A SET OF CAPACITY ZERO AND THE EQUATION $\Delta u = Pu$

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In our previous papers we introduced a method of classification of Riemann surfaces in terms of the solutions of the differential equation of elliptic type

$$\text{(A)} \qquad \Delta u = Pu.$$

Let F be an open Riemann surface and $P|dz|^2$ be a non-negative invariant expression whose coefficient P has continuous first derivatives on F. We shall always assume that P is positive except at most on a set of two-dimensional measure zero.

We recall briefly the classification scheme in terms of the equation (A) and the harmonic case. We denote by O_G the class of surfaces which have no harmonic Green function, and denote by $F \in O_{PB}$ and $F \in O_{PD}$ when there are no bounded solutions and no solutions with finite energy integral on F, respectively. Then it is known that

$$O_G \subset O_{PB} \subset O_{PD}.$$

If the integral of P on F is of finite value, that is,

$$\iint_F P(p)\, d\sigma_p < \infty,$$

then $O_{PB} = O_{PD}$ holds. We remark that this is the case for any subsurface F obtained from a closed Riemann surface W by deleting a closed set E, when $P(p)$ is well defined on the whole W. If K is a compact subsurface containing E, then a necessary and sufficient condition in order that any bounded solutions of (A) on $K-E$ are prolongable onto the whole K in the sense of (A) is that $F \in O_{PB}$. By this theorem we see easily that, if E is of logarithmic capacity zero, then E is removable for bounded solutions of (A) around E. These results have been already given in [4]. In particular, the last statement has been given explicitely in [3] and recently in [7]. If the coefficient P is not smooth at E, then the situation is not so simple. In the case where E consists of only one point, Brelot [1] discussed the matter in detail.

Let F be an abstract open Riemann surface. Its exhaustion in the usual sense will be denoted by $\{W_n\}$. Let $\Omega_n(p)$ be a bounded solution of (A) on $W_n - \overline{W}_1$ such that

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$$\Omega_n(p)=\begin{cases}1 & \text{on } \partial W_1,\\ 0 & \text{on } \partial W_n,\end{cases}$$

and $\omega_n(p)\equiv\omega(p,\ \partial W_n,\ W_n-\overline{W}_1)$ be the harmonic measure. It is known that the limit functions $\Omega(p)=\lim_{n\to\infty}\Omega_n(p)$ and $\omega(p)=\lim_{n\to\infty}\omega_n(p)$ exist and that they satisfy

$$0\leqq\Omega(p)\leqq 1-\omega(p)\leqq 1\,.$$

LEMMA 1. (Mori [2]) *If* $F\notin O_G$, *then* $\sup_{F-\overline{W}_1}\omega(p)=1$.

LEMMA 2. *If D is a compact region and u is a bounded solution of* (A) *whose boundary value on ∂D is strictly positive and if $F\in O_{PB}$, then u is positive in D.*

It is easy to prove this by making use of the so-called Harnack's inequality for a non-negative solution of (A). See [5].

LEMMA 3. (Myrberg [3]) *On any surface F, there exists the Green function $G(p, q)$ for* (A) *which satisfies an inequality*

$$\iint_F G(p,\ q)P(p)\,d\sigma_p\leqq 2\pi.$$

THEOREM 1. *If* $\inf_{F-\overline{W}_1}\Omega(p)>0$, *then* $F\in O_G$.

Proof. Suppose $F\notin O_G$, then we have

$$\sup_{F-\overline{W}_1}\omega(p)=1$$

by Lemma 1. Therefore $0\leqq\Omega(p)\leqq 1-\omega(p)$ implies that

$$\inf_{F-\overline{W}_1}\Omega(p)=0,$$

which is absurd.

This theorem has been already established in [5] under some inessential assumptions on a given surface F.

LEMMA 4. ([5]) *If F belongs to the class O_{PB} and*

$$\iint_F P(p)\,d\sigma_p=\infty,$$

then $\inf_{F-\overline{W}_1}\Omega(p)=0$.

The proof is simple if Lemma 3 is applied.

COROLLARY 1. *If* $\inf_{F-\overline{W}_1}\Omega(p)>0$, *then*

$$\iint_F P(p)\,d\sigma_p<\infty.$$

Proof. By Theorem 1 we have $F \in O_G$ and hence $F \in O_{PB}$. Therefore by Lemma 4 we have the desired fact.

