ON SOME LOCAL PROPERTIES OF FIBRED SPACES

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One of the most fruitful ideas in differential geometry is the idea exploited by E. Cartan of attaching a space to every point of a certain base space B. The attached space in Cartan's work is usually a homogeneous space F such that every point of F is equivalent to any other point under the action of a certain (structure) group G which operates transitively in F. The notion of connection as developed by Cartan consists of the establishment of a correspondence between the spaces F attached to two infinitely near points and the connection is called euclidean, affine, or projective according as the group G is the orthogonal, affine or the projective group. This conception of Cartan's has led to the modern notion of a fibre bundle developed mainly by Ehresmann, Chern and Lichnerowicz to whose fundamental works we refer. The homogeneous space F_{P} attached to a certain point P of the base space B is called the fibre. The spaces F_{P} attached to points of the base space are all homeomorphic to a certain type fibre F. The so-called bundle space E to which this leads is a leaved manifold whose dimension is the sum of the dimensions of the base space and of the fibre. Compound manifolds of a very similar kind have also been extensively treated by Wagner [25] but his point of view is somewhat different. In fibre bundle theory the three spaces E, B and F are differentiable manifolds. The fibres homeomorphic to the type fibre F are holonomic subspaces of the bundle space E and in local coordinates can be expressed by finite equations satisfied by the local coordinates of E in that region. The tangent space to E at any point can then be decomposed into two complementary spaces, one of which is tangent to the fibre and the other is a non-holonomic subspace (or a non-integrable distribution in the terminology of Chevalley [4]) transversal to the fibre. It has now become customary to refer to a vector tangent to the fibre as a 'vertical' vector, and a vector belonging to the complementary transversal distribution as a 'horizontal' vector. The notion of connection is now often formulated in terms of these complementary subspaces of E.

In this paper the authors take the general space E to be a Riemannian space, or a space with a euclidean connection, or a space of paths. The fibres are differentiable subspaces of E which can be expressed locally in the form $f(\xi)=x$ where ξ are local coordinates in E. If a geometric object defined in E can be expressed locally in terms of x only that geometric object will be said to be *induced* in the base space.

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The paper as a whole is written in the classical tradition with systematic use made of the ideas and techniques of the Tensor Calculus. A general reference may be made to Schouten's book Ricci-Calculus [21] for the techniques used.

In the first paragraph we examine the conditions in order that the base vectors of the horizontal distribution at any point shall be invariant for displacement along the fibre. This is followed by the investigation of the conditions in order that a tensor field and an affine connection may be induced in the base space. In the fourth and fifth paragraphs the fibred space E is supposed to be endorsed with a system of paths and the conditions for induction are given. This is followed by the cooresponding investigation for the metric tensor of E and for motions in E. We examine in the seventh paragraph some special results which may be obtained by taking a privileged system of coordinates in E which enable us to obtaine some interesting sidelights on the theory of connections and of the holonomy group. Conditions are also given in order that the fibres at two infinitely near points shall be isometric. This part of the theory leads naturally to the discussion of spaces in which the fundamental tensors are dependent not only on position but also upon a certain element of support. Accordingly in the final paragraph we give a treatment of Finsler spaces as a fibred space when we assume that E is the tangent bundle of the base space. We obtain the euclidean connection in Finsler space as the connection induced in the horizontal distribution where the E is supposed to be a metric space with torsion.

§1.

Let E be a differentiable manifold X_{m+n} of dimension $m+n$, and of class C^r $(r \geqq 4)$. Let an equivalence relation R divide E into equivalence classes $F(\mathrm{P})$ (the fibres), and let E/R be the base space X. The X_{m+n} is assumed to be covered by a system of coordinate neighbourhoods U_A with local coordinates ξ^{κ}_{A} 1) where A belongs to a set M. For an arbitrary point P_0 in X_{m+n} there exists a neighbourhood $U(\mathrm{P}_0)$ and a subset of coordinate neighbourhoods $U_B(\xi^{\kappa}_{B})$, $B \subset N \subset M$ such that the union $\cup F(\mathrm{P})$ for $\mathrm{P} \in U(\mathrm{P}_0)$ is covered by the union $\cup U_B(\xi^{\kappa}_{B})$ and such that if a point P lies in the intersection $U_{B_1} \cap U_{B_2}$ of two coordinate neighbonrhoods of the set U_B then the portion of the fibre $F(\mathrm{P})$ in $U_{B_1} \cap U_{B_2}$ is represented by n independent equations

$$x^{h}_{B_1} = f^{h}_{B_1}(\xi^{\kappa}_{B_1}) \quad \text{and} \quad x^{h}_{B_2} = f^{h}_{B_2}(\xi^{\kappa}_{B_2}) \tag{1.1}$$

of class C^r in the respective coordinate neighbourhoods and such that there exist relations

$$x^{h}_{B_2} = g^{h}_{B_2}(x^{i}_{B_1}) \tag{1.2}$$

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1) The convention with regard to indices will be as follows: Greek indices run from 1 to $m+n$, Latin indices a, b, c, d, e run from 1 to m, and Latin indices h, i, j, k, l run from $m+1$ to $m+n$.

of class C^r with a non-vanishing Jacobian in the domain considered.

We rewrite (1.1) and (1.2) in the forms

$$x^h = f^h(\xi^\kappa) \quad \text{and} \quad x^{h'} = f^{h'}(\xi^{\kappa'}) \tag{1.3}$$

and

$$x^{h'} = x^{h'}(x). \tag{1.4}$$

Now the fibres are m-dimensional submanifolds determined by n equations

$$x^h = x^h(\xi) = f^h(\xi), \tag{1.5}$$

where $f^h(\xi)$ are of class C^r and the rank of the matrix whose elements are

$$C^h{}_\lambda = \partial_\lambda x^h = \frac{\partial x^h}{\partial \xi^\lambda} \tag{1.6}$$

is n.

Since the the rank of the matrix $(C^h{}_\lambda)$ is n, we may regard $C^h{}_\lambda$ as n linearly independent covariant vectors in X_{m+n}, and we choose m covariant vectors $B^a{}_\lambda$, which, together with $C^h{}_\lambda$, form a base for covariant vectors in the whole fibred space X_{m+n}. This will determine a dual base of $m+n$ contravariant vectors which we denote by $(B_a{}^\kappa, C_i{}^\kappa)$. Between these two bases there exist well known relations

$$\begin{gathered} B_b{}^\lambda B^a{}_\lambda = \delta^a_b, \quad B_b{}^\lambda C^h{}_\lambda = 0, \quad C_i{}^\lambda B^a{}_\lambda = 0, \quad C_i{}^\lambda C^h{}_\lambda = \delta^h_i, \\ B_a{}^\kappa B^a{}_\mu + C_i{}^\kappa C^i{}_\mu = \delta^\kappa_\mu. \end{gathered} \tag{1.7}$$

If we define

$$B^\kappa_\mu = B_a{}^\kappa B^a{}_\mu, \qquad C^\kappa_\mu = C_i{}^\kappa C^i{}_\mu, \tag{1.8}$$

we have two tensors defined in the whole space and which are called 'projection tensors' (See Schouten [21], Walker [24], Yano [31]).

The vectors of the base $(B_a{}^\kappa, C_i{}^\kappa)$ are so chosen that $B_a{}^\kappa$ are tangent to the fibre F_x. In view of current terminology in fibre bundle theory, we shall refer to a vector in the tangent space to F_x as 'vertical' and to a vector in the non-integrable 'distribution' spanned by the $C_i{}^\kappa$ as 'horizontal'.

