A NOTE ON TYPICAL FUNCTIONS OF SUMS OF NON-NEGATIVE INDEPENDENT RANDOM VARIABLES

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1. Consider a sequence of independent random variables

$$X_1,\ X_2,\ \cdots$$

and let $\sigma_n(x)$ be the distribution function of $S_n = \sum_{i=1}^n X_i$. The existence of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n E\{X_i\} = m$$

with some conditions on distributions of X_i, ensures

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=1}^{N(h)} \Phi_{\sigma_n}(h) = \frac{\pi}{2m},$$

where

$$\Phi_{\sigma_n}(h) = \int_{-\infty}^{\infty} \frac{h^2}{h^2+x^2}\, d\sigma_n(x)$$

and $N(h)$ is any integral valued function such that

$$\frac{N(h)}{h} \to \infty \qquad \text{as } h\to\infty.$$

This property on the typical functions $\Phi_{\sigma_n}(h)$ has been proved by T. Kawata [2]. In this short paper, we shall note that assuming the existence of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n a_i = a$$

and some further conditions, we have

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=1}^{N(h)} a_n \Phi_{\sigma_n}(h) = \frac{\pi a}{2m}.$$

2. In the first place, we shall prepare some lemmas.

Lemma 1. *Let $\{X_i\}$ be a sequence of non-negative independent random variables, and suppose that*

$$0 < m_i = E\{X_i\} \quad (i=1,\ 2,\ \cdots), \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n m_i = m > 0,$$

and

$$\lim_{A\to\infty} \int_A^{\infty} x\, dF_n(x) = 0$$

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holds uniformly with respect to n, where $F_n(x)$ is the distribution function of X_n. Moreover, let $\{a_i\}$ be a sequence of non-negative numbers such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i=a.$$

Then we have

$$\sum_{n=1}^{\infty}a_n\varphi_n(s)<\infty \quad \textit{for} \quad s>0$$

and

$$\lim_{s\downarrow 0}s\sum_{n=1}^{\infty}a_n\varphi_n(s)=\frac{a}{m},$$

where

$$\varphi_n(s)=\int_0^\infty e^{-sx}d\sigma_n(x) \quad \textit{and} \quad \sigma_n=F_1*F_2*\cdots*F_n.$$

LEMMA 2. *Besides the hypotheses of Lemma 1, we further suppose that*

$$F_i(x)-F_i(0)\leqq Ax^p \quad \textit{for} \quad 0<x<\delta,$$

where δ and A are constants independent of $i\geqq 1$ and $p>1$. Then there exist s_0 and B independent of i such that

$$f_n(s)\equiv\int_0^\infty e^{-sx}dF_n(x)<Bs^{-q}$$

for $0<s_0<s$ and $n\geqq 1$, q being any positive number less than p, and there exists a continuous function $\theta(s)$ such that

$$\theta(s)<1$$

and

$$-\varphi_n'(s)\leqq\frac{1}{s}n\theta(s)^{n-1} \quad \textit{for} \quad 0<s.$$

Lemma 1 and Lemma 2 have been shown in [1] and [2], respectively.

LEMMA 3. *Under the conditions of Lemma 2, $\sum_{n=1}^{\infty}a_n\varphi_n(s)$ is integrable in (δ,∞), $\delta>0$ and $\sum_{n=1}^{\infty}a_n\varphi_n'(s)$ is uniformly convergent in every finite interval not containing the origin.*

Proof. Since $a_n=o(n)$ as $n\to\infty$, we have by the first half of Lemma 2 that

$$0\leqq+\sum_{n=1}^{\infty}a_n\varphi_n(s)\leqq\sum_{n=1}^{\infty}Cn\cdot B^n s^{-nq}=\frac{C\cdot Bs^{-q}}{(1-Bs^{-q})^2}\leqq As^{-q}$$

for some positive constants A, C and sufficiently large s where $q>1$. And this inequality shows with Lemma 1 that $\sum_{n=1}^{\infty}a_n\varphi_n(s)$ is integrable in (δ,∞), $\delta>0$. Secondly, by the latter half of Lemma 2, we have

$$0 \leqq -\sum_{n=1}^{\infty} a_n \varphi_n'(s) \leqq \frac{C}{s} \sum_{n=1}^{\infty} n^2 \theta(s)^{n-1},$$

which implies that $\sum_{n=1}^{\infty} a_n \varphi_n'(s)$ is uniformly convergent in every finite interval not containing the origin.

