ISOMORPHISMS BETWEEN COMMUTATIVE BANACH ALGEBRAS WITH AN APPLICATION TO RINGS OF ANALYTIC FUNCTIONS

BY MASAO NAGASAWA

1. Introduction.

Chevalley and Kakutani showed that if D_1 and D_2 are plane domains with no AB-removable boundary points, then they are conformally equivalent if and only if the rings $B(D_1)$ and $B(D_2)$ of all bounded analytic functions on D_1 and D_2, respectively, are algebraically isomorphic (cf. [5]).

It is well known that two compact Hausdorff spaces are homeomorphic if and only if the Banach algebras of all real valued continuous functions on these spaces are isometric as Banach spaces, where the norm is defined by "sup" (the Theorem of Banach-Stone, cf. [1]), and also if and only if these Banach algebras are isomorphic as algebras (the Theorem of Gelfand-Kolmogoroff, cf. [4]).

If we define a norm in $B(D_j)$ by "sup", then $B(D_j)$ appear as Banach spaces with the unit ($j = 1, 2$). This suggests us an analogous question for the above mentioned rings of analytic functions: If $B(D_1)$ and $B(D_2)$ are isometric as Banach spaces, are D_1 and D_2 conformally equivalent?

This question will be solved in general form as the main Theorem of this paper which will be stated in terms of Banach algebras, say, let B_1 (resp. B_2) be a commutative Banach algebra with the unit satisfying a norm condition $\| x^2 \| = \| x \|^2$ for all $x \in B_1$ (resp. B_2), then B_1 and B_2 are algebraically isomorphic if and only if they are isometric as Banach spaces. Then a Banach-Stone type theorem for analytic functions is a direct corollary of the main theorem. In the course of the proof, the representation theory of Banach algebras and the Krein-Milman's theorem which asserts that any weakly* compact convex subset of the conjugate space of a Banach space has sufficiently many extreme points, play naturally essential role.

The author wishes to express his hearty thanks to Professor H. Umegaki, Mr. M. Takesaki and Professor M. Ozawa for their many suggestive discussions in presenting this paper.

2. Main theorems.

In the present paper we assume that any algebra and its subalgebras have always the unit.

Received June 27, 1959.

THEOREM 1. *Let B_1 and B_2 be commutative Banach algebras with the unit satisfying a norm condition* $\| x^2 \| = \| x \|^2$ *for all* $x \in B_j$ $(j=1, 2)$, *then* B_1 *and* B_2 *are algebraically isomorphic if and only if* B_1 *and* B_2 *are isometric as Banach spaces.*

As is well known, each maximal ideal M of B_1 (resp. B_2) uniquely determins an algebraic homomorphism χ_M of B_1 (resp. B_2) onto the complex number field. Furthermore the space of all χ_M, which is weak* compact and will be called the *character space* of B_1 (resp. B_2), is homeomorphic to the *maximal ideal space* of B_1 (resp. B_2). Therefore, in the following we identify the maximal ideal space and the character space of B_1 (resp. B_2) and denote it by Γ_1 (resp. Γ_2) (cf., e.g. [6]).

Now by the norm condition we have for any $b \in B_j$ $(j=1, 2)$

$$\| b \| = \lim_{n\to\infty} (\| b \|^{2^n})^{2^{-n}} = \lim_{n\to\infty} (\| b^{2^n} \|)^{2^{-n}} = \sup\{ |\chi(b)| : \chi \in \Gamma_j \}.$$

If ϕ is the algebraic isomorphism of B_1 onto B_2, then there is one-to-one mapping ϕ' of Γ_2 onto Γ_1 defined by $\chi(\phi b) = \phi'\chi(b)$ for all $b \in B_1$ and all $\chi \in \Gamma_2$, and therefore

$$\begin{aligned} \| \phi b \| &= \sup\{\chi(\phi b) : \chi \in \Gamma_2\} = \sup\{\phi'\chi(b) : \chi \in \Gamma_2\} \\ &= \sup\{\chi'(b) : \chi' \in \Gamma_1\} = \| b \|. \end{aligned}$$

Thus the necessity of the theorem is proved.