THEOREM 2. *Let F be a subsurface obtained from a closed Riemann surface W by deleting a closed set E. Let K be a subregion of W containing E, and suppose that $P(p)$ is defined on the whole W. Then a necessary and sufficient condition in order that all the bounded solutions of* (A) *on $K-E$ are prolongable onto the whole K in the sense of* (A) *is that $F \in O_G$, namely, E is of capacity zero.*

Proof. Sufficiency has been already explained. So we shall prove its necessity. We may put $F-\overline{K}$ as the first member of the exhaustion of F. Then $\Omega(p)$ is a non-negative bounded solution of (A) on $K-E$. Therefore $\Omega(p)$ is prolongable onto the whole K as a bounded non-negative solution of (A). Since $\Omega(p)$ is constant 1 on ∂K, $\Omega(p)$ is strictly positive in K by Lemma 2. Therefore

$$\inf_{K-E} \Omega(p) = \inf_{K} \Omega(p) > 0,$$

which implies that $F \in O_G$.

REMARK. If $P(p)$ is continuous without its smoothness on W, more generally around E, then the corresponding statement cannot be expressed in terms of the ordinary Laplacian operator. However, the result also holds and can be expressed in terms of the so-called generalized Laplacian operator. If $P(p)$ has the Hölder continuity, the theorem 2 remains valid with the ordinary Laplacian operator.

Now we shall enter into the corresponding theorem for the energy-finite solutions of (A).

LEMMA 5. ([4]) *Let F be an open Riemann surface. If $F \in O_{PD}$, then there is no non-constant energy-finite solution of* (A) *on F-$\overline{W}_1$ with vanishing boundary value on ∂W_1.*

LEMMA 6. *Let u_n be a positive solution of* (A) *on $W_n - \overline{W}_1$ satisfying a boundary condition*

$$u_n = \begin{cases} f(>0) & on\ \partial W_1, \\ 0 & on\ \partial W_n \end{cases}$$

and $u = \lim_{n\to\infty} u_n$, then u is of finite energy on F-$\overline{W}_1$.

Proof. By Green's formula we have

$$E_n(u_n) \equiv \iint\limits_{W_n - \overline{W}_1} \left(\left(\frac{\partial}{\partial x} u_n \right)^2 + \left(\frac{\partial}{\partial y} u_n \right)^2 + P u_n^2 \right) dxdy = -\int\limits_{\partial W_1} f \frac{\partial}{\partial \nu} u_n ds.$$

Evidently $u_n \leqq u_m$ for $n < m$ and $0 \leqq u_n \leqq \max_{\partial W_1} f$, so that

$$0\leqq-\frac{\partial}{\partial\nu}u_m\leqq-\frac{\partial}{\partial\nu}u_n$$

on ∂W_1 for $n<m$. Therefore we have

$$\lim_{n\to\infty}E_n(u_n)=E_{F-\overline{W}_1}(u)=-\int_{\partial W_1}f\frac{\partial}{\partial\nu}u\,ds<\infty.$$

THEOREM 3. *If F, W, E and K are the same as in Theorem 2, then a necessary and sufficient condition in order that all the energy-finite solutions of* (A) *on $K-E$ are prolongable onto the whole K in the sense of* (A) *is that E is of capacity zero.*

Proof. By Lemma 6, $\Omega(p)$ is non-negative and of finite energy, a fortiori, of finite Dirichlet integral on $F-\overline{W}_1\equiv K-E$. By the assumption $\Omega(p)$ is prolongable onto the whole K. By the same reasoning as in Theorem 2, we have $F\in O_G$ and $\mathrm{cap}(E)=0$. The proof of necessity part is now complete. To prove the sufficiency part let us remark that $F\in O_G$ implies $F\in O_{PD}$. Let u be an energy-finite solution of (A) on $F-\overline{W}_1$ and let v be a solution of (A) on K such that $v=u$ on $\partial W_1\equiv\partial K$. Then we have $E_K(v)<\infty$ by the smoothness of $P(p)$ on W. On denoting by $U=u-v$, we see that $E_{K-E}(U)<\infty$ by the facts that $E_{K-E}(u)>\infty$ and $E_K(v)<\infty$. Since $U=0$ an ∂K, by Lemma 5, $F\in O_{PD}$ implies $U\equiv 0$ on $K-E$, whence follows that $u\equiv v$ on $K-E$. However, v satisfies the differential equation (A) on the whole K. Thus u is prolongable onto K in the sense of (A).

REMARK. In our Theorem 3, the assumption of the necessity part can be considerably weakened, that is, the energy-finiteness can be replaced by the finiteness of Dirichlet integral, since the function $\Omega(p)$ is of finite Dirichlet integral. For the sufficiency part Royden [7] gave its alternative proof under the assumption of Dirichlet-finiteness. Thus the Theorem 3 holds for the Dirichlet-finite solutions.