THEOREM 1. *Let $\Phi_{\sigma_n}(h)$ be the typical function of $\sigma_n(x)$. If conditions in Lemma 3 are satisfied, then we have*

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=1}^{\infty} a_n \Phi_{\sigma_n}(h) = \frac{a\pi}{2m}.$$

Proof. The proof can be easily obtained by considering $a_n \varphi_n(s)$ in place of $\varphi_n(s)$ in the course of the proof of Theorem 1 in [2] using Lemmas 1 and 3.

THEOREM 2. *Let $N(h)$ be any integral valued function such that*

$$\frac{N(h)}{h} \to \infty \quad \text{as } h \to \infty.$$

Then under the condition of Theorem 1, we have

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=1}^{N(h)} a_n \Phi_{\sigma_n}(h) = \frac{a\pi}{2m}.$$

For the proof, we show some lemmas.

LEMMA 4. *Under the conditions of Theorem 1, there exists a $\theta = \theta(\delta, A)$ less than 1, such that*

$$\sum_{i=n+1}^{\infty} a_i \varphi_i(s) \leqq C_1 n \theta^n \quad \textit{for } \delta \leqq s \leqq A,$$

where δ and A are any positive constants and C_1 is a constant independent of n.

Proof. There exists a $\theta_1 = \theta_1(s)$ such that

$$f_i(s) = \int_0^{\infty} e^{-sx} dF_i(x) < \theta_1.$$

$\theta_1(s)$ is a continuous function of s and $\theta_1(s) < 1$ for $\delta \leqq s \leqq A$. This facts have been proved in the proof of Lemma 3 in [2]. Since $\varphi_n(s) = f_1(s) \cdots f_n(s)$, we have

$$\sum_{i=n+1}^{\infty} a_i \varphi_i(s) \leqq C \sum_{i=n+1}^{\infty} i\theta^i \leqq \frac{C(n+1)\theta^{n+1}}{(1-\theta)^2} \leqq C_1 n \theta^n$$

where $\theta = \theta(\delta, A) = \max_{\delta \leqq s \leqq A} \theta_1(s)$.

LEMMA 5. *Under the conditions of Lemma 1, there exist positive constants m_1 and D such that*

$$\varphi_n(s) < e^{-snm_1}$$

and

$$\sum_{i=n+1}^{\infty} \varphi_i(s) \leqq Ds^{-1}e^{-snm_1}$$

for sufficiently large n and sufficiently small $s > 0$.

This has been proved in Lemma 6 of [2].

Now we shall prove Theorem 2. Put

$$J_n(h) = \int_0^\infty \sin sh \sum_{i=n+1}^{\infty} a_i\varphi_i(s)\,ds$$
$$= \int_0^\delta + \int_\delta^A + \int_A^\infty = I_1 + I_2 + I_3.$$

We can take a constant A such that

$$A^q > B \quad \text{and} \quad \varphi_i(s) \leqq B^i s^{-qi} \quad \text{for} \quad s \geqq A,$$

where $B>0$ and $q>1$ are some constants. This fact may be proved by Lemma 2. Then we have

$$|I_3| \leqq \int_A^\infty \sum_{i=n+1}^{\infty} a_i\varphi_i(s)ds \leqq C\int_A^\infty \frac{(n+1)B^{n+1}s^{-q(n+1)}}{(1-Bs^{-q})^2}\,ds$$
$$\leqq \frac{C}{(1-BA^{-q})^2}\,\frac{(n+1)B^{n+1}}{\{q(n+1)-1\}A^{q(n+1)-1}} = o(1) \quad \text{as } n\to\infty.$$

Next, Lemma 4 shows

$$|I_2| \leqq \int_A^\delta \sum_{i=n+1}^{\infty} a_i\varphi_i(s)ds \leqq C_1(A-\delta)n\theta^n = o(1) \quad \text{as } n\to\infty.$$

