call $C(K, L)$ the *Green capacity of* K *with respect to* $R-L$, since the function $C(K, L)$ depends on the choice of $\{K_0, L_0\}$ only up to a constant factor. When $X(t, w)$ is the

1) A different proof is obtained in Maruyama and Tanaka [7], independently of the present author, by making use of an ergodic theorem.

2-dim. Brownian motion and K is contained in one component of $R-L$, $C(K, L)$ is the classical Green capacity of K with respect to the component, and $m=m_{K_0, L_0}$ is the Lebesgue measure, where $K_0=\{z \mid |z| \leqq 1\}$ and $L_0=\{z \mid |z| \geqq e\}$ with $e=2.718\cdots$. Note that $C(K, L)^{-1}$ increases when K and L shrink, or get far apart, and decreases when K and L expand, or get near, a little like a distance. In the case of 2-dim. Brownian motion this coincides with the *extremal* distance[2] of K and L with respect to the domain between K and L.

We can prove that $m=m_{K_0, L_0}$ takes a positive value for a non-empty open set and a finite value for a compact set, and that every $G^{R-K}(x, \cdot)$ is absolutely continuous relative to m. Fix a version of the densities $\{g^{R-K}(x, y)\}$ of $G^{R-K}(x, \cdot)$ relative to m, and call it the system of *Green kernels* induced by $\{X(t, w)\}$. In the case of 2-dim. Brownian motion we can take the classical Green kernel $g_{R-K}(x, y)$ for $g^{R-K}(x, y)$. But in general case $g^{R-K}(x, y)$ is not necessarily symmetric in x and y.

W. Feller [3] obtained a representation

$$\boldsymbol{G} f=\frac{d}{dm} \frac{d^{+}}{ds} f$$

of the generator of linear diffusion processes, which has an intrinsic meaning for the behavior of the process. This was generalized by Ito and McKean [6] in the case of multidimensional diffusion processes with Brownian hitting measures, by making use of the classical Green kernels. We note that the formal use of the kernels defined above leads to a representation of the generator $\boldsymbol{G}$ of $X(t, w)$, that is, for each f in the domain $\mathfrak{D}(\boldsymbol{G})$ of $\boldsymbol{G}$, there is a unique sigma-finite measure m_f on R satisfying

$$\boldsymbol{G} f=\frac{dm_f}{dm} \quad \text{(in the sense of Radon-Nikodym)},$$

$$\int_K h^K(x, dy) f(y)-f(x)=U^{m_f}{}_{g^{R-K}}(x)=\int_R g^{R-K}(x, y) m_f(dy).$$

But we note that it is necessary to check, in which case this formal representation keeps the intrinsic meaning of the original definition cited above.

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§1. Notations and Assumptions.[3]

Let R be a separable locally compact space containing at least two points and satisfying

(R. 1) For each point $x \in R$, we can take a base of neighborhoods $\mathfrak{U}_x$ of x consisting of arcwise connected open sets,

2) Defined by Ahlfors and Beurling [1].

3) The general set up in this section, excepting the special assumptions (X. 1), (X. 2), (X. 3), (X. 5) are due to the Lectures given by Ito and McKean in 1957~8, and to Watanabe [10]. The propositions stated without proofs in this section are to be contained with proofs in K. Ito and McKean [6] and Watanabe [10].

(*R.* 2) R is connected.

We denote by $\boldsymbol{B}$ the topological Borel field of the subsets of R. For a formal convenience, we add an extra point ω to R as an isolated one and get a topological space $\bar{R}=R\cup\{\omega\}$ and the corresponding Borel field $\bar{\boldsymbol{B}}$.[4)]

For a measurable function $w(t)$ from $[0, +\infty]$ to $\bar{R}$, we define

$$\begin{aligned}\sigma_\omega(w) &= \inf\{t\geqq 0 \mid w(t)=\omega\}, &&\text{if such } t \text{ exists,}\\ &=\infty, &&\text{otherwise.}\end{aligned}$$

Let W be the space of all functions $w(t)$'s satisfying

(*W.* 1) $\qquad W(t)=\omega \qquad$ for $\quad t\geqq\sigma_\omega(w)$,[5)]

(*W.* 2) $w(t)$ is right continuous and has left limits at each

$$0\leqq t<\sigma_\omega(w).$$

We call W the *space of path functions.* $X(t, w)$ (or simply $X(t)$) is a function on W defined by $X(t, w)=w(t)$, $t\geqq 0$. The *hitting time* σ_A for a set $A\in\boldsymbol{B}$ is defined by

$$\begin{aligned}\sigma_A(w) &= \inf\{t\geqq 0 \mid w(t)\in A\}, &&\text{if such } t \text{ exists,}\\ &=\infty, &&\text{otherwise.}\end{aligned}$$

We denote by $\tilde{\mathfrak{B}}$, the smallest Borel field of subsets of W containing $\{w \mid X(t, w)\in A\}$ for all $A\in\bar{\boldsymbol{B}}$ and $t\geqq 0$. In order to use hitting times for closed sets freely, we use another Borel field $\mathfrak{B}$ generated by all elements of $\tilde{\mathfrak{B}}$ and the sets $\{w \mid \sigma_A(w)>t\}$ for all closed sets A and $0\leqq t\leqq\infty$.[6)] For a $\mathfrak{B}$-measurable function (*random time*) σ taking values in $[0, \infty]$, we define *the stopped path* w_σ^+ and the *shifted path* w_σ^- by

$$\begin{aligned}w_\sigma^-(t) &= X(\min(t, \sigma(w)), w), && 0\leqq t<\infty,\\ &=\omega, && t=\infty^{7)},\\ w_\sigma^+(t) &= X(t+\sigma(w), w), && 0\leqq t\leqq\infty.\end{aligned}$$

Let $\mathfrak{B}_\sigma$ be the smallest Borel field containing $\{w \mid w_\sigma^-\in B\}$ for all $B\in\mathfrak{B}$. It can be proved that the mapping $w\to w_\sigma^-$ and $w\to w_\sigma^+$ are $\mathfrak{B}$-measurable, and hence that $\mathfrak{B}_\sigma$ is a subfield of $\mathfrak{B}$.

A random time σ is called a *Markov time,* if

$$\{w \mid \sigma(w)<t\}\in\mathfrak{B}_t, \qquad \text{for} \quad 0\leqq t\leqq\infty.$$

It can be proved that a hitting time σ_A for a closed or open A is a Markov

4) Since the process in this paper is recurrent, we can make the following discussions without adding ω. The main reason to use this formulation is the reference to [10].

5) This condition means that the path does not come from ω into R once it arrives at ω.

6) This aritificial set up owes to the fact that σ_A for open A is $\tilde{\mathfrak{B}}$-measurable and satisfies necessary regularities, bnt for closed A this is not known.

7) This is to guarantee $W_\sigma^-\in W$.

time,[8] and that $X(\sigma)$ is $\mathfrak{B}_{\sigma+}$-measurable for a Markov time σ, where $\mathfrak{B}_{\sigma+} = \bigcap_{n=1}^{\infty}\mathfrak{B}_{\sigma+1/n}$.

Let $\{P_x(\cdot),\ x\in \bar{R}\}$ be a system of probability measures on $\mathfrak{B}$ satisfying

$(P.\ 1)$ $P_x(E)$ is a $\bar{\boldsymbol{B}}$-measurable function of x for each $E\in\mathfrak{B}$,

$(P.\ 2)$ $P_x(\{w \mid X(0, w)=x\})=1$ for each $x\in\bar{R}$,

$(P.\ 3)$ if $\{\sigma_n\}$ is a sequence of Markov times increasing monotonely with P_x-probability 1, then we have

$$P_x(\{w \mid \lim_{n\to\infty} X(\sigma_n(w), w) = X(\sigma_\infty(w), w),\ \sigma_\infty(w)<\sigma_\omega(w)\}) = P_x(\{w \mid \sigma_\infty(w)<\sigma_\omega(w)\}),$$

where $\sigma_\infty(w)=\lim_{n\to\infty}\sigma_n(w)$,

$(P.\ 4)$ *Markov property*: for any bounded $\mathfrak{B}$-measurable function $F(w)$ and $x\in\bar{R}$, we have

$$E_x(F(w_t^+)\mid \boldsymbol{B}_t) = E_{x(t,w)}(F) \qquad \text{with } P_x\text{-probability } 1,$$

where $E_x(\cdot)$ is the expectation of $\cdot$ with respect to P_x, and $E_x(\cdot\mid\mathfrak{B}_\sigma)$ is the conditional expectation with respect to the Borel field $\mathfrak{B}_\sigma$.

We call the system $\{W, \mathfrak{B}, P_x(\cdot)\}$ a *Markov process on R* and sometimes denote it by $\{X(t)\}$. We use the abbreviation $P_x(\Lambda)$ instead of $P_x(L)$ for $L=\{w \mid w$ satisfies the condition $\Lambda\}$.

$B(K)$ is the space of all bounded functions defined on $K\in\bar{\boldsymbol{B}}$, measurable with respect to the topological Borel field of K, and taking the value 0 at ω when $\omega\in K$. $C(K)$ is the space of all bounded functions on K, continuous with respect to the relative topology and taking the value 0 at ω when $\omega\in K$. The norm on both these spaces is given by $\|f\|=\sup_{x\in K}|f(x)|$. We write

$$\begin{aligned} P(t, x, A) &= P_x(X(t, w)\in A), && \text{for } A\in\bar{\boldsymbol{B}}, \\ T_t f(x) &= E_x(f(X(t, w))), && f\in B(\bar{R}), \\ G_\alpha f(x) &= E_x\left(\int_0^\infty e^{-\alpha t} f(X(t, w))\,dt\right) \\ &= \int_0^\infty e^{-\alpha t} T_t f(x)\,dt, && f\in B(\bar{R}),\ \alpha>0. \end{aligned}$$

Now, we introduce assumptions on $\{X(t)\}$.

$(X.\ 1)$ *Recurrence. The process hits any set $A\in\boldsymbol{B}$ containing an inner point with probability* 1, *i.e.*

$$P_x(X(t, w)\in A, \textit{ for some } 0\leqq t<\infty)=1, \textit{ for any } x\in R.$$

Since σ_A is $\mathfrak{B}$-measurable when A is closed or open, we define the hitting measure $h^A(x, \cdot)$ for such $A\subset R$ by

$$h^A(x, S) = P_x(X(\sigma_A(w), w)\in S,\ \sigma_A(w)<\infty),\ x\in R,\ S\in\boldsymbol{B}.$$

We sometimes use the restriction

$$h_{A'}{}^A(x, S) = h^A(x, S\frown A'), \qquad A'\subset A,\ A'\in\boldsymbol{B}.$$

8) The conclusion for closed A depends upon the artificial definitions of $\mathfrak{B}$ and $\mathfrak{B}_t$ etc.

We write

$$h^A f(x) = \int_{\bar{A}} h^A(x, dy) f(y), \qquad f \in B(R) \quad \text{or} \quad f \in B(\bar{A}).$$

(X. 2) *For any $f \in C(K)$, $h^K f(x)$ is continuous in $R-K$, where K is a closed set in R containing an inner point.*

(X. 3) *Maximum principle. For any non-negative $f \in B(K)$, $h^K f(x)$ is either strictly positive, or 0 for all points x of any one componet of $R-K$, that is, $h^K(x, \cdot)$ are equivalent for all x of any one component of $R-K$, where K is a closed set in R containing an inner point.*[9]

(X. 4) *G_α maps $C(\bar{R})$ into $C(\bar{R})$ for each $\alpha > 0$, that is, $G_\alpha f(x)$ is continuous on R and 0 at ω, if f is in $C(\bar{R})$.*

(X 5.) *There is no point of positive holding time, that is, there is no such $x \in R$ that*

$$P_x(\sigma > 0) > 0, \qquad \sigma = \inf\{t \geqq 0 \mid X(t) \in R - \{x\}\}.$$

The assumptions (X. 1)–(X. 4) are assumed in §§2, 3 and 4, and (X. 5) is assumed in §4.

We prepare the following implications of (X. 4):

1. The process $\{X(t)\}$ has the *strong Markov property*, that is,

(1.1) $$E_x(F(w_\sigma^+) \mid \mathfrak{B}_{\sigma+}) = E_{x(\sigma)}(F) \qquad \text{with } P_x\text{-probability } 1,$$

for any $\mathfrak{B}$-measurable function F and a Markov time σ.

2. The *generator* $\boldsymbol{G}$ of $\{X(t)\}$ is defined as follows: By (X. 4) we can consider G_α as an operator on $C(R)$. Since the resolvent equation

$$G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0$$

holds, the range and the null space of G_α do not depend on the choice of α. We denote these spaces by $\mathfrak{R}$ and $\mathfrak{N}$ respectively. Since we can prove that $\mathfrak{N} = \{0\}$ from the right continuity of path functions, G_α is a one to one mapping from $C(R)$ onto $\mathfrak{R}$. Hence, the generator $\boldsymbol{G}$ of $\{X(t)\}$ is defined by

$$\boldsymbol{G} f = (\alpha - G_\alpha^{-1}) f, \qquad \text{for } f \in \mathfrak{R}$$

independently of the choice of α. We have

(1.2) $$(\alpha - \boldsymbol{G}) G_\alpha f = f, \qquad \text{for } f \in C(R).$$

We write $\mathfrak{D}(\boldsymbol{G})$ for $\mathfrak{R}$.