Defining

$$\partial_\lambda = \partial/\partial\xi^\lambda, \quad X_a = B_a{}^\lambda \partial_\lambda, \quad X_i = C_i{}^\lambda \partial_\lambda, \tag{1.9}$$

we can consider the effect of interchanging the order of these operators. If we put

$$\Omega_{cb}{}^a = (X_c B_b{}^\lambda - X_b B_c{}^\lambda) B^a{}_\lambda = -B_c{}^\mu B_b{}^\lambda (\partial_\mu B^a{}_\lambda - \partial_\lambda B^a{}_\mu), \tag{1.10}$$

$$\Omega_{cb}{}^h = (X_c B_b{}^\lambda - X_b B_c{}^\lambda) C^h{}_\lambda = -B_c{}^\mu B_b{}^\lambda (\partial_\mu C^h{}_\lambda - \partial_\lambda C^h{}_\mu), \tag{1.11}$$

we have, for any funtion $f(\xi^\kappa)$

$$(X_c X_b - X_b X_c) f = \Omega_{cb}{}^a X_a f + \Omega_{cb}{}^h X_h f \tag{1.12}$$

with corresponding results for the interchanging of operators corresponding to indices c and i, and j and i (See Yano and Davies [32]).

In our particular case we have, in view of the definition (1.6) of $C^h{}_\lambda$

$$\Omega_{cb}{}^h = 0, \qquad \Omega_{ci}{}^h = 0, \qquad \Omega_{ji}{}^h = 0, \tag{1.13}$$

so that

$$\begin{cases} (X_c X_b - X_b X_c)f = \Omega_{cb}{}^a X_a f, \\ (X_c X_i - X_i X_c)f = \Omega_{ci}{}^a X_a f, \\ (X_j X_i - X_i X_j)f = \Omega_{ji}{}^a X_a f, \end{cases} \tag{1.14}$$

where

$$\begin{cases} \Omega_{ci}{}^a = (X_c C_i{}^\lambda - X_i B_c{}^\lambda) B^a{}_\lambda, \\ \Omega_{ji}{}^a = (X_j C_i{}^\lambda - X_i C_j{}^\lambda) B^a{}_\lambda. \end{cases} \tag{1.15}$$

The first of the equations (1.13) shows that the system of partial differential equations

$$X_a f = 0 \tag{1.16}$$

is completely integrable, with the n independent solutions $x^h = f^h(\xi)$ considered in (1.5).

Any function of the x^h only is therefore a solution of the system (1.16) and any solution of (1.16) is expressible in terms of x^h only (Goursat [13]).

We shall need the Lie derivatives (See Yano [30]) of the base vectors with respect to the vectors $B_c{}^\kappa$. From the definition of the Lie derivative we have immediately

$$\begin{cases} \underset{B_c}{\pounds}\, B_b{}^\kappa = B_c{}^\mu \partial_\mu B_b{}^\kappa - B_b{}^\mu \partial_\mu B_c{}^\kappa = \Omega_{cb}{}^a B_a{}^\kappa, \\ \underset{B_c}{\pounds}\, C_i{}^\kappa = B_c{}^\mu \partial_\mu C_i{}^\kappa - C_i{}^\mu \partial_\mu B_c{}^\kappa = \Omega_{ci}{}^a B_a{}^\kappa \end{cases} \tag{1.17}$$

and using (1.7)

$$\begin{cases} \underset{B_c}{\pounds}\, B^a{}_\lambda = -\Omega_{cb}{}^a B^b{}_\lambda - \Omega_{ci}{}^a C^i{}_\lambda, \\ \underset{B_c}{\pounds}\, C^h{}_\lambda = 0. \end{cases} \tag{1.18}$$

We note therefore that the $C_i{}^\kappa$ forming a base for the horizontal distribution will have its Lie derivative zero for any vector of the base $B_a{}^\kappa$ of the fibre provided that $\Omega_{ci}{}^a = 0$.

Hence

The horizontal distribution is invariant for any displacement along the fibre if $\Omega_{ci}{}^a = 0$.

§2.

In this paragraph, we examine the conditions under which a tensor field in the fibre space X_{m+n} induces a tensor field in the base space X_n.

If $f(\xi)$ is a scalar defined in X_{m+n}, we know that it induces a scalar in X_n if and only if

$$X_c f = 0.$$

For our purpose it is convenient to write this in the form

$$\underset{B_c}{\pounds}\, f = 0. \tag{2.1}$$

Passing to the case of a contravariant vector field $v^\kappa(\xi)$ we know that it has a component in the horizontal distribution given by $v^h = C^h{}_\lambda v^\lambda$.

Under a transformation of ξ in X_{m+n}, the v^h undergoes the transformation

$$v^{h'} = \frac{\partial x^{h'}}{\partial x^h} v^h$$

and consequently v^κ induces a contravariant vector field in the base space X_n if and only if

$$X_c v^h = \underset{B_C}{\pounds}\, v^h = 0.$$

Since the Lie derivative of $C^h{}_\lambda$ vanishes, we can say that v^κ induces a contravariant vector field in X_n if and only if

$$\underset{B_C}{\pounds}\, v^h = C^h{}_\lambda(\underset{B_C}{\pounds}\, v^\lambda) = 0. \tag{2.2}$$

If we assume $\Omega_{ci}{}^a = 0$ in which case the Lie derivative of $C_i{}^\kappa$ also vanishes, the argument just used can apply to a tensor of any order. The order of the operation of Lie derivative and projection on the horizontal distribution can be interchanged and hence we can state:— *The necessary and sufficient condition that a tensor field such as* $T_\lambda{}^\kappa(\xi)$ *in the fibred space* X_{m+n} *induces tensor field* $T_i{}^h$ *in the base space* X_n, *is that*

$$\underset{B_C}{\pounds}\, T_i{}^h = C_i{}^\lambda C^h{}_\kappa(\underset{B_C}{\pounds}\, T_\lambda{}^\kappa) = 0. \tag{2.3}$$

In particular consider the exterior differential form

$$w = w_{\lambda_1 \cdots \lambda_p} d\xi^{\lambda_1} \wedge d\xi^{\lambda_2} \wedge \cdots \wedge d\xi^{\lambda_p}. \tag{2.4}$$

By writing down the differentials $d\xi^\kappa$ in terms of their components $(dx)^a = B^a{}_\lambda d\xi^\lambda$ and $dx^h = C^h{}_\lambda d\xi^\lambda$ where a bracket round the dx indicates that it is not an exact differential, we can write (2.4) in a form

$$w = \text{terms containing } (dx)^a + w_{i_1 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

where

$$w_{i_1 \cdots i_p} = C_{i_1}{}^{\lambda_1} \cdots C_{i_p}{}^{\lambda_p} w_{\lambda_1 \cdots \lambda_p}.$$

We may therefore state that the form w in X_{m+n} induces a form w in the base space X_n if the coefficients $w_{i_1 i_2 \cdots i_p}$ are functions of x^h only which we express as

$$\underset{B_C}{\pounds}\, w_{i_1 \cdots i_p} = C_{i_1}{}^{\lambda_1} \cdots C_{i_p}{}^{\lambda_p}(\underset{B_C}{\pounds}\, w_{\lambda_1 \cdots \lambda_p}) = 0. \tag{2.5}$$

§3.

If an affine connection with coefficients $\Pi^\kappa_{\mu\lambda}(\xi)$ is defined in the fibred space X_{m+n} the projection tensors B and C enable us to define connections in the fibre F and in the horizontal distribution X^n_{m+n} respectively. The well known method (Yano and Davies [32]) can be used to define four sets of connection parameters as well as four sets of Euler-Schouten curvature tensors relating to F and X^n_{m+n} as follows.—

The covariant differential of

$$v^\kappa = B_a{}^\kappa v^a + C_h{}^\kappa v^h$$

will be

$$\begin{aligned}\delta v^\kappa = {} & B_a{}^\kappa[(dx)^c(X_c v^a + \Gamma^a_{cb}v^b + \Gamma^a_{ci}v^i) + dx^j(X_j v^a + \Gamma^a_{jb}v^b + \Gamma^a_{ji}v^i)] \\ & + C_h{}^\kappa[(dx)^c(X_c v^h + \Gamma^h_{cb}v^b + \Gamma^h_{ci}v^i) + dx^j(X_j v^h + \Gamma^h_{jb}v^b + \Gamma^h_{ji}v^i)],\end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \Gamma^a_{cb} &= -B_c{}^\mu B_b{}^\lambda \nabla_\mu B^a{}_\lambda, & \Gamma^a_{jb} &= -C_j{}^\mu B_b{}^\lambda \nabla_\mu B^a{}_\lambda, \\ \Gamma^h_{ci} &= -B_c{}^\mu C_i{}^\lambda \nabla_\mu C^h{}_\lambda, & \Gamma^h_{ji} &= -C_j{}^\mu C_i{}^\lambda \nabla_\mu C^h{}_\lambda \end{aligned} \tag{3.2}$$

are connection paramters, and

$$\begin{aligned} \Gamma^a_{ci} &= -B_c{}^\mu C_i{}^\lambda \nabla_\mu B^a{}_\lambda, & \Gamma^a_{ji} &= -C_j{}^\mu C_i{}^\lambda \nabla_\mu B^a{}_\lambda, \\ \Gamma^h_{cb} &= -B_c{}^\mu B_b{}^\lambda \nabla_\mu C^h{}_\lambda, & \Gamma^h_{jc} &= -C_j{}^\mu B_c{}^\lambda \nabla_\mu C^h{}_\lambda \end{aligned} \tag{3.3}$$

are Euler-Schouten curvature tensors.