Next, let $B_j(\Gamma_j)$ be the Gelfand representations of B_j $(j=1, 2)$ and $C(\Gamma_j)$ be the B^*-algebras of all complex valued continuous functions on Γ_j, then B_j, identified with $B_j(\Gamma_j)$, are subalgebras of $C(\Gamma_j)$ $(j=1, 2)$. Furthermore, $C(\Gamma_j)$ are isometrically isomorphic as B^*-algebras to certain commutative C^*-algebras with the unit on some Hilbert spaces, respectively (cf. e.g. [7]). Therefore, the proof of sufficiency is reduced to the following theorem:

THEOREM 2. *Let* A_1 *(resp.* A_2*) be a commutative* C^**-algebra with the unit on some Hilbert space* H_1 *(resp.* H_2*) and* B_1 *(resp.* B_2*) be uniformly closed but not necessarily self-adjoint subalgebra of* A_1 *(resp.* A_2*) also with the unit. If* B_1 *and* B_2 *are isometric as Banach spaces, then* B_1 *and* B_2 *are algebraically isomorphic.*[1]

Before passing to the proof of the Theorem 2, we shall give several lemmas for operator algebras in next section and the proof of the theorem will be given in section 4.

3. Some lemmas.

Let A be C^*-algebra with the unit I, B a closed linear subspace of A with the unit but not necessarily self-adjoint, and A^* (resp. B^*) the conjugate space of A (resp. B) as Banach space. Let "U_A (resp. U_B)" denote the unit

1) Here, the induced isomorphism is in general different from the given isometry.

sphere of A^* (resp. B^*) and "E_A (resp. E_B)" the set of all extreme points of U_A (resp. U_B). An element σ of U_A (resp. U_B) is said to be the *state* of A (resp. B), if σ satisfies the condition that $\sigma(I)=1$. Let "S_A (resp. S_B)" be the set of all states of A (resp. B), which will be called the *state space* of A (resp. B), then S_A (resp. S_B) is convex and weakly* compact and hence by the Krein-Milman's theorem S_A (resp. S_B) contains extreme points. Denote by "Ω_A (resp. Ω_B)" the set of all extreme points of S_A (resp. S_B). An element of Ω_A (resp. Ω_B) is called a *pure state* and Ω_A (resp. Ω_B) the *pure state space* of A (resp. B).

REMARK 1. *Any state $\sigma \in S_B$ is a restriction of certain state $\sigma' \in S_A$ on B.* Indeed, by the Hahn-Banach's extention theorem σ can be extended to σ' on A and σ' satisfies the condition that $\|\sigma'\|=1$ and $\sigma'(I)=1$, and hence by the Bohnenblust-Karlin's theorem [2], σ' is a positive linear functional in A with $\|\sigma'\|=1$, i.e. a state of A in the usual sense.

LEMMA 1. *If μ is any element of E_B, then there exists an element $\hat{\mu}$ of E_A such that the restriction of $\hat{\mu}$ on B is μ.*

Proof. For any element $\mu \in E_B$, let $F_\mu = \{\sigma : \sigma \in U_A$, and $\sigma = \mu$ *on* $B\}$. By the Hahn-Banach's theorem we extend $\mu \in E_B$ to $\mu' \in U_A$, which belongs to F_μ, so that F_μ is non-empty. Furthermore, F_μ is clearly a weakly* compact and convex subset of A^*, hence there exists an extreme point $\hat{\mu}$ of F_μ by the Krein-Milman's theorem. We now show that $\hat{\mu}$ is an extreme point of U_A i.e. an element of E_A. Assume that $2\hat{\mu} = \nu + \rho$, where ν and ρ are elements of U_A not belonging to F_μ. Let $\hat{\mu}$ be restricted to B and ν_B, ρ_B denote the restrictions of ν and ρ on B, then $2\mu = \nu_B + \rho_B$ holds on B, but μ is an element of E_B, so μ, ν_B and ρ_B must coincide in view of the extremity of μ. It contradicts the assumption that $\nu, \rho \notin F_\mu$. (Similarly, the case where one of ν, ρ is in F_μ.) Thus $\hat{\mu}$ is an element of E_A. This concludes the proof.

LEMMA 2. *If ω is an element of Ω_B, then there exists an element $\hat{\omega}$ of Ω_A whose restriction on B is ω.*

Proof. For any element $\omega \in \Omega_B$, let $F_\omega = \{\mu : \mu$ *is a state of A which coincides with ω on B*$\}$, and following the same process as in the proof of Lemma 1, we can complete the proof.

In the following, we assume that the C^*-algebra A considered is commutative.

LEMMA 3. *Any element $\mu \in E_A$ is represented as $\mu = \lambda\omega$ by a complex number λ with $|\lambda|=1$ and $\omega \in \Omega_A$, and therefore it satisfies $|\mu(I)|=1$.*

Proof. This follows from the proof of Arens-Kelley's lemma (cf. Lemma 3.2 [1]) under a little modification for the present complex case. We give another proof in footnote.[2)]

LEMMA 4. *If A is commutative and B is its subalgebra, then any element ω of Ω_B is multiplicative, i.e. $\omega(xy)=\omega(x)\omega(y)$ for any $x, y \in B$.*

Proof. If ω is extended to $\hat{\omega}$ as Lemma 2, then $\hat{\omega}$ is a pure state of A, which is multiplicative. Thus ω, the restriction of $\hat{\omega}$ on B, is multiplicative. This concludes the proof.