3. For a Markov time σ, we have *Dynkin's formula*

(1.3) $$E_x(e^{-\alpha\sigma} f(X(\sigma))) - f(x) = -E_x\left\{\int_0^\sigma e^{-\alpha t} (\alpha - \boldsymbol{G}) f(X(t))\, dt\right\}$$

for $f \in \mathfrak{D}(\boldsymbol{G})$ and $\alpha > 0$. When σ has the finite expectation $E_x(\sigma) < \infty$, we have

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9) We say that measures μ and μ' are equivalent, if they are absolutely continuous relative to each other.

$$E_x(f(X(\sigma)))-f(x)=E_x\left\{\int_0^{\sigma} Gf(X(t))\,dt\right\}, \qquad f\in\mathfrak{D}(G). \tag{1.4}$$

It can be proved from $(X.\ 1)$ and $(X.\ 4)$ that

$$P_k(\sigma_\omega(w)=\infty)=1 \qquad \text{for any} \quad x\in R,$$

using Watanabe [10, Th. 4. 1]. Hence, we mean $A\subset R$ when we say a set A, unless specifically mentioned.

Now, we introduce

$$G^{R-K}(x,\ A)=E_x\left(\int_0^{\sigma_K}\chi_A(X(t))\,dt\right) \qquad \text{for} \quad x\in R,\ A\in \boldsymbol{B},$$

for any closed set K containing an inner point, where χ_A is the characteristic function of the set A. We call this, considered as a measure in A, the *Green measure* of $R-K$ with starting point $x\in R$. This is actually a sigma-finite measure on R according to

LEMMA 1.1. *$(X.\ 1)$ and $(X.\ 4)$ imply*

$$G^{R-K}(x,\ A)\leqq M(A,\ K)<\infty, \qquad x\in R \tag{1.5}$$

for any closed set K containing an inner point and any $A\in\boldsymbol{B}$ with compact closure.

Proof.[10] Take a closed set $K_0\subset K$, which contains an inner point and consists of inner points of K. Let $g\in C(R)$ satisfy $g(x)\geqq 1$ on R, $g(x)=1$ on $\bar{K}^c$ and $g(x)>1$ on K_0.[11] Write $f=G_\alpha g\in\mathfrak{D}(G)$ for some fixed $\alpha>0$. Then, we have

$$\begin{aligned} Gf(x)&=(\alpha-G_\alpha{}^{-1})G_\alpha g(x)=\alpha G_\alpha g(x)-g(x)=E_x\left\{\alpha\int_0^\infty e^{-\alpha t}g(X(t))\,dt-g(x)\right\}\\ &=\alpha E_x\left\{\int_0^\infty e^{-\alpha t}(g(X(t))-1)\,dt\right\}>0, \qquad \text{for} \quad x\in\bar{K}^c, \end{aligned}$$

where the last inequality follows from $(X.\ 1)$. Since $A\subset K$ implies $G^{R-K}(x,\ A)=0$, we assume $A-K\neq\phi$. $\overline{A-K}$ being compact, we can define $f_0(x)=\{\inf_{x\in\overline{A-B}}Gf(x)\}^{-1}\cdot f(x)$ which satisfies $f_0(x)>0$ on R, $Gf_0(x)>0$ on $\bar{K}^c$ and $Gf_0(x)\geqq 1$ on $\overline{A-K}$. Noting that $P_x(Gf_0(X(t))>0,\ 0\leqq t\leqq\sigma_K)=1$ for any $x\in\bar{K}^c$, we have (1.4) for $\sigma=\sigma_K$ and $f=f_0\in\mathfrak{D}(G)$ by letting $\alpha\downarrow 0$ in (1.3). Hence, we have

$$\begin{aligned} G^{R-K}(x,\ A)&\leqq G^{R-K}(x,\ \bar{A})=G^{R-K}(x,\ \overline{A-K})=E_x\left(\int_0^{\sigma_K}\chi_{\overline{A-K}}(X(t))\,dt\right)\\ &\leqq E_x\left(\int_0^{\sigma_K}Gf_0(X(t))\,dt\right)=E_x(f_0(X(\sigma_K))-f_0(x))\\ &\leqq \|f_0\|-f_0(x)\leqq 2\|f_0\|<\infty, \end{aligned} \tag{1.6}$$

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10) The proofs of Lemma 1.1–2 are slight modifications of the techniques in Ito and McKean [6].

11) We sometimes write $A^c=R-A$.

for each $x \in R$, completing the proof.

LEMMA 1.2.[12] *(X. 1), (X. 4) and the continuity of path functions imply (X. 5).*

Proof. It is sufficient to prove that there is no point of positive holding time when the above conditions are satisfied. Take any point $x_0 \in R$ and $f \in \mathfrak{D}(G)$ satisfying $Gf(x_0) > 0$. Putting $\sigma = \sigma_{\{x_0\}^c}$ and $x = x_0$ in (1.3) and repeating the same procedure in the proof of Lemma 1.1, we have $E_{x_0}(\sigma) < \infty$. Continuity of path functions implies $P_{x_0}(X(\sigma) = x_0) = 1$ and hence

$$0 = f(x_0) - f(x_0) = E_{x_0}(f(X(\sigma))) - f(x_0) = E_{x_0}\left(\int_0^\sigma Gf(X(t))\,dt\right) = Gf(x_0)\cdot E_{x_0}(\sigma),$$

that is, $E_{x_0}(\sigma) = 0$, completing the proof.

REMARK. *(X. 1)–(X. 4) do not imply (X. 5).* In fact, take $R = [0, 1]$ and put

$$T_t f(x) = e^{-t} f(x) + (1 - e^{-t}) \int_0^1 f(x)\,dx, \qquad t \geqq 0.$$

$\{T_t, t \geqq 0\}$ is a semigroup on $C(R)$, strongly continuous at $t = 0$. It can be proved that this is realized by a Markov process, where a particle at any point x at time $t = 0$ stays there for an exponential holding time σ with $E_x(\sigma) = 1$, and then jumps into $R - \{x\}$ with uniform distribution, or Lebesgue measure on $[0, 1]$. It is easy to prove (X. 1)–(X. 4), but (X. 5) is not satisfied.

§2. Fundamental lemmas.

Let $\mathfrak{F}$ be the family of all $\{K, L\}$ satisfying the following conditions:

($\mathfrak{F}$. 1) K and L are closed subsets of R and contain inner points,

($\mathfrak{F}$. 2) At least one of K and L is compact,

($\mathfrak{F}$. 3) At least one of following a) or b) holds. a) K is contained in one component of $R - L$. b) L is contained in one component of $R - K$.

For a pair $\{K, L\} \in \mathfrak{F}$, define $T^K(x, \cdot)$ for $x \in K$ by

$$T^K(x, E) = \int_R h^L(x, dy) h^K(y, E), \qquad x \in K,\ E \in \boldsymbol{B},\ E \subset K.$$

$\{T^K(x, \cdot), x \in K\}$ is a system of transition probabilities on K, since $T^K(x, K) \equiv 1$ by (X. 1). $T^K(x, \cdot)$ defines an operator T^K on both $B(K)$ and the space $B^*(K)$ of sigma-finite measures with supports contained in K, by

$$\begin{aligned} T^K f(x) &= \int_R T^K(x, dy) f(y), \qquad && x \in K,\ f \in B(K), \\ \varphi T^K(A) &= \int_R \varphi(dx) T^K(x, A), && A \subset K,\ A \in \boldsymbol{B}. \end{aligned} \tag{2.1}$$

12) The result of Ito and McKean [6] is sharper: the holding time at any point $x \in R$ is 0 or ∞ with $P_x(\cdot)$-probability 1 under the assumption (X. 4) and the continuity of path functions. This implies the conclusion trivially.

T^K *maps* $C(K)$ *into* $C(K)$, since by (X. 2) $h^K f(x)$ is continuous in $R-K$ for $f \in C(K)$ and hence $T^K f(x)$ is continuous in $R-L$. We note that $T^K(x, \cdot)$ *is equivalent for all* $x \in K$, since $h^L(x, \cdot)$ are equivalent for all $x \in K$ by (X. 3) if a) in ($\mathfrak{F}$. 3) holds and $h^K(x, \cdot)$ are equivalent for all $x \in L$ otherwise. T^L is defined similarly. Applying an ergodic theorem to T^K or T^L we have

LEMMA 2.1.[13] *For each* $\{K, L\} \in \mathfrak{F}$ *there is a unique pair of measures* $\mu_L{}^K$ *and* $\mu_K{}^L$ *with total mass* 1 *on* K *and* L *respectivily, satisfying*

$$\begin{aligned} \mu_L{}^K(\cdot) &= \mu_K{}^L h^K(\cdot) = \int_L \mu_K{}^L(dx) h^K(x, \cdot), \\ \mu_K{}^L(\cdot) &= \mu_L{}^K h^L(\cdot) = \int_K \mu_L{}^K(dx) h^L(x, \cdot). \end{aligned} \tag{2.2}$$

Proof. To fix the notation, let K be compact. We say that φ is an invariant measure on K, if it is a measure on K with total mass 1 and is invariant by T^K. Yosida [11, pp. 100–101] proved that a necessary and sufficient condition for the existence of an invariant measure on K is

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} (T^K)^m f(x) > 0, \qquad \text{for some} \quad x \in K, \ f \in C(K). \tag{2.3}$$

The left hand side of (2.3) is identically 1 for $f \equiv 1 \in C(K)$ by (X. 1), and there is an invariant measure.

To prove the uniqueness we use the decomposition of K in [11, p. 102]: there is a Borel subset $K' \subset K$ and a system of invariant measures $\{\varphi_x(\cdot), x \in K'\}$ on K satisfying $\varphi_x(K') = 1$. Every invariant measure φ on K is represented by

$$\varphi(\cdot) = \int_{K'} \varphi(dx) \varphi_x(\cdot). \tag{2.4}$$

The system of measures $\{\varphi_x\}$ determines a classification $\{E_\lambda, \lambda \in \Lambda\}$ of K' by the relation $\varphi_x(\cdot) = \varphi_y(\cdot)$, and we have

$$\varphi_x(E_\lambda) = \varphi_x(K') = 1 \qquad \text{for} \quad x \in E_\lambda.$$

For each E_λ there is a Borel subset $\hat{E}_\lambda$ satisfying

$$T^K(x, \hat{E}_\lambda) = 1 \qquad \text{for} \quad x \in \hat{E}_\lambda. \tag{2.5}$$

In our case there is only one class $E_\lambda = K'$. In fact, if there are two classes E_{λ_1} and E_{λ_2}, $T^K(x, \hat{E}_{\lambda_1}) = 1$ on $\hat{E}_{\lambda_1}$ and 0 on $\hat{E}_{\lambda_2}$ by (2.5), which is impossible by the equivalence of $T^K(x, \cdot)$, $x \in K$. Hence, there is only one T^K-invariant measure $\mu_L{}^K$ on K by (2.4). Since any T^K-invariant measure μ induces a T^L-invariant measure μh^L and conversely, we have a unique T^L-invariant measure $\mu_K{}^L = \mu_L{}^K h^L$ satisfying (2.2).

13) H. P. McKean advised the author to check the connection between Nelson [7] and this proposition in a private communication. The result of the check is contained in Appendix I.

LEMMA 2.2 *If $\{K, L\}$, and $\{K', L\}$ belong to $\mathfrak{F}$ with $K' \subset K$, then*

$$\nu(\cdot) = \mu_L{}^K h_{K'}{}^{K' \cup L}(\cdot) = \int_K \mu_L{}^K(dx) h_{K'}{}^{K' \cup L}(x, \cdot) \tag{2.6}$$

is a non-trivial measure on K' satisfying

$$\nu(\cdot) = \nu(K') \cdot \mu_L{}^{K'}(\cdot). \tag{2.7}$$

The counterpart ν' on L of ν is given by

$$\nu'(\cdot) = \nu h^L(\cdot) = \mu_K{}^L(\cdot) - \mu_L{}^K h_L{}^{K' \cup L}(\cdot). \tag{2.8}$$

Proof. To see the invariance of ν under the transformation $(h^{K'} h^L)(x, \cdot)$ we note, for $E \subset L$,

$$\begin{aligned}
h^L(x, E) &= P_x(X(\sigma_L \in E)) \\
&= P_x(X(\sigma_L) \in E,\ \sigma_L < \sigma_{K'}) + P_x(X(\sigma_L) \in E,\ \sigma_L > \sigma_{K'}) \\
&= P_x(X(\sigma_{L \cup K'}) \in E, \sigma_L < \sigma_{K'}) + P_x(X(\sigma_L) \in E,\ \sigma_L > \sigma_{L \cup K'}) \\
&= h^{L \cup K'}(x, E) + E_x(P_x(X(\sigma_L) \in E / \mathfrak{B}_{\sigma_{L \cup K'}}),\ \sigma_L > \sigma_{L \cup K'}) \\
&= h_L{}^{L \cup K'}(x, E) + E_x(P_{x(\sigma_{L \cup K'})}(X(\sigma_L) \in E),\ \sigma_L > \sigma_{L \cup K'}) \\
&= h_L{}^{L \cup K'}(x, E) + \int_{K'} h_{K'}{}^{L \cup K'}(x, dy) h^L(y, E) \\
&= h_L{}^{L \cup K'}(x, E) + (h_{K'}{}^{L \cup K'} h^L)(x, E).
\end{aligned}$$

Hence, by (2.6), we have

$$\begin{aligned}
\mu_K{}^L(E) &= \mu_L{}^K h^L(E) = \mu_L{}^K h_L{}^{L \cup K'}(E) + \mu_L{}^K h_{K'}{}^{L \cup K'} h^L(E) \\
&= \mu_L{}^K h_L{}^{L \cup K'}(E) + (\nu h^L)(E),
\end{aligned} \tag{2.9}$$

which implies

$$\mu_K{}^L h^{K'}(\cdot) = \mu_L{}^K h_L{}^{L \cup K'} h^{K'}(\cdot) + \nu h^L h^{K'}(\cdot). \tag{2.10}$$

Similar calculations for $h^{K'}(\cdot)$ shows

$$h^{K'}(x, \cdot) = h_{K'}{}^{L \cup K'}(x, \cdot) + h_L{}^{L \cup K'} h^{K'}(x, \cdot).$$

This implies

$$\begin{aligned}
\mu_K{}^L h^{K'}(\cdot) &= \mu_K{}^L h^K h^{K'}(\cdot) = \mu_L{}^K h^{K'}(\cdot) \\
&= \mu_L{}^K h^{L \cup K'}(\cdot) + \mu_L{}^K h_L{}^{L \cup K'} h^{K'} = \nu(\cdot) + \mu_L{}^K h_L{}^{L \cup K'} h^{K'}(\cdot).
\end{aligned} \tag{2.11}$$

Comparing (2.10) and (2.11) we have the invariance of ν. (2.9) shows that the counterpart of ν on L is given by (2.8). By the uniquess of the $h^L h^{K'}(x, \cdot)$-invariant measure up to a constant factor we have (2.7).