Assuming that the contravariant vector field $v^\kappa(\xi)$ in X_{m+n} induces a contravariant vector field $v^h(x)$ in the base space X_n, we wish to examine the condition in order that the connection defined in the fibred space can define a connection in the base space X_n leading to a set of connection parameters depending only upon the variables x^h.

We first assume that the contravariant vector field v^κ is in the horizontal distribution so that $v^a = 0$. We further assume that the displacement $d\xi^\kappa$ is also in the horizontal distribution so that $(dx)^a = 0$. In that case, we immediately deduce from (3.1) that

$$C^h{}_\lambda \delta v^\lambda = (X_j v^h + \Gamma^h_{ji}v^i)dx^j \tag{3.4}$$

where Γ^h_{ji} are defined in (3.2) in terms of the connection parameters of X_{m+n} and of the projection tensors.

Under a transformation of the coordinates ξ in X_{m+n} the $\Gamma^h_{ji}(\xi)$ undergo the transformation

$$\Gamma^{h'}_{j'i'}(\xi') = \frac{\partial x^{h'}}{\partial x^h}\left(\frac{\partial^2 x^h}{\partial x^{j'}\partial x^{i'}} + \Gamma^h_{ji}(\xi)\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^i}{\partial x^{i'}}\right) \tag{3.5}$$

and consequently $\Gamma^h_{ji}(\xi)$ induces an affine connection in the base space if and only if

$$X_c \Gamma^h_{ji} = \underset{B_c}{\pounds}\, \Gamma^h_{ji} = 0. \tag{3.6}$$

But we have

$$\underset{B_c}{\pounds}\, \Gamma^h_{ji} = -\underset{B_c}{\pounds}\,[C_j{}^\mu C_i{}^\lambda(\nabla_\mu C^h{}_\lambda)] = -C_j{}^\mu C_i{}^\lambda(\underset{B_c}{\pounds}\, \nabla_\mu C^h{}_\lambda)$$

by virtue of (1.18). On the other hand, we have

$$\underset{B_c}{\pounds}\, \nabla_\mu w_\lambda - \nabla_\mu \underset{B_c}{\pounds}\, w_\lambda = -(\underset{B_c}{\pounds}\, \Pi^\kappa_{\mu\lambda})w_\kappa \tag{3.7}$$

for a general covariant vector w_λ (Yano [30]). Thus

$$\underset{B_C}{\pounds}\,\Gamma^h_{ji} = -C_j{}^\mu C_i{}^\lambda[\nabla_\mu \underset{B_C}{\pounds}\, C^h{}_\lambda - (\underset{B_C}{\pounds}\, \Pi^\kappa_{\mu\lambda})C^h{}_\kappa]$$

from which

$$\underset{B_C}{\pounds}\,\Gamma^h_{ji} = C_j{}^\mu C_i{}^\lambda C^h{}_\kappa(\underset{B_C}{\pounds}\, \Pi^\kappa_{\mu\lambda}) \tag{3.8}$$

by virtue of (1.18). Thus

An affine connection $\Pi^\kappa_{\mu\lambda}(\xi)$ *of the fibred space for a vector and displacement both in the horizontal distribution induces an affine connection in the base space if and only if*

$$\underset{B_C}{\pounds}\,\Gamma^h_{ji} = C_j{}^\mu C_i{}^\kappa C^h{}_\kappa(\underset{B_C}{\pounds}\, \Pi^\kappa_{\mu\lambda}) = 0. \tag{3.9}$$

Similarly we get from (3.3)

An affine connection $\Pi^\kappa_{\mu\lambda}(\xi)$ *of the fibred space for a general vector* v^κ *and a displacement in the horizontal distribution induces an affine connection in the base space if and only if*

$$\Gamma^h_{jb} = 0, \qquad \underset{B_C}{\pounds}\,\Gamma^h_{ji} = C_j{}^\mu C_i{}^\lambda C^h{}_\kappa(\underset{B_C}{\pounds}\, \Pi^\kappa_{\mu\lambda}) = 0. \tag{3.10}$$

An affine connction $\Pi^\kappa_{\mu\lambda}(\xi)$ *of the fibred space for a vector* v^κ *in the distribution and a general displacement induces an affine connction in the base space if and only if*

$$\Gamma^h_{ci} = 0, \qquad \underset{B_C}{\pounds}\,\Gamma^h_{ji} = C_j{}^\mu C_i{}^\lambda C^h{}_\kappa(\underset{B_C}{\pounds}\, \Pi^\kappa_{\mu\lambda}) = 0. \tag{3.11}$$

An affine connction $\Pi^\kappa_{\mu\lambda}(\xi)$ *of the fibred space for a general vector* v^κ *and a general displacement induces an affine connction in the base space if and only if*

$$\Gamma^h_{jb} = 0, \quad \Gamma^h_{ci} = 0, \quad \underset{B_C}{\pounds}\,\Gamma^h_{ji} = C_j{}^\mu C_i{}^\lambda C^h{}_\kappa(\underset{B_C}{\pounds}\, \Pi^\kappa_{\mu\lambda}) = 0. \tag{3.12}$$

§ 4.

We next consider that there is given in X_{m+n} a system of paths defined by the system of equations

$$\frac{d^2\xi^\kappa}{dt^2} + \Gamma^\kappa_{\mu\lambda}(\xi)\frac{d\xi^\mu}{dt}\frac{d\xi^\lambda}{dt} = 0 \tag{4.1}$$

in which the coefficients $\Gamma^\kappa_{\mu\lambda}(\xi)$ are symmetrical and t is an affine parameter on the path.

The solutions $\xi^\kappa = \xi^\kappa(t)$ of (4.1) induce curves

$$x^h = f^h(\xi(t)) = x^h(t) \tag{4.2}$$

in the base space X_n, and the question arises whether the curves induced in X_n are paths in X_n.

From (4.2), on using the fact that

$$\frac{dx^h}{dt} = C^h{}_\lambda \frac{d\xi^\lambda}{dt}$$

and differentiating, we have

$$\frac{d^2x^h}{dt^2}=(\nabla_\mu C^h{}_\lambda)\frac{d\xi^\mu}{dt}\frac{d\xi^\lambda}{dt}. \tag{4.3}$$

Expressing $d\xi^\lambda/dt$ in terms of its two components

$$\frac{d\xi^\lambda}{dt}=B_a{}^\lambda\frac{(dx)^a}{dt}+C_h{}^\lambda\frac{dx^h}{dt}$$

(4.3) gives us

$$\frac{d^2x^h}{dt^2}+\Gamma^h_{cb}\frac{(dx)^c}{dt}\frac{(dx)^b}{dt}+\Gamma^h_{ci}\frac{(dx)^c}{dt}\frac{(dx)^i}{dt}+\Gamma^h_{jb}\frac{dx^j}{dt}\frac{(dx)^b}{dt}+\Gamma^h_{ji}\frac{dx^j}{dt}\frac{dx^i}{dt}=0. \tag{4.4}$$

Since we are concerned with finding the conditions in order that we may obtain an induced system of paths in X_n, we need to take account of the fact that the parameter t will not in general be a privileged parameter on the paths in X_n, so that we must write down the conditions under which (4.4) takes on the form

$$\frac{d^2x^h}{dt^2}+{}'\Gamma^h_{ji}(x)\frac{dx^j}{dt}\frac{dx^i}{dt}=\phi\frac{dx^h}{dt}. \tag{4.5}$$

We must therefore express the condition in order that (4.4) may have the form (4.5).