REMARK 2. As is well known, the maximal ideal space Γ_A of a commutative C^*-algebra A coincides with the pure state space Ω_A of A. Therefore, noticing that Γ_A is defined by algebraical terms and that Ω_A is defined by metrical terms, we may be convinced, glancing at the relation $\Gamma_A=\Omega_A$, that the commutative C^*-algebras are algebraically isomorphic if and only if they are isometric as Banach spaces. But for a uniformly closed subalgebra B of A which is not self-adjoint we have in general $\Omega_B \subsetneqq \Gamma_B$.[3)] Therefore, there is no such simple relation for not self-adoint subalgebras as in the case of C^*-algebras.

By above mentioned lemmas, we are able to concider another representation of B different from the Gelfand's one, if B is a not necessarily self-adjoint uniformly closed subalgebra of a commutative C^*-algebra A. Let $C(\bar{\Omega}_B)$ be the algebra of all complex valued continuous functions on $\bar{\Omega}_B$, where $\bar{\Omega}_B$ is the weak* closure of Ω_B in Γ_B, then B is imbedded into $C(\bar{\Omega}_B)$ in isometrically isomorphic manner, which will be called the *Ω_B-representation* of B. When B is self-adjoint, Ω_B-representation of B coincides with the usual Gelfand's representation.

In this paper, when Ω_B-representation of B will be considered, B and its Ω_B-representation will be always identified.

---

2) Let $\hat{\Omega}_A=\{\lambda\omega : |\lambda|=1, \omega \in \Omega_A\}$, then the bipolar $\hat{\Omega}_A{}^{00}$ of $\hat{\Omega}_A$ coincides with U_A, where the definitions of polar and bipolar are in the sense of Bourbaki [3]. For, the polar $\hat{\Omega}_A{}^0$ of $\hat{\Omega}_A$ is Σ_A, the unit sphere of A, and $\Sigma_A{}^0=U_A$, which imply $\hat{\Omega}_A{}^{00}=U_A$. Furthermore, $\hat{\Omega}_A{}^{00}=K(\hat{\Omega}_A)$, the closed convex hull of $\hat{\Omega}_A$, (cf. chap. IV, §1, Prop. 3 [3]). Therefore $E_A=\hat{\Omega}_A$, since $\hat{\Omega}_A$ is weak* compact (cf. chap. II, Prop. 4 [3]). This concludes the proof.

3) For example, let A be the ring of all functions that are analytic inside of the unit sphere $\{z; |z| \leqq 1\}$ of complex plane and continuous on the unit sphere. Let B be the ring of all functions $\hat{f}$ which are restrictions of $f \in A$ on the unit circle $\{z; |z|=1\}$ as their domain. Then $B \subset C$, where C is the ring of all complex valued continuous functions on the unit circle. Clearly C is a C^*-algebra and B is its uniformly closed and not self-adjoint subalgebra. Therefore, Ω_C and Γ_C coincide with the unit circle, but Γ_B is the unit sphere (cf. pp. 182-183 [7]), hence $\Omega_C \subsetneqq \Gamma_B$. Any element of Ω_B can be extended on C but any element of $\Gamma_B - \Omega_C$ is not the extension of any element of Ω_B. Thus $\Omega_B \subsetneqq \Gamma_B$.

4. Proof of Theorem 2.

First we prove the case where the isometry θ of B_1 onto B_2 preserves the unit. Under the same notations as in Theorem 2 we have:

THEOREM 3. *If B_1 and B_2 are isometric as Banach spaces and the isometry θ preserves the unit, i.e. $\theta(I)=I$, then θ is also an algebraical isomorphism between B_1 and B_2.*

Proof. If we define the transpose θ' of θ by $\mu(\theta x)=\theta'\mu(x)$ for all $x\in B_1$ and $\mu\in B_2^*$, then θ' is an isometry of B_2^* onto B_1^* and θ' carries the pure state space Ω_2 of B_2 onto Ω_1 of B_1 in one-to-one manner, because θ preserves the unit. By Lemma 4 any element of Ω_j $(j=1, 2)$ is multiplicative, so that

$$\begin{aligned}\omega(\theta(xy)) &= \theta'\omega(xy)=\theta'\omega(x)\theta'\omega(y)\\ &= \omega(\theta x)\omega(\theta y)=\omega(\theta x\theta y)\end{aligned}$$

for any $\omega\in\Omega_2$ and any x, $y\in B_1$ Therefore, considering Ω_2-representation, we have $\theta(xy)=\theta(x)\theta(y)$, which implies multiplicativity of θ. Thus θ is an algebraical isomorphism, because θ preserves the unit by assumption. This concludes the proof.