To prove that ν is non-trivial, assume the contrary, or $\nu(K') = 0$. Applying $(X.\ 3)$ for L and

$$\int_K h^K(x, dy) h^{K' \cup L}(y, K') = \int_K h^K(x, dy) P_y(\sigma_{K'} < \sigma_L)$$

in the case a) in ($\mathfrak{F}$. 3), we know

$$T^K h^{K'\cup L}(x, K') = \int_K T^K(x, dy) h^{K'\cup L}(y, K') = \int_L h^L(x, dy) \int_K h^K(y, dz) h^{K'\cup L}(z, K')$$

is positive for all $x \in K$ or $\equiv 0$ on K. Since, in the case of b),

$$\int_K h^K(x, dy) h^{K'\cup L}(y, K')$$

is positive for all $x \in L$ or $\equiv 0$ on L by the application of (X. 3) to K and $h^{K'\cup L}(y, K')$ on K, we have the same conclusion for $T^K h^{K'\cup L}(x, K')$ on K. But

$$\begin{aligned} \nu(K') &= \mu_L{}^K h^{K'\cup L}(K') = (\mu_L{}^K T^K) h^{K'\cup L}(K') = \mu_L{}^K (T^K h^{K'\cup L})(K') \\ &= \int_K \mu_L{}^K(dx) (T^K h^{K'\cup L})(x, K') \end{aligned} \tag{2.12}$$

implies the latter case, i.e. $T^K h^{K'\cup L}(x, K) \equiv 0$ on K. (2.12) also shows that

$$\int_K h^K(x, dy) h^{K'\cup L}(y, K') = 0$$

for a set of $x \in L$ with $\mu_K{}^L$-measure one. Fix such a point $x_0 \in L$.

Define the following sequence of hitting times.

$$\begin{aligned} \sigma_1 &= \sigma_K, \\ \tau_n &= \inf\{t > \sigma_n \mid X(t) \in L\}, && \text{when such } t < \infty \text{ exists,} \\ &= \infty, && \text{otherwise,} \\ \sigma_{n+1} &= \inf\{t > \tau_n \mid X(t) \in K\}, && \text{when such } t < \infty \text{ exists,} \\ &= \infty, && \text{otherwise.} \end{aligned}$$

By the assumptions on the path functions the definition is possible, and using (X. 1), we find that

$$\sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots,$$
$$P_x(\lim_{n\to\infty} \sigma_n = \infty) = P_x(\lim_{n\to\infty} \tau_n = \infty) = 1.$$

Then, for x_0 above, we have

$$\begin{aligned} 1 &= P_{x_0}(\sigma_{K'} < \infty) = \sum_{n=1}^{\infty} P_{x_0}(\sigma_n \leqq \sigma_{K'} < \tau_n < \infty) \\ &= E_{x_0}\{P_{x(\sigma_1)}(\sigma_{K'} < \sigma_L)\} + \sum_{n=2}^{\infty} E_{x_0}\{P_{x(\sigma_{n-1})}(\tau_1 < \sigma_2 \leqq \sigma_{K'} < \tau_2),\ \sigma_{n-1} < \sigma_{K'}\} \\ &= \int_K h^K(x_0, dy) h^{K'\cup L}(y, K') + \sum_{n=2}^{\infty} E_{x_0}[E_{x(\sigma_{n-1})}\{P_{x(\tau_1)}(\sigma_K \leqq \sigma_{K'} < \tau_2), \\ &\qquad\qquad \tau_1 < \sigma_{K'}\},\ \sigma_{n-1} < \sigma_{K'}] \\ &\leqq 0 + \sum_{n=1}^{\infty} E_{x_0}[E_{x(\sigma_n)}\{E_{x(\sigma_L)}) P_{x(\sigma_K)}(\sigma_{K'} < \sigma_L))\},\ \sigma_n < \sigma_{K'}] \\ &= \sum_{n=1}^{\infty} E_{x_0}\left\{\int_K h^K(X(\sigma_n), dx) \int_L h^L(x, dy) \int_K h^K(y, dz) P_z(\sigma_{K'} < \sigma_L),\ \sigma_n < \sigma_{K'}\right\} \\ &= \sum_{n=1}^{\infty} E_{n_0}\left\{\int_K h^K(X(\sigma_n), dx)(T^K h^{K'\cup L}(x, K')),\ \sigma_n < \sigma_{K'}\right\} = 0, \end{aligned}$$

contradicting (X. 1). This completes the proof.

For a pair $\{K, L\} \in \mathfrak{F}$, define

$$m_{K,L}(\cdot) = (\mu_L{}^K G^{R-L})(\cdot) + (\mu_K{}^L G^{R-K})(\cdot) = \int_K \mu_L{}^K(dx) G^{R-L}(x, \cdot) + \int_L \mu_K{}^L(dx) G^{R-K}(x, \cdot). \tag{2.13}$$

This is a sigma-finite measure on $(R, \boldsymbol{B})$ by Lemma 1.1. We note for a later use that there is a compact set K_0 for which $0 < m_{K,L}(K_0) < \infty$, because $G^{R-L}(x, R) = E_x(\sigma_L) > 0$ for each $x \in K$ by the right continuity of paths and so $0 < \mu_L{}^K G^{R-L}(R) \leqq m_{K,L}(R) \leqq \infty$, which, together with the existence of a sequence of compact sets $K_n \uparrow R$, implies the existence of such $K_0 = K_{n_0}$.

LEMMA 2.3. *If K, L, K', ν and ν' are the same as in Lemma 2.2, then*

$$m_{K,L}(\cdot) = \nu(K') m_{K',L}(\cdot). \tag{2.14}$$

Proof. For $x \in K$ and $E \in \boldsymbol{B}$, we have

$$\begin{aligned} G^{R-L}(x, E) &= E_x\left(\int_0^{\sigma_L} \chi_E(X(t))\,dt\right) \\ &= E_x\left(\int_0^{\sigma_L} {''}, \sigma_L \leqq \sigma_{K'}\right) + E_x\left(\int_0^{\sigma_{K'}} {''}, \sigma_L > \sigma_{K'}\right) + E_x\left(\int_{\sigma_{K'}}^{\sigma_L} {''}, \sigma_L > \sigma_{K'}\right)^{14)} \\ &= E_x\left(\int_0^{\sigma_{L \cup K'}} {''}\right) + E_x\left(\int_{\sigma_{K'}}^{\sigma_L} {''}, \sigma_L > \sigma_{K'}\right) \\ &= E_x\left(\int_0^{\sigma_{L \cup K'}} {''}\right) + E_x\left(E_x\left(\int_{\sigma_{K'}}^{\sigma_L} {''} / \mathfrak{B}_{\sigma_{K'}+}\right), \sigma_L > \sigma_{K'}\right) \\ &= E_x\left(\int_0^{\sigma_{L \cup K'}} {''}\right) + E_x\left(E_{x(\sigma_{K'})}\left(\int_0^{\sigma_L} {''}\right), \sigma_L > \sigma_{K'}\right) \\ &= G^{R-(L \cup K')}(x, E) + \int_{K'} h^{L \cup K'}(x, dy) G^{R-L}(y, E). \end{aligned} \tag{2.15}$$

Similarly we have

$$E_x\left(\int_0^{\sigma_{K'}} \chi_E(X(t))\,dt\right) = G^{R-(L \cup K')}(x, E) + \int_L h^{L \cup K'}(x, dy)\, G^{R-K'}(y, E). \tag{2.15'}$$

(2.6) and (2.15) imply

$$\begin{aligned} \int_K \mu_L{}^K(dx)\, G^{R-L}(x, E) &= \int_K \mu_L{}^K(dx)\, G^{R-(L \cup K')}(x, E) \\ &\quad + \int_K \mu_L{}^K(dx) \int_{K'} h^{L \cup K'}(x, dy)\, G^{R-L}(y, E) \\ &= \int_K \mu_L{}^K(dx)\, G^{R-(L \cup K')}(x, E) + \int_{K'} \nu(dy)\, G^{R-L}(y, E). \end{aligned} \tag{2.16}$$

14) We use the abbreviation $E_x\left(\int_\sigma^\tau {''}, \Lambda\right)$ or $E_x\left(\int_\sigma^\tau {''} / \mathfrak{B}_{\sigma+}\right)$ etc. for $E_x\left(\int_\sigma^\tau \chi_A(X(t))dt, \Lambda\right)$ or $E_x\left(\int_\sigma^\tau \chi_A(X(t))\,dt / \mathfrak{B}_{\sigma+}\right)$ etc. when no confusion is expected, where σ and τ are Markov times and Λ is a $\mathfrak{B}$-measuaable condition.

On the other hand we have

$$
\begin{aligned}
G^{R-K'}(x, E) &= E_x\Big(\int_0^{\sigma_{K'}} \chi_E(X(t))\,dt\Big) = E_x\Big(\int_0^{\sigma_K} {}''\Big) + E_x\Big(\int_{\sigma_K}^{\sigma_{K'}} {}''\Big)\\
&= E_x\Big(\int_0^{\sigma_K} {}''\Big) + E_x\Big(E_{x(\sigma_K)}\Big(\int_0^{\sigma_{K'}} {}''\Big)\Big) \qquad (2.17)\\
&= G^{R-K}(x, E) + \int_K h^K(x, dy) E_y\Big(\int_0^{\sigma_{K'}} \chi_E(X(t))\,dt\Big).
\end{aligned}
$$

(2.8), (2.15′) and (2.17) imply

$$
\begin{aligned}
&\int_L \nu'(dx) G^{R-K'}(x, E)\\
&= \int_L \mu_K{}^L(dx) G^{R-K'}(x, E) - \int_K \mu_L{}^K(dx) \int_L h^{L\cup K'}(x, dy) G^{R-K'}(y, E)\\
&= \int_L \mu_K{}^L(dx) G^{R-K}(x, E) + \int_L \mu_K{}^L(dx) \int_K h^K(x, dy) \Big\{ G^{R-(L\cup K')}(y, E) \qquad (2.18)\\
&\quad + \int_L h^{L\cup K'}(y, dz) G^{R-K'}(z, E)\Big\} - \int_L \mu_L{}^K(dx) \int_L h^{L\cup K'}(x, dy) G^{R-K'}(y, E)\\
&= \int_L \mu_K{}^L(dx) G^{R-K}(x, E) + \int_K \mu_L{}^K(dx) G^{R-(L\cup K')}(x, E).
\end{aligned}
$$

Hence, by (2.16) and (2.18), we have

$$\int_{K'} \nu(dx) G^{R-L}(x, E) + \int_L \nu'(dx) G^{R-K'}(x, E) = m_{K,L}(E),$$

where the left hand side is clearly $\nu(K') m_{K',L}(E)$ by (2.7).

Now, we introduce, without proof, an important property of $m_{K,L}$ thus constructed:

THEOREM. (G. Maruyama and H. Tanaka) *$m_{K,L}$ is an invariant measure for the process $X(t)$, that is,*

$$m_{K,L}(\cdot) = \int_R m_{K,L}(dx) P(t, x, \cdot), \qquad \textit{for each} \quad t > 0.$$

The proof is the same as that given in [7] which is based on a slightly stronger condition than (X. 2). An intuitive explanation of the invariance is found in the proof of the corresponding result for temporally discrete case. The reader, if interested, can consult Appendix II.

For an application in §5, we consider now the convergence of $\mu(T^K)^n$, $n = 1, 2, \cdots$ to $\mu_L{}^K$ for an arbitrarily given measure μ on K with total mass 1. In the place of (X. 2) in §1, we use here a stronger condition

(X. 2′). *If $\{f_\lambda, \lambda \in \Lambda\}$ is a uniformly bounded class of functions of $B(K)$, $\{h^K f_\lambda, \lambda \in \Lambda\}$ is equicontinuous on each fixed compact subset of $R-K$, where K is a closed set ($\neq R$) with an inner point.*

Note that this is satisfied in the case of the Brownian motion.

PROPOSITION 2.1. *In order that the measure $h^K(x, \cdot)$ varies continuously in $x \in R-K$ with respect to the norm of total variation, it is necessary and sufficient that (X. 2′) holds.*

Proof. Since the necessity is clear, we prove the sufficiency. Assume the contrary, i.e. assume the existence of a sequence $x_n \in R-K$, $n=0, 1, 2, \cdots$ and an $\varepsilon_0>0$ with $\lim_{n\to\infty} x_n = x_0$ and $\| h^K(x_n, \cdot) - h^K(x_0, \cdot) \| \geqq \varepsilon_0$ for $n \geqq 1$. Fix a compact neighborhood $L \subset R-K$ of x_0 and assume without the loss of generality that all x_n are in L. Let K_n^+ and K_n^- be the sets of positive part and the negative part of K with respect to the Hahn-decomposition of $h^K(x_0, \cdot) - h^K(x_n, \cdot)$, $n \geqq 1$ respectively. Define χ_n on K by $\chi_n(x)=1$ for $x \in K_n^+$ and $\chi_n(x)=-1$ for $x \in K_n^-$, and f_n on L by $f_n(x)=h^K \chi_n(x)$.