The term $\Gamma^h_{cb}(dx)^c/dt\cdot(dx)^b/dt$ could *not* contribute to a term of the form $\phi\, dx^h/dt$ since the $(dx)^a/dt$ are arbitrary, we deduce that $\Gamma^h_{cb}=0$ identically, which means that the fibres must be totally geodesic.

Considering further the terms

$$\Gamma^h_{ci}\frac{(dx)^c}{dt}\frac{dx^i}{dt}+\Gamma^h_{jb}\frac{dx^j}{dt}\frac{(dx)^b}{dt}$$

they can provide a term $\phi\, dx^h/dt$ provided ϕ is of the form $2\phi_c(dx)^c/dt$, so that we must have

$$\begin{aligned}\Gamma^h_{ci}&=\Gamma^h_{ic}\\&=\varphi_c\delta^h_i\end{aligned}\qquad\text{(by means of the symmetry of }\Gamma^\kappa_{\mu\lambda}).$$

Finally since the ${}'\Gamma^h_{ji}$ and the Γ^h_{ji} must be coefficients relating to the same paths, they must be related by a relation

$${}'\Gamma^h_{ji}(x)=\Gamma^h_{ji}(\xi)+\delta^h_j p_i+\delta^h_i p_j$$

in which we assume that ${}'\Gamma$ depends only on the x^h while the functions occurring on the right will depend on ξ^κ.

The fact that the ${}'\Gamma^h_{ji}$ are to be functions of x^h only can therefore be expressed in the form

$$\underset{B_c}{\pounds}\,{}'\Gamma^h_{ji}=0$$

or equivalently

$$\underset{B_c}{\pounds}\,\Gamma^h_{ji}=-\delta^h_j\underset{B_c}{\pounds}\,p_i-\delta^h_i\underset{B_c}{\pounds}\,p_j.$$

Hence, *the system of paths in the fibred space* X_{m+n} *induces a system of paths in the base space if and only if*

$$\Gamma^h_{cb} = 0, \qquad \Gamma^h_{ci} = \varphi_c \delta^h_i, \qquad \Gamma^h_{jb} = \varphi_b \delta^h, \tag{4.6}$$
$$C_j{}^\mu C_\iota{}^\lambda C^h{}_\kappa (\underset{B_c}{\pounds}\, \Gamma^\kappa_{\mu\lambda}) = \delta^h_j p_{\iota c} + \delta^h_i p_{jc}.$$

It will be convenient for future purposes to express these conditions under a different form.

By expressing the $\nabla_\mu C^h{}_\lambda$ in terms of its components obtained by contracting with the B and C tensors, using the conditions expressed in (4.6) and writing $\varphi_\lambda = B^c{}_\lambda \varphi_c$, we have

$$\nabla_\mu C^h{}_\lambda = \varphi_\mu C^h{}_\lambda + \varphi_\lambda C^h{}_\mu + C^j{}_\mu C^i{}_\lambda \Gamma^h_{ji}. \tag{4.7}$$

The covariant derivative of the equation $B_a{}^\lambda C^h{}_\lambda = 0$ will lead in a similar way to an expression for $\nabla_\mu B_a{}^\kappa$ in the form

$$\nabla_\mu B_a{}^\kappa = \varphi^c_{\mu a} B_c{}^\kappa - \delta^\kappa_\mu \varphi_a \tag{4.8}$$

which is an equivalent form of the condition of the first line of (4.6), for if $\nabla_\mu B_a{}^\kappa$ has the form (4.8), we have

$$B_b{}^\lambda \nabla_\mu C^h{}_\lambda = \varphi_b C^h{}_\mu$$

from which we can conclude $\Gamma^h_{cb} = 0$ and $\Gamma^h_{jc} = \Gamma^h_{cj} = \varphi_c \delta^h_j$.

From (4.8) by further covariant derivation and using relations already obtained, it is possible to express the Lie derivative of the connection coefficients in the form

$$\underset{B_c}{\pounds}\, \Gamma^\kappa_{\mu\lambda} = \delta^\kappa_\mu \varphi_{\lambda c} + \delta^\kappa_\lambda \varphi_{\mu c} + B_a{}^\kappa \varphi^a_{\mu\lambda}. \tag{4.9}$$

Conversely if the covariant derivative of $B_a{}^\kappa$ has the form (4.8) and the Lie derivative of $\Gamma^\kappa_{\mu\lambda}$ has the form (4.9), the equation (4.6) is also satisfied. Thus (4.8) and (4.9) are necessary and sufficient condition in order that a system of paths in X_{m+n} induce a system of paths in the base space X_n.

Some interesting interpretations can be given to these conditions in the case where the fibre is one-dimensional. Let B^κ denote the unique vector $B_a{}^\kappa$ so that our frame now becomes

$$(B^\kappa, C_i{}^\kappa) \quad \text{and} \quad (B_\lambda, C^h{}_\lambda).$$

The conditions (4.8) and (4.9) for the induction of paths in the base space then become

$$\nabla_\mu B^\kappa = \alpha \delta^\kappa_\mu + \varphi_\mu B^\kappa, \qquad \underset{B}{\pounds}\, \Gamma^\kappa_{\mu\lambda} = \delta^\kappa_\mu p_\lambda + \delta^\kappa_\lambda p_\mu + B^\kappa p_{\mu\lambda} \tag{4.10}$$

where α is a scalar, φ_μ and p_μ vectors and $p_{\mu\lambda}$ a tensor.

We can give a geometrical interpretation to the first of equations (4.10) as follows.

The point $\xi^\kappa - \alpha^{-1} B^\kappa$ lies in the tangent space to X_{m+n} at the point ξ^κ. Its absolute differential is by definition

$$\delta(\xi^\kappa - \alpha^{-1} B^\kappa) = d\xi^\kappa + \alpha^{-2} d\alpha B^\kappa - \alpha^{-1} \delta B^\kappa$$

which, on using (4.10) becomes

$$B^\kappa(\alpha^{-2}d\alpha + \alpha^{-1}\varphi_\mu d\xi^\mu)$$

so that the vector field B^κ is tangent to the locus of the point $\xi^\kappa - \alpha^{-1}B^\kappa$. Therefore the field B^κ is then said to be 'torse-forming' (Yano [27]).

We may also interpret the second equation of (4.10). Consider a curve in the fibred space X_{m+n} whose osculating plane contains the direction B^κ. It is called a subpath with respect to the vector field B^κ (Yano [28]). Subpaths are given by differntial equations of the form

$$\frac{d^2\xi^\kappa}{dt^2} + \Gamma^\kappa_{\mu\lambda}(\xi)\frac{d\xi^\mu}{dt}\frac{d\xi^\lambda}{dt} = \alpha\frac{d\xi^\kappa}{dt} + \beta B^\kappa. \tag{4.11}$$

If we consider an infinitesimal transformation $'\xi^\kappa = \xi^\kappa + \varepsilon B^\kappa$ and express the fact that this transformation transforms subpths into subpaths, we obtain the second of equations (4.10).

§5.

An important special case of equations (4.10) has already been the object of study by Schouten and his collaborators (Schouten and Haantjes [22]). It is the case in which they reduce to

$$\nabla_\mu B^\kappa = \alpha\delta^\kappa_\mu + \beta_\mu B^\kappa, \qquad \underset{B}{\pounds}\, \Pi^\kappa_{\mu\lambda} = 0. \tag{5.1}$$

In this case the infinitesimal transformation $'\xi^\kappa = \xi^\kappa + \varepsilon B^\kappa$ becomes an infinitesimal collineation.