Proof of Theorm 2. Let θ be the given isometry of B_1 onto B_2. If we can construct from the given isometry θ an isometry θ_0 of B_1 onto B_2 which preserves the unit, then by Theorem 3 the proof will be completed.

First we show that $\theta(I)$ is unitary. Let θ' be the transpose of θ, then θ' is an isometry of B_2^* onto B_1^*, and therefore carries the pure state space Ω_2 of B_2 into the extreme point space E_1 of B_1. So, we have by Lemma 2 and Lemma 3,

$$|\omega(\theta(I))|=|\theta'\omega(I)|=1,$$

for any $\omega\in\Omega_2$. Considering in Ω_2-representation, $\theta(I)$ is unitary.

In the following part we consider B_2 in Ω_2-representation. Put $u=\theta(I)$, $\theta_0=u^{-1}\theta$ and $B_3=u^{-1}B_2$. Then B_3 is a uniformly closed linear subspace of $C=C(\bar{\Omega}_2)$ and has the unit. θ_0 is an isometry of B_1 onto B_3 preserving the unit. If we show that $B_3=B_2$, the proof is finished.

We shall prove that if ω_1, $\omega_2\in\Omega_C$ are such that $\omega_1=\omega_2$ on B_3, then $\omega_1\equiv\omega_2$. Notice that any element $x\in C$ has the form

$$x=\lim_\nu\textstyle\sum_{i=1}^{m_\nu}x_{i,\nu}^*y_{i,\nu}=\lim_\nu\sum_{i=1}^{m_\nu}(u^{-1}x_{i,\nu})^*(u^{-1}y_{i,\nu}),$$

where $x_{i,\nu}$, $y_{i,\nu}\in B_2$ and the limit is of uniform sense. Then,

$$\omega_1(x)=\lim_\nu\textstyle\sum_{i=1}^{m_\nu}\omega_1(x_{i,\nu}^*y_{i,\nu})=\lim_\nu\sum_{i=1}^{m_\nu}\overline{\omega_1(u^{-1}x_{i,\nu})}\omega_1(u^{-1}y_{i,\nu}),$$

$$\omega_2(x)=\lim_\nu\textstyle\sum_{i=1}^{m_\nu}\omega_2(x_{i,\nu}^*y_{i,\nu})=\lim_\nu\sum_{i=1}^{m_\nu}\overline{\omega_2(u^{-1}x_{i,\nu})}\omega_2(u^{-1}y_{i,\nu}).$$

Since $u^{-1}x_{i,\nu}$, $u^{-1}y_{i,\nu}\in B_3$ and $\omega_1=\omega_2$ on B_3, the right hand sides of the above equalities are equal. Thus $\omega_1=\omega_2$ in Ω_C, which implies that $\omega\in\Omega_3$, the pure state space of B_3, has unique extension property to an element of Ω_C. Hence we can regard as $\Omega_3 \subset \Omega_C$.

For any element $\omega \in \Omega_2$, regarded as an element of Ω_C, $\omega(u^{-1}x) = \overline{\omega(u)}\omega(x)$ on B_3, and therefore ω, restricted on B_3, is a state of B_3. Moreover, we shall show that ω is a pure state of B_3. Considering ω as an element of Ω_C, we have for any $x \in B_2$,

$$(*) \qquad \omega(x) = \omega(xu^{-1}u) = \omega(xu^{-1})\omega(u) = \omega(u^{-1}x)\overline{\omega(u^{-1})}.$$

We assume that ω is not a pure state of B_3, then $2\omega = \sigma + \sigma'$, where σ and σ' are states of B_3, and hence by (*) for any $x \in B_2$,

$$\begin{aligned} 4\omega(x) &= 2\omega(u^{-1}x)\overline{2\omega(u^{-1})} \\ &= [\sigma(u^{-1}x) + \sigma'(u^{-1}x)][\overline{\sigma(u^{-1})} + \overline{\sigma'(u^{-1})}] \\ &= \sigma(u^{-1}x)\overline{\sigma(u^{-1})} + \sigma(u^{-1}x)\overline{\sigma'(u^{-1})} \\ &\quad + \sigma'(u^{-1}x)\overline{\sigma(u^{-1})} + \sigma'(u^{-1}x)\overline{\sigma'(u^{-1})}. \end{aligned}$$