Since $\| \chi_n \| \leqq 1$, $\{f_n(x), n=1, 2, \cdots\}$ is a class of equicontinuous functions in $C(L)$ by (X. 2′), and hence there is a subsequence $\{f_{n'}\}$ and a continuous function f_0 on L with

$$\lim_{n'\to\infty} \|f_{n'} - f_0\| = 0,$$

according to the Ascoli-Arzelà theorem. This, with the continuity of f_0, implies

$$\begin{aligned} 0 < \varepsilon_0 &\leqq \lim_{n'\to\infty} \| h^K(x_{n'}, \cdot) - h^K(x_0, \cdot) \| = \lim_{n'\to\infty} \int_K \{h^K(x_{n'}, dy) - h^K(x_0, dy)\} \chi_{n'}(y) \\ &= \lim_{n'\to\infty} (f_{n'}(x_{n'}) - f_0(x_0)) \leqq \lim_{n'\to\infty} \{ |f_{n'}(x_{n'}) - f_0(x_{n'})| + |f_0(x_{n'}) - f_0(x_0)| \} \\ &\leqq \lim_{n'\to\infty} \{ \|f_{n'} - f_0\| + |f_0(x_{n'}) - f_0(x_0)| \} = 0, \end{aligned}$$

which is a contradiction.

PROPOSITION 2.2. *Assume that (X. 2′) holds. For any measure μ on K and ν on L with total mass 1, $\mu(T^K)^n$, $\nu(T^L)^n$, $n=1, 2, \cdots$, converge to the unique limit $\mu_L{}^K$ and $\mu_K{}^L$ respectively, with respect to the norm of total variation, and exponentially fast.*

Proof. Since $h^K(x, \cdot)$ and $h^L(x, \cdot)$ vary continuously in x by Proposition 2.1, so do $T^K(x, \cdot)$ and $T^L(x, \cdot)$. In fact,

$$\begin{aligned} &\|T^K(x, \cdot) - T^K(y, \cdot)\| = \sup_{f \in B_1(K)} \int_K \{T^K(x, dz) - T^K(y, dz)\} f(z) \\ (2.19) \qquad &= \sup_{f \in B_1(K)} \int_L \{h^L(x, dz) - h^L(y, dz)\} h^K f(z) \\ &\leqq \| h^L(x, \cdot) - h^L(y, \cdot) \| \cdot \sup_{f \in B_1(K)} \{ \sup_{z \in L} | h^K f(z) | \} \leqq \| h^L(x, \cdot) - h^L(y, \cdot) \|, \end{aligned}$$

where $B_1(K) = \{ f \in B(K) \mid \|f\| \leq 1 \}$. To fix the notation, let K be compact. We now prove

$$Q(T^K) = \frac{1}{2} \sup_{x, y \in K} \| T^K(x, \cdot) - T^K(y, \cdot) \| < 1, \tag{2.20}$$

which, by [9, pp. 454–5], implies the conclusion for $\{\mu(T^K)^n\}$. If we assume the contrary, there are two sequences $\{x_n \in K\}$ and $\{y_n \in K\}$ with

$\lim_{n\to\infty} \| T^K(x_n, \cdot) - T^K(y_n, \cdot) \| = 2$. Since K is compact, there are subsequences $\{x_{n'}\}$ and $\{y_{n'}\}$ with limits x_0 and y_0 in K. By the continuity of $T^K(x, \cdot)$ in x we have

$$\| T^K(x_0, \cdot) - T^K(y_0, \cdot) \| = \lim_{n'\to\infty} \| T^K(x_{n'}, \cdot) - T^K(y_{n'}, \cdot) \| = 2.$$

By the Hahn decomposition of $T^K(x_0, \cdot) - T^K(y_0, \cdot)$ we have two mutually disjoint subsets K^+ and $K^- \in \boldsymbol{B}$ with $K^+ \cap K^- = R$, for which we have

$$T^K(x_0, K^+) = T^K(y_0, K^-) = 1, \ T^K(x_0, K^-) = T^K(y_0, K^+) = 0, \tag{2.21}$$

contradicting the equivalence of $T^K(x_0, \cdot)$ and $T^K(y_0, \cdot)$. For the convergence of $\{\nu(T^L)^n\}$, we can prove that $Q(T^L)^2) \leqq Q(T^K)$ by a computation similar to (2.19), and then infer the proposition from [9, pp. 454–5].

§3. The system of Green capacities and the measure m.[15]

For later use we say that $\{K, L\} \in \mathfrak{F}$ belongs to $\mathfrak{F}_0$ if $\{K, L\}$ satisfies both a) and b) in ($\mathfrak{F}$. 3) of §2. Given $\{K, L\}$ and $\{K, L'\}$ in $\mathfrak{F}$, we write

$$\{K, L\} \leftrightarrow \{K, L'\} \qquad \text{when} \quad \{K, L \cup L'\} \in \mathfrak{F}. \tag{3.1}$$

If, for $\{K, L\}$ and $\{K', L'\}$ in $\mathfrak{F}$, there is a sequence $\alpha = (\{K_1, L_1\}, \{K_2, L_2\}, \cdots, \{K_n, L_n\})$ of $\mathfrak{F}$ satisfying $\{K, L\} \leftrightarrow \{K_1, L_1\} \leftrightarrow \cdots \leftrightarrow \{K_n, L_n\} \leftrightarrow \{K', L'\}$, we write

$$\{K, L\} \underset{\alpha}{\leftrightarrow} \{K', L'\}. \tag{3.2}$$

LEMMA 3.1 *For any two elements $\{K, L\}$ and $\{K', L'\}$ of $\mathfrak{F}$ there is a sequence $\alpha = (\{K_1, L_1\}, \{K_2, L_2\}, \cdots, \{K_n, L_n\})$ consisting of elements of $\mathfrak{F}_0$ satisfying* (3.2).

Proof. Since a) or b) holds for $\{K, L\}$, let a) hold to fix the notation and take one component V of $R - K$ which contains an inner point of L and put $L_1 = L \cap V$. Then, $\{K, L_1\}$ clearly belongs to $\mathfrak{F}_0$ and satisfies $\{K, L\} \leftrightarrow \{K, L_1\}$. By the same consideration applied to $\{K', L'\}$, we know that it is sufficient to prove the lemma only for $\{K, L\}$ and $\{K', L'\}$ in $\mathfrak{F}_0$. Hence, we assume this condition in the following.

Let K be a closed set in R and let $\{V_\alpha\}$ be the family of all connected components of $R - K$, we note that for each V_α, there exist two points $x \in V_\alpha$ and $y \in K$ and an arc, contained in $K \cup V_\alpha$, joining x and y. To show this let $\cup' V_\alpha$ be the union of all V_α, for which two points and the arc cited above exist. Since $R' = K \cup (\cup' V_\alpha)$ is a non-empty closed set and R is connected, it is sufficient to prove that R' is open, or $R = R'$. If $x \in R'$ is in $\cup' V_\alpha$, it is an

15) The author owes main part of this section to Keniti Sato; more precisely, the topological set up and the idea of using m to establish the system of capacities are his. His help ameliorates the condition on the topology of R and simplifies the author's original proof, which was tedious and restricted to processes on the union of a domain in R^n ($n \geqq 2$) and measurable subset of its boundary.

inner point of $\cup' V_\alpha$ and hence of R'. When $x \in R'$ is in an arcwise connected neighborhood V of x. Then, V is contained in R'. In fact, if a point $x_0 \in V \frown K^c$ exists, then there is an arc $w = \{f(t), 0 \leqq t \leqq 1\}$ in V with $f(0) = x_0$ and $f(1) = x$. For $t_1 = \inf\{t \mid f(t) \in K\}$, $x_1 = f(t_1) \in K$ and $\{f(t), 0 \leqq t < t_1\}$ lie in one component V_α, implying $x_0 \in R'$. Hence R' is open.

First, assume $K = K'$ and that L and L' are in the same component of $R - K$. Taking a connected open set $G \subset K$ and a compact subset K_0 thereof with an inner point, we have $\{K_0, L \smile L'\} \in \mathfrak{F}_0$, and hence $\{K, L\} \leftrightarrow \{K_0, L\} \leftrightarrow \{K_0, L'\} \leftrightarrow \{K, L'\}$.

Secondly, assume $K = K'$ and that L and L' are in different components of $R - K$. For a compact subset L_0 of L with an inner point, we have clearly $\{L_0, L'\} \in \mathfrak{F}$. Moreover, we see that K and L_0 are in the same component of $R - L'$. In fact, for the connected component V of $R - K$ containing L_0, there is an arc in $K \smile V$ joining a point of V and a point of K as we have shown above. This means that L_0 and K are in the same connected component of $R - L'$, and hence there is a sequence α_1 consisting of elements of $\mathfrak{F}_0$ with $\{L_0, L'\} \underset{\alpha_1}{\leftrightarrow} \{K, L'\}$ as we have discussed above. Similarly, K and L' are in the same component of $R - L_0$, and we can find a sequence α_2 consisting of $\mathfrak{F}_0$ with $\{K, L_0\} \underset{\alpha_2}{\leftrightarrow} \{L_0, L'\}$. Hence, we have $\{K, L\} \leftrightarrow \{K, L_0\} \underset{\alpha_2}{\leftrightarrow} \{L_0, L'\} \underset{\alpha_1}{\leftrightarrow} \{K, L'\}$.

Now, consider the general case. Take inner points x_1 and x_2 ($x_1 \neq x_2$) of K and K' respectively, and let V_1, V_2 be connected open neighborhoods of x_1 and x_2 respectively with $V_1 \frown V_2 = \phi$, and then fix two compact neighborhoods $K_1 \subset V_1$ and $K_2 \subset V_2$ of x_1 and x_2 respectively. Clearly $\{K_1, L\}$, $\{K_1, K_2\}$ and $\{K_2, L'\}$ are in $\mathfrak{F}_0$. Hence, by combining above two special cases, we can find two sequences α_1 and α_2 consisting of elements of $\mathfrak{F}_0$ with

$$\{K, L\} \leftrightarrow \{K_1, L\} \underset{\alpha_1}{\leftrightarrow} \{K_1, K_2\} \underset{\alpha_2}{\leftrightarrow} \{K_2, L'\} \leftrightarrow \{K', L'\},$$

which completes the proof.

To describe the relations among $\{\mu_L{}^K\}$ and $\{m_{K,L}\}$ for $\{K, L\} \in \mathfrak{F}$, we make the following definitions for $\{K, L\}$ and $\{K', L'\}$ in $\mathfrak{F}$, using Lemma 2.2.

$$\begin{aligned} &0 < C_{(K,L)}(K', L) = (\mu_L{}^K h^{K' \smile L})(K') \qquad \text{and} \\ &C_{(K',L)}(K, L) = C_{(K,L)}(K', L)^{-1}, \qquad \text{when} \quad K' \subset K. \end{aligned} \tag{3.3}$$

$$\begin{aligned} &0 < C_{(K,L)}(K', L) = C_{(K,L)}(K \smile K', L) \cdot C_{(K \smile K', L)}(K', L), \\ &\qquad\qquad \text{when} \quad \{K, L\} \leftrightarrow \{K', L\}. \end{aligned} \tag{3.3'}$$

$$0 < C^\alpha{}_{(K,L)}(K', L') = C_{(K,L)}(K_1, L_1) \cdot C_{(K_1, L_1)}(K_2, L_2) \cdots C_{(K_n, L_n)}(K', L'), \tag{3.3''}$$

when, $\{K, L\} \underset{\alpha}{\leftrightarrow} \{K', L'\}$ with $\alpha = (\{K_1, L_1\}, \{K_2, L_2\}, \cdots, \{K_n, L_n\})$. Note that the order of sets, for instance, K and L in $C_{(K,L)}(\cdot, \cdot)$ or $C_{(\cdot,\cdot)}(K, L)$ does not make any difference.

LEMMA 3.2. *$C^\alpha{}_{(K,L)}(K', L')$ does not depend on the choice of α. Hence, we can define a positive valued function*

$$C_{(K,L)}(K', L') \equiv C^{\alpha}{}_{(K,L)}(K', L'),$$

by any fixed choice of α *with* $\{K, L\} \underset{\alpha}{\leftrightarrow} \{K', L'\}$. *This function satisfies, by definition,*

$$C_{(K_1,L_1)}(K_2, L_2)C_{(K_2,L_2)}(K_3, L_3) = C_{(K_1,L_1)}(K_3, L_3) \tag{3.4}$$

$$C_{(K_1,L_1)}(K_2, L_2) = C_{(K_2,L_2)}(K_1, L_1)^{-1}, \tag{3.5}$$

and depends only on the system of hitting measures. Moreover, we have

$$m_{K,L}(\cdot) = C_{(K,L)}(K', L')m_{K',L'}(\cdot) \tag{3.6}$$

which characterizes this function.