In [4] we discussed the relations between the maximum principle for any bounded solutions of the equation (A) and the class O_{PD}. Now we shall discuss a maximum principle for the energy-finite solutions of (A) and the class O_{PD}.

LEMMA 7. ([4]) *If $F\notin O_{PD}$, then there is the so-called reproducing kernel $K(p,q)$ of* (A) *on F for the Hilbert space consisting of energy-finite solutions of* (A) *on F. $K(p,q)$ satisfies*

$$K(p,q)=K(q,p)\quad and\quad 0<K(p,q)<M<\infty$$

on F. Therefore $O_{PB}\subset O_{PD}\equiv O_{PBD}$.

LEMMA 8. *Every energy-finite non-negative solution u of* (A) *on $F-\overline{W}_1$ can be decomposed into two energy-finite non-negative solutions u_1 and u_2 of*

(A) *on* $F-\overline{W}_1$ *in such a way that* $u=u_1+u_2$ *on* $F-\overline{W}_1$, $u_1=u$ *and* $u_2=0$ *on* ∂W_1 *and* u_1 *satisfies the maximum principle on* $F-\overline{W}_1$. *Here the maximum principle means that*

$$\sup_{F-\overline{W}_1} u(p) = \sup_{\partial W_1} u(p).$$

Proof. Let u_{1n} be a solution of (A) on $W_n-\overline{W}_1$ such that

$$u_{1n}=\begin{cases} u & \text{on } \partial W_1, \\ 0 & \text{on } \partial W_n, \end{cases}$$

then $u_1 \equiv \lim_{n\to\infty} u_{1n}$ exists and is non-negative. By Lemma 6 u_1 is of finite energy, so that $u_2=u-u_1$ is also of finite energy and has the desired properties.

THEOREM 4. *If* $F\in O_{PD}$, *then the maximum principle holds for any positive energy-finite solutions of* (A) *on* $F-\overline{W}_1$, *and vice versa.*

Proof. By Lemma 8 we have $u=u_1+u_2$, $E_{F-\overline{W}_1}(u_1)<\infty$, $E_{F-\overline{W}_1}(u_2)<\infty$ and $u_1=u$ and $u_2=0$ on ∂W_1. By Lemma 5, $u_2\equiv 0$ on $F-\overline{W}_1$, whence follows $u\equiv u_1$ on $F-\overline{W}_1$. On the other hand, $u_{1n}\leqq \sup_{\partial\overline{W}_1} u$ on $W_n-\overline{W}_1$ which shows that $\sup_{F-\overline{W}_1} u=\sup_{F-\overline{W}_1} u_1 \leqq \sup_{\partial W_1} u$. Conversely, if $F\notin O_{PD}$, then there is at least one non-constant non-negative energy-finite solution $K(p, q)$ of (A) on F by Lemma 7. The maximum principle implies that

$$\sup_{F-\overline{W}_1} K(p, q)=\sup_{\partial W_1} K(p, q) \quad \text{and} \quad \sup_{\overline{W}_1} K(p, q)=\sup_{\partial W_1} K(p, q).$$

This is absurd, since no positive solution has its maximum in an inner point of the domain.

SUPPLEMENTARY NOTE. We shall mention here a correction to [4]. In that paper we claimed the following theorem ([4], Theorem 5.2).

Let G be a non-compact connected subregion of F with an analytic boundary C. If there exists a non-constant solution $U(z)$ of (A) on G, such that $U=0$ on C and $D_G(U)<\infty$, then $F\notin O_D$. Conversely, if $F\notin O_D$, then we can find such a domain G and a solution $U(z)$ of (A).

Notations $D_G(u)$ and O_D in [4] coincide with the energy integral $E_G(u)$ and O_{PD} in the present paper, respectively.

The proof of the latter half should be corrected as follows: $F\notin O_{PD}$ implies the existence of the reproducing kernel $K(p, q)$ (>0) of (A) on F. Let G be a domain $F-\overline{W}_1$, Let $u_n(p)$ be a solution of (A) on $W_n-\overline{W}_1$ such that $u_n(p)=K(p, q)$ on ∂W_1 and $u_n(p)=0$ on ∂W_n. Then $u(p)=\lim_{n\to\infty} u_n(p)$ exists and $u(p)\leqq K(p, q)$, $u(p)\not\equiv K(p, q)$ on G. Moreover $E_G(u)<\infty$ by Lemma 6. The function $U(p)=K(p, q)-u(p)$ on G is the desired.

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