Put $\sigma_1(x) = \sigma(u^{-1}x)\overline{\sigma(u^{-1})}$, $\sigma_2(x) = \sigma(u^{-1}x)\overline{\sigma'(u^{-1})}$, $\sigma_3(x) = \sigma'(u^{-1}x)\overline{\sigma(u^{-1})}$ and $\sigma_4(x) = \sigma'(u^{-1}x)\overline{\sigma'(u^{-1})}$ for any $x \in B_2$, then σ_j $(j = 1, 2, 3, 4)$ are elements of the unit sphere of B_2^*, moreover states of B_2 and $4\omega = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ on B_2, which contradicts the extremity of ω. Therefore $\omega \in \Omega_3$, and $\Omega_2 \subset \Omega_3$. Analogously we have $\Omega_3 \subset \Omega_2$, hence $\Omega_3 = \Omega_2$.

Now if we take $x, y \in B_1$, then for any $\omega \in \Omega_3$,

$$\begin{aligned} \omega(\theta_0(xy)) &= \theta_0'\omega(xy) = \theta_0'\omega(x)\theta_0'\omega(y) \\ &= \omega(\theta_0 x)\omega(\theta_0 y) = \omega(\theta_0(x)\theta_0(y)), \end{aligned}$$

where θ_0' is the transpose of θ_0, and hence

$$\theta_0(xy) = \theta_0(x)\theta_0(y) \in B_3.$$

Thus B_3 is an algebra and so $u^{-1}xu^{-1}y \in B_3$, which implies $xu^{-1}y \in B_2$. In particular putting $x = y = I$, $u^{-1} \in B_2$ and hence $B_2 = B_3$. This concludes the proof.

5. An application to rings of analytic functions.

Let D_1 (resp. D_2) be a plane domain with no AB-removable boundary points and $B(D_1)$ (resp. $B(D_2)$) be the ring of all bounded analytic functions in D_1 (resp. D_2), then the Theorem of Chevalley and Kakutani [5] is stated as follows: *D_1 and D_2 are conformally equivalent if and only if $B(D_1)$ and $B(D_2)$ are algebraically isomorphic.*

Moreover, if we define a norm of f in $B(D_j)$ by $\|f\| = \sup\{|f(z)|: z \in D_j\}$ $(j = 1, 2)$, then $B(D_j)$ are commutative Banach algebras satisfying the norm condition $\|f^2\| = \|f\|^2$. Therefore, by Theorem 1 $B(D_1)$ and $B(D_2)$ are algebraically isomorphic if and only if they are isometric as Banach spaces. Thus we have obtained the following:

THEOREM 4. *The following assertions are mutually equivalent:*

(1) *D_1 and D_2 are conformally equivalent,*
(2) *$B(D_1)$ and $B(D_2)$ are algebraically isomorphic,*
(3) *$B(D_1)$ and $B(D_2)$ are isometric as Banach spaces.*

REMARK 3. When D_1 (resp. D_2) is not assumed to satisfy the above condition, let D_1' (resp. D_2') be the smallest plane domain with no AB-removable boundary points containing D_1 (resp. D_2), then by Rudin [8] the assertion (1) in the above theorem is to be replaced by

(1') *D_1' and D_2' are conformally equivalent.*

REFERENCES

[1] ARENS, R. F., AND J. L. KELLEY, Characterization of the space of continuous functions over a compact Hausforff spaces. Trans. Amer. Math. Soc. 62 (1947), 499–508.
[2] BOHNENBLUST, H. F., AND S. KARLIN, Geometrical properties of the unit sphere of Banach algebras. Ann. of Math. (2) 62 (1955), 217–229.
[3] BOURBAKI, N., Espaces vectoriels topologiques. Paris, Hermann, (1953), (1955).
[4] GELFAND, I., AND A. N. KOLMOGOROFF, On rings of continuous functions on topological spaces. C. R. (Doklady) Acad. Sci. URSS 22 (1939), 11–15.
[5] KAKUTANI, S., Rings of analytic functions. Lectures on functions of a complex variable. Univ. of Michigan Press, Ann Arbor (1955), 71–83.
[6] LOOMIS, L. H., An introduction to abstract harmonic analysis. New-York, (1953).
[7] Наймарк, М. А., Нормированные кольца. Москва (1956).
[8] RUDIN, W., Some theorems on bounded analytic functions. Trans. Amer. Math. Soc. 78 (1955), 333–342.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.