Proof. First, we prove (3.6) for some special choice of α, i.e. $m_{K,L} = C^{\alpha}{}_{(K,L)}(K', L')m_{K',L'}$. By Lemma 2.3 we have $m_{K,L} = C^{\alpha}{}_{(K,L)}(K', L)m_{K',L}$ in the case (3.3). In the case (3.3′) we also have the result, since

$$\begin{aligned} m_{K,L} &= C_{(K,L)}(K' \smile K, L)m_{K'\smile K,L} = C_{(K,L)}(K' \smile K, L)C_{(K'\smile K,L)}(K', L)m_{K',L} \\ &= C_{(K,L)}(K', L)m_{K'L}. \end{aligned}$$

Now, applying the result repeatedly for α in (3.3″), we have

$$\begin{aligned} m_{K,L} &= C_{(K,L)}(K_1, L_1)m_{K_1,L_1} = C_{(K,L)}(K_1, L_1)C_{(K_1,L_1)}(K_2, L_2)m_{K_2,L_2} \\ &= \cdots \\ &= C_{(K,L)}(K_1, L_1)\cdot C_{(K_1,L_1)}(K_2, L_2)\cdots C_{(K_n,L_n)}(K', L')m_{K',L'} \\ &= C^{\alpha}{}_{(K,L)}(K', L')m_{K',L'}. \end{aligned} \tag{3.7}$$

Take a set K_0 with $0 < m_{K,L}(K_0) < \infty$, which really exists by the note just before Lemma 2.3. By applying (3.7) for two sequences α_i with $\{K, L\} \underset{\alpha_i}{\leftrightarrow} \{K', L'\}$ $(i = 1, 2)$, we have

$$0 < m_{K,L}(K_0) = C^{\alpha_i}{}_{(K,L)}(K', L')\cdot m_{K',L'}(K_0) < \infty \qquad (i = 1, 2),$$

which necessarily imply $C^{\alpha_1}{}_{(K,L)}(K', L') = C^{\alpha_2}{}_{(K,L)}(K', L')$.

LEMMA 3.3. *$m_{K,L}$ takes a positive value for any set in $\boldsymbol{B}$ with an inner point.*

Proof. If V is such a set, we can take an open subset V_0 of V and a compact set K' which is contained in one component of V_0 and has an inner point. Then, $\{K', L'\} \in \mathfrak{F}$ for $L' = V^c$. By the right continuity of paths, we have $P_x(\sigma_{L'} = \sigma_{V_0{}^c} > 0) = 1$ for each $x \in K'$, and hence

$$0 < E_x(\sigma_{L'}) = E\left(\int_0^{\sigma_{L'}} \chi_{V_0}(X(t))dt\right) = G^{R-L'}(x, V_0) \leqq G^{R-L'}(x, V), \quad \text{for } x \in K',$$

which implies

$$m_{K,L}(V) = C_{(K,L)}(K', L')m_{K',L'}(V) \leqq C_{(K,L)}(K', L')\int_{K'} \mu_{L'}{}^{K'}(dx)G^{R-L'}(x, V) > 0.$$

Now, fix any $\{K_0, L_0\} \in \mathfrak{F}$ and call $C(K, L) = C_{(K_0,L_0)}(K, L)$ the *Green capa-*

city of K and of L with respect to $R-L$ and $R-K$ respectively. By (3.4) and (3.5) such a function $C(K, L)$ depends on the choice of $\{K_0, L_0\}$ only up to a constant factor. Writing

$$\nu_L{}^K(\cdot)=C(K, L)\mu_L{}^K(\cdot),$$

we sum up the main results in §2 and §3 for later use.

THEOREM 3.1. *For each $\{K, L\}$ in $\mathfrak{F}$, there is a pair of measures $\nu_L{}^K$ and $\nu_K{}^L$ on K and L respectively, satisfying*

$$\nu_L{}^K h^L=\nu_K{}^L, \qquad \nu_K{}^L h^K=\nu_L{}^K \tag{3.8}$$

$$\nu_L{}^{K'}=\nu_L{}^K h_{K'}{}^{L\cup K'}, \quad and \quad \nu_{K'}{}^L=\nu_K{}^L-\nu_L{}^K h_L{}^{L\cup K'}, \tag{3.9}$$

$$when \quad K'\subset K,\ \{K, L\},\ \{K, L'\}\in\mathfrak{F}.$$

Such a system is unique up to the constant factor, and depends only on the system of hitting measures.

The right hand side of

$$m(\cdot)=\int_K \nu_L{}^K(dx)G^{R-L}(x, \cdot)+\int_L \nu_K{}^L(dx)G^{R-K}(x, \cdot) \tag{3.10}$$

does not depend on the choice of $\{K, L\}\in\mathfrak{F}$. The measure m is sigma-finite on $(R, \boldsymbol{B})$, takes positive value for each Borel set with inner points, and is an invariant measure of the process $\{X(t)\}$.

§4. Green potentials.

THEOREM 4.1. *Every Green measure $G^{R-K}(x, \cdot)$ is absolutely continuous relative to m.*

Proof. At first we prove that $G^{R-K}(x_0, A)>0$ implies $m(A)>0$ for any $x_0\in R-K$ and $A\subset R-\{x_0\}$. Choose a monotone sequence of connected open neighborhoods $V_n\subset R-K$ of x_0 converging to x_0. Since $G^{R-K}(x_0, A\cap V_n{}^c)$ converges to $G^{R-K}(x_0, A)>0$, there is an $n=n_0$ with $G^{R-K}(x_0, A\cap V_n{}^c)>0$. This shows that it is sufficient to prove the statement for $A\subset R-V$, where V is a connected open neighborhood of x_0. According to

$$\begin{aligned}0<G^{R-K}(x_0, A)&=E_{x_0}\left(\int_0^{\sigma_K}\chi_A(X(t)\,dt)\right)=E_x\left(\int_0^{\sigma_{V^c}}{}''+\int_{\sigma_{V^c}}^{\sigma_K}{}''\right)\\&=E_{x_0}\left(\int_{\sigma_{V^c}}^{\sigma_K}{}''\right)=E_{x_0}\left(E_{X(\sigma_{V^c})}\left(\int_0^{\sigma_K}{}''\right)\right)=\int_{V^c}h^{V^c}(x_0, dy)G^{R-K}(y, A),\end{aligned} \tag{4.1}$$

we have $h^{V^c}(x_0, E)>0$, where $E=\{x\in V^c\mid G^{R-K}(x, A)>0\}$. Then, by the equivalence of $h^{V^c}(x, \cdot)$ for $x\in V$, we have

$$h^{V^c}(x, E)>0, \qquad \text{for} \quad x\in V.$$

Hence, by putting $x=x_0$ in (4.1), we have

$$G^{R-K}(x, A)=\int_{V^c}h^{V^c}(x, dy)G^{R-K}(y, A)>0, \qquad \text{for} \quad x\in V.$$

Then, taking an $L' \subset V$, such that $\{K, L'\} \in \mathfrak{F}$, we have

$$0 < \int_{L'} \nu_K{}^{L'}(dx) G^{R-K}(x, A) \leqq m(A).$$

To complete the proof, it is sufficient to prove $m(\{x_0\}) > 0$ from

$$G^{R-K}(x_0, \{x_0\}) > 0. \tag{4.2}$$

Take the sequence $\{V_n\}$ used above. Considering

$$G^{R-K}(x, \{x_0\}) = E_x\left(\int_0^{\sigma_{V_n{}^c}} \chi_{\{x_0\}}(X(t))\, dt\right) + \int_{V_n{}^c} h^{V_n{}^c}(x, dy) G^{R-K}(y, \{x_0\}) \tag{4.3}$$

we have two cases:

$$\lim_{n\to\infty} E_{x_0}\left(\int_0^{\sigma_{V_n{}^c}} \chi_{\{x_0\}}(X(t))\, dt\right) = 0 \qquad \text{(case 1)}, \tag{4.4}$$

$$E_{x_0}\left(\int_0^{\sigma_{V_n{}^c}} \chi_{\{x_0\}}(X(t))\, dt\right) \geqq \varepsilon > 0 \qquad \text{(case 2)}. \tag{4.5}$$

Case 1. Putting $x = x_0$ in (4·3), we have an $n = n_0$ such that the second summand of (4.3) is positive for $x = x_0$. Since this implies

$$h^{V_{n_0}{}^c}(x_0, E') > 0, \qquad E' = \{y \in V_{n_0}{}^c \mid G^{R-K}(y, \{x_0\}) > 0\},$$

we have, by the equivalence of $h^{V_{n_0}{}^c}(x, \cdot)$ for $x \in V_n$,

$$0 < \int_{V^c{}_{n_0}} h^{V_{n_0}{}^c}(x, dy) G^{R-K}(y, \{x_0\}) \leqq G^{R-K}(x, \{x_0\}) \qquad \text{for} \quad x \in V_{n_0},$$

by (4.3). Taking $L' \subset V_{n_0}$ with $\{L', K\} \in \mathfrak{F}_0$, we have

$$0 < \int_{L'} \nu_K{}^{L'}(dx) G^{R-K}(x, \{x_0\}) \leqq m(\{x_0\}).$$

Case 2. Since $\sigma_{V_n{}^c}$ converges monotonically to $\sigma = \sigma_{R-\{x_0\}}$, we have

$$\begin{aligned} E_{x_0}(\sigma) &= \lim_{n\to\infty} E_{x_0}(\sigma_{V_n{}^c}) = \lim_{n\to\infty} E_{x_0}\left(\int_0^{\sigma_{V^c}} 1 dt\right) \\ &\geqq \lim_{n\to\infty} E_{x_0}\left(\int_0^{\sigma_{V_n{}^c}} \chi_{\{x_0\}}(X(t))\, dt\right) \geqq \varepsilon > 0, \end{aligned}$$

implying that x_0 is a point of positive holding time. Such a case is omitted by (X. 5).

Now there is a density function of $G^{R-K}(x, \cdot)$ with respect to m. Fix a version of this density function and denote it by

$$g^{R-K}(x, y), \qquad x \in R. \tag{4.9}$$

To discribe a rough but general situation concerning the potentials with kernel g^{R-K}, we give a rather rough

DEFINITION 1. We say that a real-valued function f defined on R is harmonic in an open set D, if it is $\boldsymbol{B}$-measurable, $h^{V^c}(x, \cdot)$-integrable for each domain V having closure $\overline{V}(\neq R)$ contained in D and $x \in V$, and satisfies

$$f(x)=\int_{V^c} h^{V^c}(x, dy)f(y), \qquad x \in V. \tag{4.10}$$

In the case of processes with continuous paths, a function *defined on* D with the above properties is called harmonic in D.

2. A $\boldsymbol{B}$-measurable function f defined on R, which takes extended real values and is bounded from below, is called *super-harmonic*[16] in an open set D, if it satisfies

$$f(x) \geqq \int_{V^c} h^{V^c}(x, dy)f(y) \qquad x \in V, \tag{4.11}$$

for V as in **1**. In the case of processes with continuous paths, a function *defined on* D with above properties is called super-harmonic in D.

LEMMA 4.1. *Let f be a non-negative, Borel measurable function defined on R, which takes extended real values. Then*

$$G^{R-K}f(x)=\int_R G^{R-K}(x, dy)f(y) \tag{4.12}$$

is superharmonic in $R-K$ for each closed $K \neq R$ with an inner point. If (4.12) *is finite on $R-\{K \smile S(f)\}$, it is harmonic there, where $S(f)$ is the support of f.*

Proof. When $f(x)$ is the characteristic function of a Borel set A, the lemma holds by the following inequality,

$$\begin{aligned} G^{R-K}f(x) &= G^{R-K}(x, A)=E_x\left(\int_0^{\sigma_K}\chi_A(X(t))dt\right)=E_x\left(\int_0^{\sigma_{V^c}} // + \int_{\sigma_{V^c}}^{\sigma_K} //\right) \\ &= E_x\left(\int_0^{\sigma_{V^c}} //\right)+E_x\left\{E_{x(\sigma_{V^c})}\left(\int_0^{\sigma_K} //\right)\right\}=G^V(x, A)+\int_{V^c} h^{V^c}(x, dy)G^{R-K}(x, A) \\ &\geqq \int_{V^c} h^{V^c}(x, dy)G^{R-K}(y, A)=\int_{V^c} h^{V^c}(x, dy)G^{R-K}f(x), \qquad \text{for} \quad x \in V, \end{aligned} \tag{4.13}$$

where V is a domain with $\bar{V} \subset R-K$. If $V \subset R-\{K \smile \bar{A}\}$, then $P_x(\sigma_A \geqq \sigma_{V^c})=1$ holds and hence $G^V(x, A)=0$ for $x \in V$. This and (4.11) imply the equality sign in (4.13).

For a general f, non-negative on R, we can choose a monotone sequence of functions $\{f_n\}$, where each f_n is a linear combination of characteristic functions of Borel sets with non-negative coefficients, satisfying $\lim_{n\to\infty} f_n(x)=f(x)$ for each $x \in R$. Then, by monotone convergence, we have

$$G^{R-K}f(x)=\int_R G^{R-K}(x, dy)f(y)=\lim_{n\to\infty}\int_R G^{R-K}(x, dy)f_n(y)=\lim_{n\to\infty} G^{R-K}f_n(x),$$

and hence

$$\int_{V^c} h^{V^c}(x, dy)G^{R-K}f(y)=\lim_{n\to\infty}\int_{V^c} h^{V^c}(x, dy)G^{R-K}f_n(y).$$

16) This definition is a little more restricted than the classical one.

These equations and (4.13) for $f_n(x)$ imply the lemma. The statement for f *defined on* D in the case of processes with continuous paths follows trivially.

THEOREM 4.2. *Let n be a sigma-finite signed measure on R, absolutely continuous relative to m. Assume that the potential*

$$U^n_{g^{R-K}}(x) = \int_R g^{R-K}(x, y)\, n(dy), \tag{4.14}$$

is well defined on R, and is $h^{V^c}(x_0, \cdot)$-integrable for any $x_0 \in V$, where $K(\neq R)$ is a closed set with an inner point and V is a domain with compact closure $\bar{V} \neq R$. In order that $U^n_{g^{R-K}}(x)$ is harmonic in an open set $D \subset R - K$, it is necessary and sufficient that $n(A) = 0$ for any Borel subset A of D.