Putting $(a_1 + \cdots + a_n)/n = a + \varepsilon_n$, we have

$$a_n = a + n\varepsilon_n - (n-1)\varepsilon_{n-1}$$

and $\varepsilon_n \to 0$ $(n\to\infty)$. So we get

$$|I_1| \leqq h\int_0^\delta s \sum_{i=n+1}^{\infty} a_i\varphi_i(s)ds$$
$$= ha\int_0^\delta s \sum_{i=n+1}^{\infty} \varphi_i(s)ds - n\varepsilon_n h\int_0^\delta s\varphi_{n+1}(s)ds + h\int_0^\delta s \sum_{i=n+1}^{\infty} i\varepsilon_i\varphi_i(s)(1-f_{i+1}(s))ds$$
$$= I_{11} + I_{12} + I_{13},$$

and

$$|I_{11}| \leqq ha\int_0^\delta s\cdot D\cdot s^{-1}e^{-snm_1}\,ds$$
$$\leqq aD\cdot h\cdot \frac{1}{nm_1} = \frac{aD}{m_1}\cdot \frac{h}{N(h)} \to 0 \quad \text{as } h\to\infty,$$

taking $n = N(h)$. Further we have

$$|I_{12}| \leqq C \cdot nh \int_0^{\delta} s e^{-s(n+1)m_1} ds$$
$$\leqq Ch \cdot n \cdot \frac{1}{(n+1)^2 m_1^2} \leqq \frac{C}{m_1^2} \cdot \frac{h}{n}$$
$$= \frac{C}{m_1^2} \cdot \frac{h}{N(h)} \to 0 \quad \text{as } h \to \infty.$$

Since

$$\lim_{A \to \infty} \int_A^{\infty} x \, dF_n(x) = 0$$

holds uniformly with respect to n,

$$m_n = \int_0^{\infty} x \, dF_n(x)$$

are bounded for $n = 1, 2, \cdots$ and

$$1 - f_n(s) = \int_0^{\infty} (1 - e^{-sx}) dF_n(x)$$
$$\leqq s \int_0^{\infty} x \, dF_n(x) \leqq Cs \quad \text{for } n = 1, 2, \cdots,$$

where C is some positive constant. Hence we have

$$\sum_{i=n+1}^{\infty} i \varepsilon_i \varphi_i(s)(1 - f_{i+1}(s)) \leqq C \cdot \sum_{i=n+1}^{\infty} i e^{-i m_1 s} \cdot s$$
$$= C \cdot s \cdot \frac{(n + 1 - n e^{-s m_1})}{(1 - e^{-m_1 s})^2} e^{-s(n+1)m_1}$$
$$\leqq C \cdot s \cdot \frac{n \cdot s m_1}{s^2} e^{-s n m_1} + C \cdot s \cdot \frac{e^{-s n m_1}}{s^2}$$
$$\leqq C \cdot n e^{-s n m_1} + C \cdot \frac{e^{-s n m_1}}{s},$$

where C's are constants which may be different on each occurrence. Therefore

$$|I_{13}| \leqq C \cdot hn \int_0^{\delta} s e^{-s n m_1} ds + C \cdot h \int_0^{\delta} s \cdot \frac{e^{-s n m_1}}{s} ds$$
$$\leqq C \cdot hn \cdot \frac{1}{n^2 m_1^2} + C \cdot h \cdot \frac{1}{n m_1}$$
$$= C \cdot \frac{h}{n} = C \cdot \frac{h}{N(h)} \to 0 \quad \text{as } h \to \infty.$$

Putting together above estimations, we get

$$J_{N(h)+1}(\mathrm{h}) \to 0 \quad \text{as } h \to \infty,$$

which with Theorem 1, shows the validity of Theorem 2, because

$$J_{N(h)+1}=\int_0^\infty \sin hs \sum_{n=N(h)+1}^{\infty} \varphi_n(s)\,ds=\frac{1}{h}\sum_{n=N(h)+1}^{\infty} \Phi_{\sigma_n}(h).$$

REMARK. Theorems 1 and 2 hold for a sequence of real numbers a_n such that

$$a_n=b_n-c_n, \quad b_n\geqq 0, \quad c_n\geqq 0 \quad (n=1,2,\cdots)$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} b_i \text{ and } \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} c_i \text{ exist.}$$

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