In this paragraph we shall consider some applications of (5.1). We consider a vector field v^κ in X_{m+n} whose component in the direction B^κ we denote by v^0, so that

$$v^\kappa = B^\kappa v^0 + (C_h{}^\kappa v^h) \tag{5.2}$$

and we suppose thrt v^κ induces a scalar v^0 and a contravariant vector field v^h in X_n, so that

$$\underset{B}{\pounds}\, v^0 = \underset{B}{\pounds}\, v^h = 0 \quad \text{and} \quad \text{consequently } \underset{B}{\pounds}\, v^\kappa = 0.$$

By writing down the special forms which the various connection parameters and Euler-Schouten curvature tensors introduced in §2 take when $m=1$, we have on using (5.1)

$$\Gamma^0_{00} = 1, \quad \Gamma^0_{j0} = 0, \quad \Gamma^h_{00} = 0, \quad \Gamma^h_{j0} = \delta^h_j. \tag{5.2}$$

The symmetry of the Γ's in the lower indices enables us to conclude further that

$$\Gamma^0_{0i} = 0, \qquad \Gamma^h_{0i} = \delta^h_i. \tag{5.3}$$

Moreover from the second of (5.1) and the results of §3, we have

$$\underset{B}{\pounds}\, \Gamma^0_{ji} = 0, \qquad \underset{B}{\pounds}\, \Gamma^h_{ji} = 0, \tag{5.4}$$

so that Γ^0_{ji} and Γ^h_{ji} are functions of x^h only and hence Γ^0_{ji} is a tensor and Γ^h_{ji} are coefficients of a symmetric affine connection in the base space X_n.

Now suppose we choose

$$\bar{B}_\lambda = B_\lambda + p_\lambda$$

for which

$$\underset{B}{\pounds}\, p_\lambda = 0 \quad \text{and} \quad B^\lambda p_\lambda = 0. \tag{5.5}$$

The matrix $(B^\kappa, \bar{C}_i{}^\kappa)$ inverse to $(\bar{B}_\lambda, C^h{}_\lambda)$ will be given by

$$\bar{C}_i{}^\kappa = C_i{}^\kappa - p_i B^\kappa \qquad \text{with} \quad p_i = C_i{}^\lambda p_\lambda. \tag{5.6}$$

Denoting by a bar the functions relating to this new frame, we have

$$\begin{aligned} \bar{\Gamma}^0_{ji} &= \Gamma^0_{ji} - \nabla_j p_i - p_j p_i, \\ \bar{\Gamma}^h_{ji} &= \Gamma^h_{ji} - p_j \delta^h_i - p_i \delta^h_j \end{aligned} \tag{5.7}$$

which indicates that Γ^0_{ji} and Γ^h_{ji} may be interpreted as components of a projective connection (Yano and Takano [34]).

§ 6.

In this section we shall assume that the fibred space X_{m+n} is a Riemannian space with a metric tensor $G_{\mu\lambda}(\xi)$ defining the distance between two near points by

$$ds^2 = G_{\mu\lambda}(\xi) d\xi^\mu d\xi^\lambda. \tag{6.1}$$

We define the covariant vectors $C^h{}_\lambda$ as in (1.6) and take the vectors $B^a{}_\lambda$ to be orthogonal with respect to the metric defined in (6.1), so that

$$G^{\mu\lambda} B^a{}_\mu C^h{}_\lambda = 0. \tag{6.2}$$

Correspondingly the vectors of the dual matrix $(B_a{}^\kappa, C_i{}^\kappa)$ will satisfy the condition of orthogonality

$$G_{\mu\lambda} B_a{}^\mu C_i{}^\lambda = 0. \tag{6.3}$$

If we write down the $d\xi^\kappa$ in terms of its components in the fibre and in the horizontal distribution as

$$d\xi^\kappa = B_a{}^\kappa \omega^a + C_i{}^\kappa dx^i$$

where $\omega^a = B^a{}_\lambda d\xi^\lambda$ is not an exact differential, then ds^2 can be written as the sum $ds_1{}^2 + ds_2{}^2$ where

$$\text{(a)} \quad ds_1{}^2 = g_{ji}(\xi) dx^j dx^i, \qquad \text{(b)} \quad ds_2{}^2 = g_{cb}(\xi) \omega^c \omega^b. \tag{6.4}$$

We shall now examine under what condition the $g_{ji} = G_{\mu\lambda} C_j{}^\mu C_i{}^\lambda$ are functions of x only and therefore can be regarded as the components of a metric tensor which has been *induced* in the base space X_n. The condition for this is evidently $X_a g_{ji} = 0$, which, when written in terms of Lie derivation can be written

$$(\underset{B_a}{\pounds}\, G_{\mu\lambda}) C_j{}^{\mu} C_i{}^{\lambda} + G_{\mu\lambda}(\underset{B_a}{\pounds}\, C_j{}^{\mu}) C_i{}^{\lambda} + G_{\mu\lambda} C_j{}^{\mu}(\underset{B_a}{\pounds}\, C_i{}^{\lambda}) = 0.$$

But if we take account of the table (1.17) and of the equation (6.3) we immediately conclude that g_{ji} will depend upon x only provided

$$(\underset{B_a}{\pounds}\, G_{\mu\lambda}) C_j{}^{\mu} C_i{}^{\lambda} = 0. \tag{6.5}$$

On using covariant derivation and the van der Waerden-Bortolotti operator (Schouten [21], p. 254) with respect to the connection parameters appropriate to Riemannian geometry, we may easily modify (6.5) to the form

$$G_{\mu\lambda} C_{(j}{}^{\mu} D_{i)} B_a{}^{\lambda} = - G_{\mu\lambda} B_a{}^{\mu} D_{(j} C_{i)}{}^{\lambda} = g_{ab} \Gamma^{b}_{(ji)} = 0. \tag*{6.6)}$$

The condition has therefore been expressed in terms of the Euler-Schouten curvature tensor Γ^{a}_{ji} introduced in table (3.3).

A metric will therefore be induced in the base space if and only if the horizontal distribution is geodesic at every point (Schouten [21], 263).

Let us now consider whether a vector field $v^{\kappa}(\xi)$ which defines a motion in the fibred space can induce a motion in the base space. The vector $v^{\kappa}(\xi)$ is therefore assumed to be a Killing vector so that $\nabla_{(\mu} v_{\lambda)} = 0$. We further assume that a vector field $v^{h}(x)$ is induced in the base space in accordance with (2.2). The vector $v^{h}(x)$ will be a Killing vector in the base space provided

$$\nabla_{(j} v_{i)} = 0. \tag{6.7}$$

On using the fact that $v_i = C_i{}^{\lambda} v_{\lambda}$ and the definition of the operator D already used, we immediately obtain

$$\nabla_{(j} v_{i)} = \Gamma^{a}_{(ji)} v_a + C_j{}^{\mu} C_i{}^{\lambda} \nabla_{(\mu} v_{\lambda)} \tag{6.8}$$

so that (6.6) immediately ensures that a Killing vector will be induced from a Killing vector in the fibred space. We may therefore state that

A motion in the fibred space X_{m+n} will induce a motion in the base space X_n provided (a) *the Killing vector $v^{\kappa}(\xi)$ induces a vector field $v^{h}(x)$ and* (b) *the horizontal distribution is totally geodesic.*

§7.