Proof. Let $f(x)$ be a version of the density of n relative to m. Write

$$f^+(x) = \max\{f(x), 0\},\ f^-(x) = \min\{f(x), 0\},$$
$$n^+(\cdot) = \int_{(\cdot)} f^+(x) m(dx),\ n^-(\cdot) = \int_{(\cdot)} f^-(x) m(dx).$$

Then, n^+ and n^- are sigma-finite measures on R, and $U^n_{g^{R-K}}(x)$ are $h^{V^c}(x, \cdot)$-integrable and

$$U^n_{g^{R-K}}(x) = U^{n^+}_{g^{R-K}}(x) + U^{n^-}_{g^{R-K}}(x), \quad \text{for} \quad x \in D.$$

Sufficiency. Since $n(A) = 0$ for any Borel set $A \subset D$, $f(x) = f^+(x) = f^-(x) = 0$ on D excepting on a set of m-measure 0. Then, by Lemma 4.2, $U^{n^\pm}_{g^{R-K}}(x)$ are harmonic in D, and hence $U^n_{g^{R-K}}(x)$ is harmonic in D.

Necessity. First we note that $U^n_{g^V}(x) = 0$ on V for each domain V with $\bar{V} \subset D$. In fact, noting that

$$\begin{aligned} G^{R-K}(x, A) &= E_x\left(\int_0^{\sigma_K} \chi_A(X(t))\, dt\right) = E_x\left(\int_0^{\sigma_{V^c}} '' + \int_{\sigma_{V^c}}^{\sigma_K} ''\right) \\ &= E_x\left(\int_0^{\sigma_{V^c}} ''\right) + E_x\left\{E_{X(\sigma_{V^c})}\left(\int_0^{\sigma_K} ''\right)\right\} \\ &= G^V(x, A) + \int_{V^c} h^{V^c}(x, dy)\, G^{R-K}(y, A) \end{aligned}$$

and using the harmonicity of U_{R-K} in D, we have

$$\begin{aligned} 0 &= U^n_{g^{R-K}}(x) - \int_{V^c} h^{V^c}(x, dy) U^n_{g^{R-K}}(y) \\ &= \int_R G^{R-K}(x, dy) f(y) - \int_{V^c} h^{V^c}(x, dy) \int G^{R-K}(y, dz) f(z) \\ &= \int_R f(y)\left\{G^{R-K}(x, dz) - \int_{V^c} h^{V^c}(x, dz)\, G^{R-K}(z, dy)\right\} = \int_R f(y) G^V(x, dy) \\ &= \int_R g^V(x, y) n(dy) = U^n_{g^V}(x) \qquad \text{for} \quad x \in V. \end{aligned} \tag{4.15}$$

Next, we will show that $U^n_{g^V}(x) \geqq 0$ on V with compact closure implies $n(V) \geqq 0$. Let $K_i = \{x \in V \mid \inf_{y \in \partial V} \rho(x,y) \geqq 1/i\}$, $i = 1, 2, \cdots$, where $\rho(x, y)$ denotes a metric compatible with the topology of R. Since K_i is compact and in a domain V, we have $\{K_i, V^c\} \in \mathfrak{F}$ for $i \geqq i_0 = \min\{i \mid K_i \neq \phi\}$. Then, we have

$$\int_{K_i} \nu_{V^c}^{K_i}(dx) \int_{K_i} G^V(x, dy) f(y) = \int_{K_i} m(dy) f(y) = n(K_i) \tag{4.16}$$

and

$$\begin{aligned} &\left| \int_{K_i} \nu_{V^c}^{K_i}(dx) \int_{V-K_i} G^V(x, dy) f(y) \right| \\ &\leqq \int_{K_i} \nu_{V^c}^{K_i}(dx) \int_{V-K_i} G^V(x, dy)(f^+(y) - f^-(y)) \\ &\leqq \int_{V-K_i} m(dy)(f^+(y) - f^-(y)) = n^+(V-K_i) - n^-(V-K_i). \end{aligned} \tag{4.18}$$

Since n^+ and n^- are also sigma-finite, $n^+(V-K_i)$ and $n^-(V-K_i)$ and hence $n(V-K_i)$ converges to 0 as i tends to infinity. Hence, by (4.16) and (4.17), we have

$$0 \leqq \lim_{n \to \infty} \int_{K_n} \nu_{V^c}^{K_n}(dx) U_{g^V}^n(x) = n(V). \tag{4.18}$$

Combining (4.15) and (4.18), we know that $n(V) = 0$ for each domain V with $\bar{V} \subset D$, and hence $n(A) = 0$ for each Borel set $\bar{A} \subset D$. Then, the same thing holds for n^+ and n^-. But $n^+(D) = \lim_{i \to \infty} n^+(D_i) = 0$ and $n^-(D) = \lim_{i \to \infty} n^-(D_i) = 0$ for $D_i = \{x \in D \mid \inf_{y \in \partial D} \rho(x, y) > 1/i\}$. This completes the proof.

Now, by making use of the kernel $g^{R-K}(x, y)$, we will give a representation of the generator $\boldsymbol{G}$ of $\{X(t)\}$ defined in §1. The following is an extension of one given by Ito and McKean [6] in the case of processes with Brownian hitting measures.

THEOREM 4.3. *For each $f \in \mathfrak{D}(\boldsymbol{G})$ there exists a sigma-finite signed measure m_f, absolutely continuous relative to m and satisfying*

$$\boldsymbol{G} f = \frac{dm_f}{dm} \tag{4.19}$$

and

$$U_{g^{R-K}}^{m_f}(x) = \int_K h^K(x, dy) f(y) - f(x) \tag{4.20}$$

where the right hand side of (4.19) *denotes a version of Randon-Nikodym's density function of m_f relative to m, and K is a closed set with an inner point and with compact $\overline{R-K}$. Such a measure m_f is determined uniquely.*

Proof. For K as above, $E_x(\sigma^K) = G^{R-K}(x, \overline{R-K}) < \infty$ by Lemma 1.1. Define

$$m_f(\cdot) = \int_{(\cdot)} \boldsymbol{G} f(x) m(dx).$$

Then, (4.20) follows trivially from (1.4), by

$$E_x\left(\int_0^{\sigma_K} \boldsymbol{G}f(X(t))dt\right)=\int_R G^{R-K}(x, dy)\boldsymbol{G}f(y)$$
$$=\int_R g^{R-K}(x, y)\boldsymbol{G}f(y)m(dy)=U_{gR-K}^{m_f}(x).$$

The uniqueness follows from Theorem 4.2.

COROLLARY. *$f\in\mathfrak{D}(\boldsymbol{G})$ is harmonic in a domain $D\subset R$, if and only if*

(4.21) $$\boldsymbol{G}f(x)=0 \qquad for \quad x\in D.$$

Proof. Let f be harmonic in D. Take any point $x\in D$ and a neighborhood V of x with compact closure $V(\neq R)\subset D$. Then, $f(x)$ and

$$\int_{V^c} h^{V^c}(x, dy)f(y)$$

are both harmonic in V, and hence the potential $U_{gV}^{m_f}(x)$ is also harmonic in V by (4.20). Since V is any open set with compact $V(\neq R)\subset D$, m_f has its carrier in D^c and hence $\boldsymbol{G}f(x)=0$ on D excepting a set of m-measure 0. The continuity of $\boldsymbol{G}f$ and Theorem 3.1 imply (4.21).

Conversely, (4.21) implies that the left hand side of (4.20) vanishes on any V of the form cited above, and hence that f is harmonic in D.

NOTE. $\boldsymbol{G}$ in (4.19)–(4.20) is really a *global* operator though it looks *local* in the representation. Even when the kernel g^{R-K} and the measure m are determined locally, the domain $\mathfrak{D}(\boldsymbol{G})$ is of global character.

§5. Example: Processes with Brownian hitting measures on R^2.

Following Ito and McKean [6][17], we say that a Markov process on R^2 has Brownian hitting measures on R^2, if it satisfies (X. 4) and

(X. B) *For any closed set $K\neq R$ with an inner point and $x\in K^c$, $h^K(x, \cdot)$ coincides with the classical harmonic measure of K viewed from x with respect to the connected component of K^c containing x.*

It can be proved that (X. B) combined with (X. 4) implies (X. 1), (X. 2′), (X. 3) and the continuity of path functions.[2] Hence, combined with (X. 4), they imply (X. 5).

It is known that the classical Green function $g_D(x, y)$ of a domain D in R with compact ∂D of positive logarithmic capacity is given by

(5.1) $$g_D(x, y)=\log\frac{1}{|x-y|}-\int_{\partial D} h^{\partial D}(x, dz)\log\frac{1}{|z-y|}+\gamma_D(x), \qquad x, y\in D,$$

where $\gamma_D(x)$ is a non-negative continuous function of x and converges to 0 if $x\in D$ converges to a regular point of the boundary ∂D. For any closed set $K\neq R$ with an inner point, we define by $g_{R-K}(x, y)$ by making use of (5.1), or

17) The original definition is given for a wider class of processes. Here, we restrict it for our present use.

$$(5.2)\qquad g_{R-K}(x, y)=\log\frac{1}{|x-y|}-\int_K h^K(x, dz)\log\frac{1}{|z-y|}+\gamma_{R-K}(x),$$

where

$$\begin{aligned}\gamma_{R-K}(x)&=\gamma_D(x), &&\text{if } x \text{ belongs to a connected component } D \text{ of } R-K,\\ &=0, &&\text{otherwise.}\end{aligned}$$

Then, we have

$$(5.3)\qquad \begin{aligned} g_{R-K}(x, y)&=g_{D_\alpha}(x, y), \text{ if } x \text{ and } y \text{ belong to the same component } D_\alpha \text{ of } R-K,\\ &=0, \text{ otherwise, excepting on the set of irregular points of } \partial K.\end{aligned}$$

Next, we denote the Green capacity of a closed set K with respect to the kernel $g_D(x, y)$ of the domain D by $C_D(K)$.

Now, take a pair $\{K_0, L_0\}\in\mathfrak{F}$ such that $R-K_0$ is connected and the classical Green capacity $C_{R-K_0}(L_0)=1$, for instance, $K_0=\{x\mid |x|\leqq 1\}$ and $L_0=\{x\mid |x|\geqq e\}$ where $e=2.718\cdots$. Define $C(K, L)=C_{(K_0, L_0)}(K, L)$ for such fixed $\{K_0, L_0\}\in\mathfrak{F}$. Then, we have

THEOREM 5.1. *For each $\{K, L\}\in\mathfrak{F}$, the following assertions hold.*

1. *If μ is a measure on K with total mass* 1, *then $\mu(T^K)^n$ converges exponentially fast to $\mu_L{}^K$ with respect to the norm of total variation.*

If K is contained in one component of $R-L$, then we have:

2. *$\mu_L{}^K$ is the equilibrium distribution of K with respect to the classical Green kernel of the component.*

3. *$C(K, L)$ is the classscal Green capacity of K with respect to the classical Green kernel of the component; i.e.*

$$(5.4)\qquad C(K, L)=C_{R-L}(K).$$

4. *If L is not in one component of $R-K$, then*

$$(5.5)\qquad C(K, L)\mu_K{}^L(\cdot)=\sum_{\lambda\in\Lambda}C(L_\lambda, D_\lambda{}^C)\mu_{D^C}{}^{L_\lambda}(\cdot),$$

where $\{D_\lambda, \lambda\in\Lambda\}$ is the family of all components of $R-K$ containing a point of L not irregular for ∂L and $L_\lambda=L\frown D_\lambda$.

Proof. 1 is only a restatement of Proposition 2.2. 2. The Green function of a domain with unbounded boundary and the harmonic measure on the boundary are obtained from those of a domain with bounded boundary by making use of a conformal mapping. Hence, we assume that the boundaries of K and L are bounded. To fix the notation we assume that K is in one component D of $R-L$, and L is contained in components D_λ, $\lambda\in\Lambda$, of $R-K$. We write $L_\lambda=L\frown D_\lambda\neq\phi$.

The path functions being continuous, $\mu_K{}^L$ and $\mu_L{}^K$ are concentrated on the boundaries ∂D and ∂D_λ, $\lambda\in\Lambda$, and hence we assume $R-L=D$ and $R-K=\cup_{\lambda\in\Lambda}D_\lambda$ without loss of generality.

Consider the potentials

$$U_{g_{R-K}}^{\mu_K^L}(y)=\int_{\partial L}\mu_K{}^L(dx)g_{R-K}(x,\ y)$$

$$(5.6)\qquad =\int_{\partial L}\mu_K{}^L(dx)\left\{\log\frac{1}{|x-y|}-\int_{\partial L}h^K(x,\ dz)\log\frac{1}{|z-y|}+\gamma_{R-K}(x)\right\}$$

$$=\int_{\partial K}\mu_K{}^L(dx)\log\frac{1}{|x-y|}-\int_{\partial K}\mu_L{}^K(dz)\log\frac{1}{|z-y|}+\int_{\partial L}\mu_K{}^L(dx)\gamma_{R-K}(x),$$

and similarly,

$$U_{g_{R-L}}^{\mu_L^K}(y)=\int_{\partial K}\mu_L{}^K(dx)g_{R-L}(x,\ y)$$

$$(5.6')\qquad =\int_{\partial K}\mu_L{}^K(dx)\left\{\log\frac{1}{|x-y|}-\int_{\partial L}h^L(x,\ dz)\log\frac{1}{|z-y|}+\gamma_{R-L}(x)\right\}$$

$$=\int_{\partial K}\mu_L{}^K(dx)\log\frac{1}{|x-y|}-\int_{L}\mu_K{}^L(dz)\log\frac{1}{|z-y|}+\int_{\partial K}\mu_L{}^K(dx)\gamma_{R-L}(x)$$

Adding (5.6) and (5.6′) we have

$$(5.7)\qquad \begin{aligned}&0<U_{g_{R-L}}^{\mu_K^L}(y)+U_{g_{R-K}}^{\mu_L^K}(y)\\ &=\int_{\partial L}\mu_K{}^L(dx)\gamma_{R-K}(x)+\int_{\partial K}\mu_L{}^K(dx)\gamma_{R-L}(x)=C<\infty.\end{aligned}$$

The convergence follows from the continuity of γ_{R-K} and γ_{R-L} on the compact sets ∂L and ∂K respectively. Since the potentials $U_{g_{R-L}}^{\mu_L^K}(x)$ and $U_{g_{R-K}}^{\mu_K^L}(x)$ are 0 on L and K excepting the sets of irregular points of K and L respectively, $U_{g_{R-L}}^{\mu_L^K}(x)=C$ on K and $U_{g_{R-K}}^{\mu_K^L}(x)=C$ on L excepting the sets of irregular points of K and L respectively. Hence, $\mu_L{}^K(\cdot)$ is the equilibrium distribution on K with respect to the kernel $g_{R-L}(x,\ y)$, and the restriction $(\mu_K{}^L)_{L_\lambda}(\cdot)$ of $\mu_K{}^L(\cdot)$ on L_λ is the equilibrium distribution on L_λ with respect to the kernel $g_{D_\lambda}(x,\ y)$ multiplied by $\mu_K{}^L(L_\lambda)$.