In this section we use a special coordinate system in X_{m+n}. Recalling that the fibres are given by $x^{h} = f^{h}(\xi)$ and that in the intersection of two coordinate neighbourhoods (ξ^{κ}) and $(\xi^{\kappa'})$ there is induced a transformation of the x coordinates given by (1.4), we now proceed to introduce m functions of class C^{r}

$$y^{a} = \varphi^{a}(\xi) \tag{7.1}$$

such that the Jacobian matrix $(\partial_{\lambda} y^{a}, \partial_{\lambda} x^{h})$ is of maximum rank $m+n$. This will enable us to express ξ^{κ} as

$$\xi^{\kappa} = \xi^{\kappa}(y^{a}, x^{h}) \tag{7.2}$$

so that on taking a fixed set of values x_0^{h} for x^{h}, we obtain the parametric equations of the fibre F_{x_0} as a subspace of dimension m of the X_{m+n}. If we can take $B_a{}^\kappa$ to be $\partial_a\xi^\kappa$ and they can be taken as m independent contravariant vectors tangent to the fibre and hence vertical vectors in the sense used. We may also take $C_h{}^\kappa$ to be $\partial_h\xi^\kappa$, and in fact the relations (1.7) are all satisfied if we take

$$B_a{}^\kappa = \partial_a\xi^\kappa, \qquad C_h{}^\kappa = \partial_h\xi^\kappa, \qquad B^a{}_\lambda = \partial_\lambda y^a, \qquad C^h{}_\lambda = \partial_\lambda x^h. \tag{7.3}$$

The same relations will also be satisfied if, for $C_h{}^\kappa$ and $B^a{}_\lambda$ we take the slightly more general expressions

$$C_i{}^\kappa = \partial_i\xi^\kappa - B_a{}^\kappa\Gamma_i{}^a, \qquad B^a{}_\lambda = \partial_\lambda y^a + C^h{}_\lambda\Gamma_h{}^a \tag{7.4}$$

where $\Gamma_i{}^a$ are functions of y and x which are not determined for the moment. Let us determine them by demanding that the $C_i{}^\kappa$ given in (7.4) are orthogonal to $B_a{}^\kappa$ with respect to the metric $G_{\mu\lambda}$, so that equation (6.3) is satisfied. We can therefore decompose $d\xi^\kappa$ into components in accordance with either of the two sets (7.3) or (7.4) as

$$d\xi^\kappa = \partial_a\xi^\kappa dy^a + \partial_i\xi^\kappa dx^i \quad \text{or} \quad d\xi^\kappa = B_a{}^\kappa\omega^a + C_i{}^\kappa dx^i$$

where

$$\omega^a = B^a{}_\lambda d\xi^\lambda = dy^a + \Gamma_i{}^a dx^i. \tag{7.5}$$

Writing down the two corresponding expressions for $G_{\mu\lambda}d\xi^\mu d\xi^\lambda$ will give

$$\Gamma_i{}^a = g^{ab}G_{\mu\lambda}\partial_b\xi^\mu\partial_i\xi^\lambda. \tag{7.6}$$

The coordinates y and x introduced in this section are employed by Muto [18] under the name of favourable coordinates. The law of transformation appropriate to them is

$$y^{a'} = y^{a'}(y, x), \qquad x^{h'} = x^{h'}(x) \tag{7.7}$$

and for this transformation of coordinates, demanding that the equation $\omega^a = 0$ has invariant significance is equivalent to having the following law of transformation for the functions $\Gamma_i{}^a$

$$\Gamma_{i'}{}^{a'} = \frac{\partial y^{a'}}{\partial y^a}\left(\frac{\partial y^a}{\partial x^{i'}} + \Gamma_i{}^a\frac{\partial x^i}{\partial x^{i'}}\right). \tag{7.8}$$

In terms of favourable coordinates the table of base vectors is taken on the special form

$$\begin{aligned} B_a{}^\kappa &= (\delta_a^b,\ 0), & C_i{}^\kappa &= (-\Gamma_i{}^a,\ \delta_i^h), \\ B^a{}_\lambda &= (\delta_b^a,\ \Gamma_i{}^a), & C^h{}_\lambda &= (0,\ \delta_i^h). \end{aligned} \tag{7.9}$$

The equation $\omega^a = 0$ is interpreted by Muto as establishing a correspondence between points in the neighbouring fibres F_x and F_{x+dx}. Both the equation $\omega^a = 0$ and the transformation (7.7) appear also in Wagner ([25], p. 159) in a similar theory. A curve $y^a = y^a(t)$, $x^h = x^h(t)$ in X_{m+n} satisfying $dy^a/dt + \Gamma_i^a dx^i/dt = 0$ is called an allowed curve by Muto. It is a curve which is normal to the fibre at every point. In Wagner's terminology the equations $\omega^a = 0$ determine a linear connection in the 'compound' manifold X_{m+n} and this connection is of zero curvature if the exterior derivative of ω^a also vanishes. In our notation the vanishing of the exterior derivative of ω^a is expressed as

$$\Omega^a_{ji} = X_{[j}\Gamma^a_{i]} = 0 \qquad \text{with} \quad X_i = \partial_i - \Gamma^b_i\partial_b. \tag{7.10}$$

If Γ^a_i is linear in y, and expressible as $\Gamma^a_i = \Gamma^a_{ib}(x)y^b$ we obtain

$$\Omega^a_{ji} = y^b R_{ji,b}{}^a \tag{7.11}$$

with R representing the usual formation from the three index Γ's. More generally if $\Gamma_i(x, y) = y^b\partial_b\Gamma^a_i = y^b\Gamma^a_{ib}$ then the equation (7.11) still holds with the R representing the combination $\partial_j\Gamma^a_{ib} - \Gamma^e_j\partial_e\Gamma^a_{ib} + \Gamma^a_{je}\Gamma^e_{ib} - j/i$ where j/i represents the terms obtained from those written down by interchanging j and i. The integrability condition of the horizontal distribution therefore depends on the curvature of the connection defined by the correspondence established by 'allowed' or 'horizontal' curves. The close relationship of all this with the group of holonomy is developed by both Muto [18] and by Ishihara [15]. Its relation to the problem of the decomposability of a Riemannian space of dimension $m+n$ has been treated by Walker [23].

An interesting particular case of the transformation (7.7) is obtained by taking

$$y^{a'} = M^{a'}_a(x)y^a, \qquad x^{i'} = x^{i'}(x). \tag{7.12}$$

In terms of this transformation, defining $\Gamma^a_{ib} = \partial_b\Gamma^a_i$ and $\Gamma_{icb}^a = \partial_b\Gamma^a_{ic}$ we easily verify that the latter is a tensor whose vanishing would imply that Γ^a_{ib} are functions of x only, and hence that Γ^a_i are linear in y^a.

The correspondence between fibres F_x and F_{x+dx} will be an isometry provided the distance (in terms of the metric assumed given in X_{m+n}) between two points in F_x is equal to the distance between the corresponding points in F_{x+dx}. Since the fibres are holonomic subspaces of X_{m+n} which in this case may be assumed to be Riemannian, we may refer to studies which have been made on the subject ([18], p. 291) in which it is proved that if ds and $d\bar{s}$ are the distances between near points in F_x and F_{x+dx} respectively, the difference is expressed in the form

$$d\bar{s}^2 - ds^2 = (\underset{C_i}{\pounds}\, g_{ba})dy^b dy^a$$

with

$$\underset{C_i}{\pounds}\, g_{ba} = -2\Gamma^i_{ba}$$

so that the fibres will be isometric provided they are geodesic subspaces of the X_{m+n}.

If the equations $X_i f = 0$ are completely integrable, the horizontal distribution is holonomic, and $\Omega_{ji}{}^a = 0$. In that case we may take the functions $y^a = \varphi^a(\xi)$ of (7.1) to be the m independent solutions of $X_i f = 0$, and it will follow that the appropriate expressions to take for $C_h{}^\kappa$ and $B^a{}_\lambda$ are those given in (7.4) rather than (7.3). In other words if the horizontal distribution is integrable we can choose a coordinate system of the fibred space in such a way that $\Gamma^a_i = 0$.

§ 8. Finsler spaces.

Let us assume the existence of a vector field $B^{\kappa}(\xi)$ at every point of X_{m+n}, and let us assume further that it is tangent to the fibre at every point, so that we may also write it as

$$B^{\kappa}=B_{a}{}^{\kappa}y^{a}. \tag{8.1}$$

Let us put

$$G_{\mu\lambda}B^{\mu}B^{\lambda}=L^{2}=2F=g_{cb}y^{c}y^{b} \tag{8.2}$$

for the square of the length of the vector at any point, and let us impose the condition that

$$\nabla_{\lambda}B^{\kappa}=B_{a}{}^{\kappa}B^{a}{}_{\lambda}=B^{\kappa}_{\lambda} \tag{8.3}$$

then we have

$$B^{\lambda}\partial_{\lambda}F=B^{\lambda}\nabla_{\lambda}F=B^{\mu}B^{\lambda}\nabla_{\mu}B_{\lambda}=2F. \tag{8.4}$$

Defining $X_{c}=B_{c}{}^{\lambda}\partial_{\lambda}$, $X_{i}=C_{i}{}^{\lambda}\partial_{\lambda}$, $\partial_{\lambda}=\partial/\partial\xi^{\lambda}$, D_{c} and D_{i} as the corresponding derivatives of van der Waerden-Bortolotti, we can write the condition (8.4) in the form

$$y^{a}X_{a}F=2F=g_{cb}y^{c}y^{b} \tag{8.5}$$

and we deduce also

$$X_{a}F=D_{a}F=g_{ab}y^{b} \tag{8.6}$$

and

$$D_{b}D_{a}F=g_{ba}=X_{b}X_{a}F-\Gamma^{d}_{ba}X_{d}F. \tag{8.7}$$

By using the definition of F and (8.3) we also obtain

$$X_{j}F=D_{j}F=0. \tag{8.8}$$

If we write $D_{c}B^{\kappa}$ in two different ways, we obtain

$$B_{c}{}^{\kappa}=C_{j}{}^{\kappa}\Gamma^{j}_{ca}y^{a}+B_{a}{}^{\kappa}\nabla_{c}y^{a}$$

from which we deduce

$$\text{(a)}\quad \nabla_{c}y^{a}=\delta^{a}_{c}, \qquad \text{(b)}\quad \Gamma^{j}_{ca}y^{a}=0. \tag{8.9}$$

Similarly by writing $D_{i}B^{\lambda}$ in two different ways we obtain

$$C_{h}{}^{\kappa}\Gamma^{h}_{ja}y^{a}+B_{a}{}^{\kappa}\nabla_{j}y^{a}=0$$

so that

$$\text{(a)}\quad \nabla_{j}y^{a}=0, \qquad \text{(b)}\quad \Gamma^{h}_{ja}y^{a}=0. \tag{8.10}$$

Finally the metric tensors g_{ba} and g_{ji} in the fibre and in the horizontal distribution respectively, satisfy the equations

$$\begin{aligned}
&\text{(a)}\quad \nabla_{c}g_{ba}=X_{c}g_{ba}-\Gamma^{d}_{cb}g_{da}-\Gamma^{d}_{ca}g_{bd}=0,\\
&\text{(b)}\quad \nabla_{i}g_{ba}=X_{i}g_{ba}-\Gamma^{d}_{ib}g_{da}-\Gamma^{d}_{ia}g_{bd}=0,\\
&\text{(c)}\quad \nabla_{c}g_{ji}=X_{c}g_{ji}-\Gamma^{l}_{cj}g_{li}-\Gamma^{l}_{ci}g_{jl}=0,\\
&\text{(d)}\quad \nabla_{k}g_{ji}=X_{k}g_{ji}-\Gamma^{l}_{kj}g_{li}-\Gamma^{l}_{ki}g_{jl}=0.
\end{aligned} \tag{8.11}$$

The different connection parameters used in the equations (8.11) can be expressed in terms of the tensors of X_{m+n} together with the various 'object of anholonomity' by inserting the appropriate indices in the general formula

$$(8.12)\quad \Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\delta}(X_{\beta}g_{\gamma\delta}+X_{\gamma}g_{\beta\delta}-X_{\delta}g_{\beta\gamma})+\Omega_{\beta\gamma}{}^{\alpha}+\Omega^{\alpha}{}_{\beta\gamma}+\Omega^{\alpha}{}_{\gamma\beta}+S_{\beta\gamma}{}^{\alpha}+S^{\alpha}_{\cdot\beta\gamma}+S^{\alpha}{}_{\gamma\beta}$$

with $\Omega^{\alpha}{}_{\beta\gamma}=g^{\alpha\delta}g_{\gamma\varepsilon}\Omega_{\delta\beta}{}^{\varepsilon}$ and correspondingly for $S^{\alpha}{}_{\beta\gamma}$.

Let us now take the particular case in which the fibred space X_{m+n} is the tangent bundle of the n-dimensional base space X_n, in which the coordinates are x^h. The coordinates in the fibre will consequently be those of the tangent vectors to X_n, which we denote by $\dot{x}^h$. The law of transformation of coordinates in the fibred space X_{2n} (since m and n are now equal) will therefore be the extended point transformation

$$(8.13)\qquad x^{h'}=x^{h'}(x^h),\qquad \dot{x}^{h'}=\frac{\partial x^{h'}}{\partial x^h}\dot{x}^h.$$

At this stage we also take the vector field B^{κ} in the fibred space which is tangent to the fibre at every point to be the field of tanget vectors to X_n. This means that y^a is identified with $\dot{x}^h$. Our index convention must therefore be modified. We shall make the following convention. Let $a+n=h$, $b+n=i$, $c+n=j$, $d+n=k$, $e+n=l$. If we put $\hat{a}$ it will be understood that n is to be added to a, so that $\hat{a}=h$. Similarly $\hat{h}=h-n=a$, and so for other letters in the two groups. It is further understood that a and $\hat{a}$ occurring in a formula will imply summation from 1 to n for a and from $n+1$ to $2n$ for $\hat{a}$.

If therefore we take for ξ^{κ} the particular interpretations

$$\begin{aligned}\xi^{\kappa}=y^a=x^a=\dot{x}^h\quad&\text{for}\quad \kappa=1,2,\cdots,n,\\ =x^h\quad&\text{for}\quad \kappa=n+1,\cdots,2n\end{aligned}$$

the equation $x^h=f^h(\xi)$ of §1 will give a very special form

$$C^h{}_{\lambda}=0\quad\text{for}\quad\lambda=a,\qquad C^h{}_{\lambda}=\delta^h_k\quad\text{for}\quad\lambda=k.$$

For the vectors $B^a{}_{\lambda}$ we take any vector, which, together with $C^h{}_{\lambda}$, can form a base for covariant vectors. For this we take

$$B^a{}_{\lambda}=\delta^a_b\quad\text{for}\quad\lambda=b,\qquad B^a{}_{\lambda}=\Gamma^a_i\quad\text{for}\quad\lambda=i.$$

We may then choose for the dual matrix any matrix which satisfies the conditions of §1.

$$B_a{}^{\lambda}B^b{}_{\lambda}=\delta^b_a,\quad C_i{}^{\lambda}C^j{}_{\lambda}=\delta^j_i,\quad B_a{}^{\lambda}C^i{}_{\lambda}=0,\quad C_i{}^{\lambda}B^a{}_{\lambda}=0.$$

We have therefore the table

$$(8.14)\qquad \begin{aligned}B_a{}^{\kappa}&=(\delta^b_a,0),\qquad & C_i{}^{\kappa}&=(-\Gamma^a_i,\delta^h_i),\\ B^a{}_{\lambda}&=(\delta^a_b,\Gamma^a_i),\qquad & C^h{}_{\lambda}&=(0,\delta^h_i).\end{aligned}$$

If we write $\partial_a=\partial/\partial y^a$, $\partial_i=\partial/\partial x^i$, $X_i=\partial_i-\Gamma^a_i\partial_a$ the consequent expressions for the various 'objects of anholonomity' will all vanish except

(8.15) $$\Omega_{j\imath}{}^{a} = X_{[j}\Gamma^{a}_{\imath]}, \quad \Omega_{bi}{}^{a} = \frac{1}{2}\partial_b \Gamma^{a}_{\imath}.$$

Before writing down the special forms taken by the general expressions (8.12) for the various indices we make the following assumptions about the torsion tensor of X_{2n}

(a) Geodesics and autoparallels coincide, so that $S^{\alpha}{}_{\beta\gamma} + S^{\alpha}{}_{\gamma\beta} = 0$,

(b) $S_{bc}{}^{a} = S_{ji}{}^{h} = 0$.