3. At first, we consider a special case. Let $\{K,\ L\}$ and $\{K,\ L'\}$ with $L'\subset L$ be in $\mathfrak{F}$, and let K be contained in one component D of $R-L$. Accordingly K is contained in one component D' of $R-L'$. We show

$$(5.10)\qquad C_D(K)\cdot C_{(K,L)}(K,\ L')=C_{D'}(K).$$

Since

$$\begin{aligned}g_{R-K}(x,\ y)&=g_{R-K-L'}(x,\ y)+\int_{K\cup L'}h^{K\cup L'}(x,\ dz)g_{R-K}(z,\ y)\\ &=g_{R-K-L}(x,\ y)+\int_{L'}h^{K\cup L'}(x,\ dz)g_{R-K}(z,\ y),\end{aligned}$$

we have

$$
\begin{aligned}
U_{g_{R-K}}^{\mu_K^L}(x) &= \int_L \mu_K{}^L(du) g_{R-K}(u, x) \\
(5.11) \qquad &= \int_L \mu_K{}^L(dn) g_{R-K-L'}(u, x) + \int_L \mu_K{}^L(du) \int_{L'} h^{K \cup L'}(u, dv) g_{R-K}(v, x) \\
&= U_{g_{R-K-L'}}^{\mu_K^L}(x) + C_{(K,L)}(K, L') \cdot U_{g_{R-K}}^{\mu_K^{L'}}(x).
\end{aligned}
$$

Since $U_{g_{R-K-L'}}^{\mu_K^L}(x)$ tends to 0 as x tends to a regular point of the boundary of any fixed component corresponding to the boundary function which takes the value $U_{g_{R-K}}^{\mu_K^L}(x)$ on $\partial L'$ and 0 on ∂K.

This shows that

$$
\begin{aligned}
C_{D'}(K)^{-1} &= \sup_{x \in K} U_{g_{R-L'}}^{\mu_{L'}^K}(x) = \sup_{x \in L' \frown D_1} U_{g_{D_1}}^{\mu_K^{L'}}(x) \\
&= C_{(K,L)}(K, L')^{-1} \sup_{x \in L' \frown D_1} U_{g_{R-K}}^{\mu_R^L}(x) = C_{(K,L)}(K, L')^{-1} \sup_{x \in K} U_{g_{R-L}}^{\mu_L^K}(x) \\
&= C_{(K,L)}(K, L')^{-1} C_D(K)^{-1},
\end{aligned}
$$

implying (5.10), where D_1 is a component of $R-K$ containing an inner point of L'.

To prove the general case, we note that for any $\{K, L\} \in \mathfrak{F}$ we can take a sequence $\{K_0, L_0\}, \{K_1, L_1\}, \cdots, \{K_{n+1}, L_{n+1}\} = \{K, L\}$ with $\{K_i, L_i\} \leftrightarrow \{K_{i+1}, L_{i+1}\}$, $i = 1, 2, \cdots, n$, in such a way that one of the following relations hold for each i,

$$
\begin{gathered}
K_i \subset K_{i+1}, \ K_i \supset K_{i+1}, \ K_i \subset L_{i+1}, \ K_i \supset L_{i+1}, \\
L_i \subset L_{i+1}, \ L_i \supset L_{i+1}, \ L_i \subset K_{i+1}, \ L_i \supset K_{i+1},
\end{gathered}
$$

and such that K_i and L_i are contained in one component of $R-L_i$ and $R-K_i$, respectively, where $\{K_0, L_0\}$ is the fixed pair selected just before the theorem. Then, we can apply the discussion of the special case above for each step and get

$$
C_D(K) = C_{R-K_0}(L_0) C_{(K_0, L_0)}(K, L) = 1 \cdot C(K, L) = C(K, L),
$$

when K is contained in one component D of $R-L$.

4. We saw in the last part of 2 that

$$
(5.12) \qquad (\mu_K{}^L)_{L_\lambda}(\cdot) = \mu_K{}^L(L_\lambda) \mu_{\partial D_\lambda}{}^{L_\lambda}(\cdot).
$$

On the other hand we have

$$
\begin{aligned}
C(K, L)^{-1} &= \sup_{x \in K} U_{g_{R-L}}^{\mu_L^K}(x) = \sup_{x \in L} U_{g_{D_\lambda}}^{\mu_K^L}(x) \\
(5.13) \qquad & \\
&= \mu_K{}^L(L_\lambda) \cdot \sup_{x \in L_\lambda} U_{g_{D_\lambda}}^{\mu_{\partial D_\lambda}^{L_\lambda}} = \mu_K{}^L(L_\lambda) C(\partial D_\lambda, L_\lambda)^{-1}.
\end{aligned}
$$

By (5.12) and (5.13) and $\mu_K{}^L(\cdot) = \sum_{\lambda \in \Lambda} (\mu_K{}^L)_{L_\lambda}(\cdot)$, we have (5.5).

NOTE. The extremal distance of K and L with respect to the domain between K and L are known to coincide with the Dirichlet integral of $U^{\mu_L^K}_{g_{R-L}}(x)$ in the domain, which can be proved to be $2\pi\cdot C(K, L)^{-1}$ using Theorem 5.1, 3.

THEOREM 5.2. *For any closed set $K\neq R$ with an inner point, we can take the classical kernel $g_{R-K}(x, y)$ for $g^{R-K}(x, y)$ in* §4, *or*

$$G^{R-K}(x, E)=\int_E g_{R-K}(x, y)m(dy), \qquad E\in B,\ x\in R. \tag{5.13}$$

Proof. It is sufficient to show this in the case that $R-K$ is a domain D. Instead of proving $g_D(x, y)=g^D(x, y)$ a.e. m directly, we use the Riesz decomposition theorem for the classical kernel g_D and then compare the measure of the potential. Let A be a Borel set with compact closure. Writing D for $R-K$ in (1.6), we have

$$G^D(x, A)\leqq E_x\{f_0(X(\sigma_{D^c}))\}-f_0(x)=\int_{\partial D}h^{D^c}(x, dy)f_0(y)-f_0(x)$$

where f_0 is in $\mathfrak{D}(\boldsymbol{G})$. Note that the first summand in the right hand side is the Dirichlet solution in D for the boundary function $f_0(y)$, $y\in\partial D$. Hence, $G^D(x, A)$ tends to 0 when x tends to a regular point of ∂D from inside D. Then, the Riesz decomposition of the super-harmonic function $G^D(x, A)$ is given by

$$G^D(x, A)=\int_D g_D(x, y)n_A(dy),$$

where $n_A(\cdot)$ is a finite measure with carrier in $D\frown\bar{A}$, since $G^D(x, A)$ is harmonic in $D-\bar{A}$. Since $G^D(x, A)$ is a sigma-finite measure for fixed x, we can write the above relation in the form

$$G^D(x, A)=\int_{D\frown A} g_D(x, y)\,n_D(dy),$$

with a measure $n_D(\cdot)$ depending on D but not on A. To show that n_D does not depend on D, it is sufficient to prove $n_D(A\frown D)=n_{D'}(A\frown D)$ when $D\subset D'$. We know

$$G^{D'}(x, A)=G^D(x, A)+\int_{D^c}h^{D^c}(x, dy)G^{D'}(y, A) \quad \text{or, what is the same,}$$

$$\int_{A\frown D}g_D(x, y)\,n_D(dy)=\int_{A\frown D}n_{D'}(dy)\left\{g_{D'}(x, y)-\int_{D^c}h^{D^c}(x, dz)g_{D'}(z, y)\right\}$$
$$=\int_{A\frown D}g_D(x, y)n_{D'}(dy).$$

Then, by the uniqueness of the measure in the Riesz decomposition, we have $n_D(\cdot)=n_{D'}(\cdot)$ on D. Hence, we have a sigma-finite measure n on R, such that

$$G^D(x, \cdot)=\int_{(\cdot)}g_D(x, y)n(dy) \qquad x\in D,$$

where D is a domain with an outer point.

To complete the proof, it is sufficient to show that $n(A)=m(A)$ for any open set in any fixed D, and it is enough to prove it for any compact subset A of D. Fix such an A and take an open $D_0 \supset A$ with compact closure $K = \bar{D}_0 \subset D$. Since $\{K, D^c\} \in \mathfrak{F}$ and

$$\int_{\partial K} \nu_{D^c}{}^K(dx) g_D(x, y) = 1, \qquad \text{for} \quad y \in A$$

by Theorem 5.1, 1, we have

$$m(A) = \int_{\partial K} \nu_{D^c}{}^K(dx) G^D(x, A) = \int_{\partial K} \nu_{D^c}{}^K(dx) \int_A g_D(x, y) n(dy)$$
$$= \int_A n(dy) \int_{\partial K} \nu_{D^c}{}^K(dx) g_D(x, y) = n(A).$$

REMARK. *In this special case, the representation* (4.19)–(4.20) *coincides with one given by Ito and Mckean* [6]. *Hence, $g^D(x, y)$ is determined only by the system of harmonic measures, and the measure m is determined locally.*

Miscellaneous Notes. 1. It is natural to expect that a Markov process is determined by two kinds of quantities corresponding to the two factors, one concerned with the road system on the state space, on which the particle moves, and the other concerned with the speed of the motion. This idea has been pointed out by Feller [3, 4] repeatedly. In fact, we find a support of the fact in Dynkin's representation of the generater

$$Gf = \lim_{U \downarrow \{x\}} \frac{h^{U^c} f(x) - f(x)}{E_x(\sigma_{U^c})}, \qquad f \in \mathfrak{D}(G).$$

But a more profound and in a sense complete result was obtained in the case of linear diffusion, in which the process is characterized by the scale s, the measure m and the boundary condition, and the probabilistic meaning of those objects are explained in terms of the two factors cited above (cf. Dynkin [2], Ito and MacKean [6]).

A part of the result is extended for the processes with Brownian hitting measures by Ito, MacKean and Tanaka: the generator of a given process with Brownian hitting measures is represented by the system of classical Green kernels and the measure m, and the process is obtained from the Brownian motion by making use of "time change" which is determined by m.

The facts suggest that the situation is similar to an extent also in general cases, that is, a Markov process is (under suitable regularities) characterized by a system of kernels which is concerned only with the road system and a measure which determines the local speed of the motion. Hence, we also intended to investigate our special case from this point of view, while the result is quite unsatisfactory. As a first approach, at least the following two problems should be answered: to show whether the system of kernels $\{g^{R-K}\}$ in §4 depends only on the system of hitting measures or not, and secondly, to find the conditions under which the measure m in §3 represents the local speed of the motion. The answer to the first problem seems to be in the affirmative.

2. The system of capacities $\{C_{(K,L)}(K', L')\}$ is proved to depend only on the system of hitting measures, while the proof depends on the measure m. But, when R is a connected manifold, or the union of a domain and a measurable subset of its boundary in R^n ($n \geqq 2$), we can obtain the system using (X. 1)–(X. 3) and the strong Markov property.

3. We assumed in (X. 1) that the process hits every neighborhood of any point starting at any point with probability one. But, if the probability of this event is assumed to be positive but not necessarily one, and the path functions are continuous at each $0 \leqq t < \sigma_\omega$, then there are only two possibilities in following

PROPOSITION. *If $X(t)$ with continuous path functions satisfies (X. 2)–(X. 4) and*

(X. 1°) $P_x(X(t, w) \in A$ *for some* $0 \leqq t < \infty) > 0$, *for any* $x \in R$, *then only one of the following two holds, where* A $(\neq \phi)$ *is an arbitrary open set.*

Case 1. (X. 1) *holds.*

Case 2. *For each compact set* $K \neq R$, *there is a point* $x \in R - K$ *satisfying*

$$P_x(\sigma_K < \infty) < 1.$$

The proofs of the above notes are tedious and omitted.

Appendix I. The uniqueness of the T^K-invariant measure.

We note that the uniqueness of the T^K-invariant measure $\mu_L{}^K$ on K (in §2) can be proved also by using a result of Nelson [8] and a simple proposition. Let S be a separable, locally compact spaec and let $\mathfrak{B}_S$ be the smallest Borel field containing all open subsets of S. $\{P(x, A)\}$ is a system of transition probabilities on $(S, \mathfrak{B}_S)$. Define

$$P^1(x, A) = P(x, A), \quad P^n(x, A) = \int_S P^{n-1}(y, A) P(x, dy), \quad n > 1.$$

A sigma-finite measure $q(\cdot)$ on $(S, \mathfrak{B}_S)$ is called an *invariant measure* for $\{P(x, A)\}$ if it satisfies

$$q(A) = \int_S q(dx) P(x, A), \qquad \text{for all } A \in \mathfrak{B}_S.$$

$\{P(x, A)\}$ is called *irreducible* if $\nu_x(\cdot) = \sum_{n=1}^{\infty} 2^{-n} \cdot P^n(x, \cdot)$ are equivalent for all $x \in S$. A set $A \in \mathfrak{B}_S$ is called a null set if $\nu_x(A) = 0$ for some $x \in S$, when $\{P(x, A)\}$ is irreducible. An invariant measure $q(\cdot)$ for irreducible $\{P(x, A)\}$ is equivalent to all $\nu_x(\cdot)$, $x \in S$. We use

PROPOSITION (E. Nelson).[18] *If an irreducible $\{P(x, \cdot)\}$ with an invariant measure $q(\cdot)$ satisfies*

(I.1) $$\sum_{n=1}^{\infty} P^n(x, A) = \infty, \; x \in S, \qquad \textit{for all non-null set } A \in \mathfrak{B}_S,$$

18) Cf. Th. 5.1 of [8]. The original result is a little more general. Here, we restrict it for our present use.

then any invariant measure for $P(x, \cdot)$ is a constant multiple of $q(\cdot)$.