With these assumptions we have from (8.12) the following

(8.16) $$\Gamma^{a}_{cb} = \frac{1}{2}g^{ad}(\partial_c g_{bd} + \partial_b g_{cd} - \partial_d g_{cb}),$$

(8.17) $$\Gamma^{a}_{jb} = \frac{1}{2}g^{ad}X_j g_{db} + \Omega_{jb}{}^{a} + \Omega^{a}{}_{jb} + S_{jb}{}^{a},$$

(8.18) $$\Gamma^{h}_{c\imath} = \frac{1}{2}g^{hk}\partial_c g_{k\imath} + \Omega^{h}{}_{\imath c} + S_{c\imath}{}^{h},$$

(8.19) $$\Gamma^{h}_{ji} = \frac{1}{2}g^{hl}(X_j g_{il} + X_i g_{jl} - X_l g_{ji}).$$

We further restrict the coefficients g_{cb} to be equal to g_{ji}, so that the four equations (8.11) reduce to the last two, and we shall need to have

(8.20) $$\Gamma^{a}_{cb} = \Gamma^{\hat{a}}_{ob} = \Gamma^{h}_{jb} \quad \text{and} \quad \Gamma_{jb} = \Gamma^{\hat{a}}_{j\hat{b}} = \Gamma^{h}_{ji}$$

so that there will be equality between the right hand sides of

(a) (8.16) and (8.18), (b) (8.17) and (8.19).

The equations (8.20) therefore serve to determine some of the mixed components of the torsion tensor as follows

(8.21) $$S_{bj}{}^{h} = g^{ad}\partial_{[c}g_{d]b} + g^{kh}g_{bd}X_{[j}\Gamma^{d}_{\imath]},$$

(8.22) $$S_{ic}{}^{a} = g^{hk}X_{[j}g_{k]i} - g^{ad}g_{e[c}\partial_{d]}\Gamma^{e}_{\imath}.$$

These will be written in a different form after making a further examination of the consequences of taking $y^{a} = \dot{x}^{h}$.

Referring to the table (8.14), we have

(8.23) $$X_a = \partial_a = \partial/\partial\dot{x}^{h} \quad \text{which we write} \quad \dot{\partial}_h,$$

(8.24) $$X_{\imath} = \partial_i - \Gamma^{a}_{i}\partial_a = \partial_i - \Gamma^{\hat{h}}_{i}\dot{\partial}_h$$

and consequently the equations (8.5) – (8.10) take on new forms

(8.5)′ $\dot{x}^{h}\dot{\partial}_h F = 2F$ which expresses the homogeneity of degree 2 of the F in $\dot{x}$,

(8.6)′ $\dot{\partial}_i F = g_{ij}\dot{x}^{j}$ which shows, in conjunction with (8.5)′ that g_{ji} is homogeneous of degree zero in $\dot{x}$,

(8.7)′ $$\dot{\partial}_j\dot{\partial}_i F - \Gamma^{k}_{ji}\dot{\partial}_k F = g_{ji},$$

(8.8)′ $$\partial_i F - \Gamma^{\hat{h}}_{i}\dot{\partial}_h F = 0,$$

(8.9)′ $$\Gamma^{h}_{cj}\dot{x}^{j} = 0,$$

(8.10)′ $\Gamma^{a}_{\imath} = \Gamma^{a}_{ib}y^{b} = \Gamma^{h}_{ij}\dot{x}^{j} = \Gamma^{h}_{i0}$ where a symbol 0 appearing as an index will indicate contraction with $\dot{x}$.

The common value of Γ^a_{bc} and Γ^h_{bj} can now be written, on taking account of of (8.23) and of $g_{cb} = g_{\hat{c}\hat{b}} = g_{ji}$ as

$$\Gamma^a_{cb} = \frac{1}{2} g^{hk}(\dot{\partial}_j g_{ik} + \dot{\partial}_i g_{jk} - \dot{\partial}_k g_{ji}) = C_{ji}{}^h = g^{hk} C_{jik} \tag{8.25}$$

where we write $C_{ji}{}^h$ in view of the indices acturally occurring. With this notation we may now rewrite some of the quations as

$$\dot{\partial}_j \dot{\partial}_i F - C_{jih}\dot{x}^h = g_{ji} \qquad \text{on using} \quad (8.6)', \tag{8.7)''}$$

$$\partial_i F - \Gamma^h_i \dot{\partial}_h F = 0 \qquad \text{on writing} \quad \Gamma^h_{i0} = \Gamma^h_i, \tag{8.8)''}$$

$$C_{ji}{}^h \dot{x}^i = 0 \qquad \text{or} \quad C_{jih}\dot{x}^i = 0. \tag{8.9)''}$$

At this stage we impose convention D of Cartan ([2], p. 10) which is equivalent to $C_{jih}\dot{x}^h = 0$, so that (8.7)$''$ gives

$$g_{ji} = \dot{\partial}_j \dot{\partial}_i F \qquad \text{and} \qquad 2C_{jih} = \dot{\partial}_j \dot{\partial}_i \dot{\partial}_h F. \tag{8.26}$$

We may also write the non-vanishing components of the torsion tensor of X_{2n} in the new forms

$$S_{bj}{}^h = g^{hk} g_{im} X_{[j} \Gamma^m_{l]}, \tag{8.21)'}$$

$$S_{ic}{}^a = g^{hk}(X_{[j} g_{k]i} - g_{l[j} \dot{\partial}_{k]} \Gamma^l_i). \tag{8.22)'}$$

We note that these components of the torsion tensor, as well as the connection coefficients given in (8.19) are expressed in terms of the function F and its derivatives except for the $\Gamma^l_i = \Gamma^l_{i0}$. But this can be expressed also in terms of F and its derivatives, for if we write (8.19) in full, we have, on writing γ^h_{ji} for the three-index symbols of Christoffel

$$\Gamma^h_{ji} = \gamma^h_{ji} - \Gamma^k_j C_{ik}{}^h - \Gamma^k_i C_{jk}{}^h + g^{hl} \Gamma^k_l C_{jik} \tag{8.27}$$

and hence

$$\Gamma^h_{j0} = \gamma^h_{j0} - \Gamma^k_{00} C_{jk}{}^h, \qquad \Gamma^h_{00} = \gamma^h_{00},$$

so that

$$\Gamma^h_j = \gamma^h_{j0} - \gamma^k_{00} C_{jk}{}^h. \tag{8.28}$$

Substitution of Γ^h_i from (8.28) in (8.27), (8.21)$'$ and (8.22)$'$ determines the connection Γ^h_{ji} in the horizontal distribution as well as the torsion in the fibred space X_{2n}.

The connection thus obtained is the connection given by E. Cartan for Finsler space. We note therefore that

The euclidean connection in Finsler space given by Cartan can be regarded as the connection induced in the horizontal distribution in a metric fibred space with torsion, where the torsion coefficients are given by (8.21)$'$ *and* (8.22)$'$.

The Γ^h_{ji} occurring in (8.27) are written $\overset{*}{\Gamma}{}^h_{ji}$ in Cartan's tract. We remark that if we attempt to obtain the connection in a horizontal distribution in a space without torsion, the two sets of components of the torsion of X_{2n}

given in (8.21)′ and (8.22)′ would have to vanish. Considering (8.21)′ we can easily verify that $X_{[j}\Gamma_{l]}{}^{m} = R_{jl,0}{}^{m}$ the vanishing of which is the condition for absolute parallelism of line elements (Cartan [2], p. 42). Further, from (8.15) it follows that $\Omega^{a}_{ji} = 0$ and the horizontal distribution becomes a holonomic subspace of X_{2n} which is now a space with a euclidean connection without torsion, i.e. a Riemannian space. So that a Finsler connection cannot be induced in a horizontal distribution from an X_{2n} without torsion.

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