If we take K, $\boldsymbol{B}(K)$, $\{T^K(x, \cdot)\}$ and $\mu_L{}^K(\cdot)$ in §§1, 2 for S, $\mathfrak{B}_S$, $\{P(x, \cdot)\}$ and $q(\cdot)$ respectively, all the assumptions in the above proposition are satisfied excepting (I.1). But, noting that the total mass of $\mu_L{}^K(\cdot)$ is finite, (I.1) is also satisfied in view of following

PROPOSITION.[19] *If $\{P(x, \cdot)\}$ is irreducible and has an invariant measure $q(\cdot)$ with finite total mass, then* (I.1) *is satisfied.*

Proof. Take a non-null set A. Since $\nu_x(\cdot)$ and $q(\cdot)$ are equivalent, we have

$$\begin{aligned} 0 < q(A) &= \int_S \left\{\frac{1}{n}\sum_{k=1}^n P^k(x, A)\right\} q(dx) \\ &= \lim_{n\to\infty} \int \left\{\frac{1}{n}\sum_{k=1}^n P^k(x, A)\right\} q(dx) \\ &\leqq \int_S \left\{\overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{k=1}^n P^k(x, A)\right\} q(dx) \end{aligned}$$

and hence $q(E) > 0$, where

$$E = \left\{x \in S \,\middle|\, \overline{\lim_{n\to\infty}} \frac{1}{n}\sum_{k=1}^n P^k(x, A) > 0\right\}.$$

Since $q(\cdot)$ and $\nu_x(\cdot)$ are equivalent, $\nu_{x_0}(E) > 0$ and hence there is an n_0 such that $P^{n_0}(x_0, E) > 0$ for any fixed $x_0 \in S$. Noting that $\sum_{k=1}^\infty P^k(x, A) = \infty$ for $x \in E$ by definition of E, we have

$$\begin{aligned} \sum_{k=1}^\infty P^k(x_0, A) &\geqq \sum_{k=n_0+1}^\infty P^k(x_0, A) = \int_S P^{n_0}(x_0, dy)\left\{\sum_{k=1}^\infty P^k(y, A)\right\} \\ &\geqq \int_E P^{n_0}(x_0, dy)\left\{\sum_{k=1}^\infty P^k(y, A)\right\} = \infty \end{aligned}$$

for any fixed $x_0 \in S$, completing the proof.

Appendix II. The invariance of m in the temporally discrete case.

We prove the invariance of a measure correspnding to $m_{K,L}$ in §2 for temporally discrete Markov processes. The assumption (X. 1) is replaced by the existence of a pair of measures corresponding to $(\mu_L{}^K, \mu_K{}^L)$, so that *the theorem can be applied also for some non-recurrent cases by a suitable choice of K and L,* for instance random walks in higher dimensions. No other regularity assumptions are needed. The proof depends upon the analogue of the

19) An irreducible $\{P(x, A)\}$ satisfying (I, 1) defines a *recurrent* Markov process "in the usual sense" (Cf. Th. 4.1 of [8]). Hence, this proposition can be expressed as: *An irreducible $\{P(x, A)\}$ with an invariant measure of finite total mass defines a recurrent Markov process "in the usual sense".* But we do not use the word *recurrence* to avoid confusion.

fact that in the Brownian motion process the harmonic measure for a closed set with sufficiently smooth boundary is obtained by a normal derivative of the Green function of the complement of the set, multiplied by a suitable constant. The weak point of the proof is that it seems to be impossible to extend it in the temporally continuous case without unnecessary restrictions.

Let R be a locally compact Hausdorff space satisfying the second axiom of countability, $\boldsymbol{B}$ the smallest Borel field containing all opens, $\mathfrak{C}$ the family of all Borel sets with compact closures, W the direct product $\prod_{n=0}^{\infty} R_n$ of a countable number of copies R_n of R, and $\mathfrak{B}$ the smallest Borel field of subsets of W containing all cylinder sets. Define the R-valued function $X(n, w)$ on $N \times W$ for $N=\{0, 1, \cdots\}$ as the n-th coordinate of $w \in W$.

Let $\{P(x, A)\}$ be a system of transition probabilities on R. Kolmogorov's extension theorem asserts that there is a Markov process with sample function $X(n, w)$ with $\{P(x, A)\}$ as the system of the transitions, i.e. there is a system of measures $\{P_x(\cdot), x \in R\}$ on $\{W, \mathfrak{B}\}$ satisfying following conditions.

1) $0 \leqq P_x(E) \leqq 1$ for $E \in \mathfrak{B}$ and $P_x(W)=1$.

2) $P_x(E)$ is $\boldsymbol{B}$-measurable as a function of x for fixed $E \in \mathfrak{B}$.

3) $P_x(X(0, w)=x)=1$

4) $P_x(w_n^+ \in F, w_n^- \in E)=E_x(P_{x(n, w)}(w \in F), w_n^- \in E)$ for $E, F \in \mathfrak{B}$,

where w_n^+ is the element of W with coordinate $X(k, w_n^+)=X(k+n, w)$, and w_n^- is one with coordinate $X(k, w_n^-)=X(k, w)$ for $k \leqq n$ and $X(k, w_n^-)=X(n, w)$ for $k>n$.

5) $P_x(X(1, w) \in A)=P(x, A)$ for $A \in \boldsymbol{B}$.

Write, as in §1,

$$\sigma_K(w)=\min\{n \geqq 0 \mid X(n, w) \in K\}, \qquad K \in \boldsymbol{B},$$

$$h^K(x, A)=P_x(X(\sigma_K) \in A), \qquad A \in \boldsymbol{B},$$

$$P_D(x, A)=P_x(X(1, w) \in A,\ \sigma_{D^c}(w)>1), \quad D \in \boldsymbol{B}. \tag{II.1}$$

$\{P_D(x, A)\}$ is a system of semi-transition probabilities on R, that is, a system of measures with properties of transition probabilities excepting the condition $P_D(x, R)=1$. Noting Kolmogorov-Chapman's equation, write

$$P^n(x, A)=P_x(X(n) \in A)$$

$$P_D^n(x, A)=\int_R P_D^{n-1}(x, dy) P_D(y, A)=P_x(X(n) \in A,\ \sigma_{D^c}>n), \qquad n \geqq 1, \tag{II. 2}$$

$$P_D^0(x, A)=P_x(X(0) \in A,\ \sigma_{D^c}>0)=\chi_{A \frown D}(x).$$

$$G^D(x, A)=E_x\left\{\sum_{n=0}^{\max(\sigma_{D^c}-1, 0)} \chi_A(X(n))\right\}=\sum_{n=0}^{\infty} P_D^n(x, A) \leqq \infty, \quad \text{for } A \in \boldsymbol{B}. \tag{II. 3}$$

Then, $G^D(x, A)$ is a Borel measurable function with extended real values for fixed $A \in \boldsymbol{B}$, and is a measure in A for fixed x.

THEOREM. *Assume that there is a pair of sets $K, L \in \boldsymbol{B}$, with $K \frown L=\phi$*

and a pair of sigma-finite mesures μ and ν on K and L respectively, satisfying the following conditions.

1) *the measures $G^{R-K}(x, \cdot)$ for $x \in L$, $G^{R-L}(x, \cdot)$ for $x \in K$, and*

$$\mu G^{R-L}(\cdot) = \int_K \mu(dx) G^{R-L}(x, \cdot), \qquad \nu G^{R-K}(\cdot) = \int_L \nu(dx) G^{R-K}(x, \cdot)$$

are all sigma-finite.

2) *the following equations hold:*

$$\mu(\cdot) = \nu h^K(\cdot) = \int_K \nu(dx) h^K(x, \cdot),$$

$$\nu(\cdot) = \mu h^L(\cdot) = \int_L \mu(dx) h^L(x, \cdot), \tag{II.4}$$

Then, the measure

$$m(\cdot) = \mu G^{R-L}(\cdot) + \nu G^{R-K}(\cdot) \tag{II.5}$$

is an invariant measure for $P(x, \cdot)$, i.e.

$$m(\cdot) = \int_R m(dx) P(x, \cdot).$$

Proof. At first, consider two cases. (i) Let $A \in \mathfrak{C}$ be contained in $R-K$. Since $P(x, A) = P_{R-K}(x, A)$ for $x \in R-K$ and $P_{R-K}(x, C) = 0$ for $x \in K$ and $C \in \boldsymbol{B}$, we have

$$\int_R \nu G^{R-K}(dy) P(y, A) = \int_L \nu(dx) \int_R G^{R-K}(x, dy) P(y, A)$$

$$= \int_L \nu(x) \sum_{n=0}^{\infty} \int_R P^n_{R-K}(x, dy) P(y, A)$$

$$= \int_L \nu(dx) \sum_{n=0}^{\infty} \int_{R-K} P^n_{R-K}(x, dy) P(y, A) = \int_L \nu(dx) \sum_{n=0}^{\infty} \int_{R-K} P^n_{R-K}(x, dy) P_{R-K}(y, A)$$

$$= \int_L \nu(dx) \sum_{n=1}^{\infty} P^n_{R-K}(x, A) = \int_L \nu(dx) \left\{ \sum_{n=0}^{\infty} P^n_{R-K}(x, A) - P^0_{R-K}(x, A) \right\}$$

$$= \nu G^{R-K}(A) - \nu(A).$$

(ii) Let $A \in \mathfrak{C}$ be contained in K. For $x \in L$, we have

$$P_x(X(\sigma_K) \in A, \sigma_K = n+1) = P_x(X(n+1) \in A, \sigma_K > n)$$

$$= E_x(P_x(X(n+1) \in A / \mathfrak{B}_n), \sigma_K > n) = E_x(P(X(n), A), \sigma_K > n)$$

$$= \int_R P^n_{R-K}(x, dx) P(y, A),$$

and hence

$$h^K(x, A) = P_x(X(\sigma_K) \in A) = \sum_{n=1}^{\infty} P_x(X(\sigma_K) \in A, \sigma_K = n)$$

$$= \sum_{n=0}^{\infty} \int_R P^n_{R-K}(x, dy) P(y, A),$$

where $\mathfrak{B}_n$ is the Borel field generated by $X(k, w)$ for $k \leqq n$. Hence, we have

$$\int_R \nu G^{R-K}(x, dy)P(y, A) = \int_L \nu(dx)\int_{R-K} G^{R-K}(x, dy)P(y, A)$$
$$= \int_L \nu(dx)\sum_{n=0}^{\infty}\int_R P^n_{R-K}(x, dy)P(y, A) = \int_L \nu(dx)h^K(x, A) = \mu(A).$$

Then, combining these two cases, we have, for general A,

$$\int_R \nu G^{R-K}(dx)P(x, A) = \int_R \nu G^{R-K}(dx)P(x, A \frown (R-K)) + \int_R \nu G^{R-K}(dx)P(x, A \frown K)$$

(II.6) $$= \nu G^{R-K}(A \frown (R-K)) - \nu(A \frown (R-K)) + \mu(A \frown K)$$
$$= \nu G^{R-K}(A) - \nu(A) + \mu(A).$$

Similarly, we have

(II.7) $$\int_R \mu G^{R-L}(dx)P(x, A) = \mu G^{R-L}(A) - \mu(A) + \nu(A).$$

By (II.5)–(II.7), we have

$$\int_R m(dx)P(x, A) = \{\mu G^{R-L}(A) + \nu(A) - \mu(A)\} + \{\nu G^{R-K}(A) + \mu(A) - \nu(A)\}$$
$$= m(A)$$

and the proof is complete.

REFERENCES

[1] AHLFORS, L., AND A. BEULING, Conformal invariants and function theoretic null-set. Acta Math. 83 (1950), 101–129.

[2] DYNKIN, E. B., Infinitesimal operators of Markov processes. Theory of Prob. and its appl. 1 (1956), 38–60.

[3] FELLER, W., On second order differential operators. Ann. of Math. 61 (1955), 90–105.

[4] FELLER, W., On boundaries and lateral conditions for the Kolmogorov differential equations. Ann. of Math. 65 (1957), 527–570.

[5] HUNT, G. A., Markov processes and potentials. I, II, III, Ill. J. Math. 1 (1957), 44–93, 316–369, 2 (1958), 151–213.

[6] ITO, K., AND H. P. MCKEAN, Diffusion. Forthcoming.

[7] MARUYAMA, G., AND H. TANAKA, Ergodic property of N-dimensional recurrent Markov processes. Mem. Facul. Sci. Kyushu Univ. 13 (1959) pp. 157–172.

[8] NELSON, E., The adjoint Markov process. Duke Math. J. 25 (1958), 671–690.

[9] UENO, T., Some limit theorems for temporally discrete Markov processes. J. Facul. Sci. Univ. Tokyo 7 (1957), 449–462.

[10] WATANABE, T., Some general properties of Markov processes. J. Math. Inst. Polytech. Osaka City Univ. **9** (9958), 9-29.

[11] YOSIDA, K., Simple Markoff process with a locally compact phase space. Math. Japonicae **1** (1948), 99-103.

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