# ON THE RANGE OF ANALYTIC FUNCTIONS WITH POSITIVE REAL PART

By Yûsaku Komatu

## 1. Introduction.

In previous papers [3; 4], considering a class of single-valued analytic functions regular and of positive real part in a circle or in an annulus, we have dealt with mean distortion theorems extending a classical theorem of Rogosinski [6]. The class in doubly-connected case has been restricted by imposing a normalization along one boundary component. However, for the present problem the most general class of similar nature obtained by rejecting this normalization will be also taken into consideration.

We shall consider namely *three classes* of functions which are defined as follows.

Let $\mathfrak{R}_0 = \{\Phi(z)\}$ be the class of analytic functions which are regular and of positive real part in the unit circle $|z|<1$ and normalized by $\Phi(0)=1$.

Let $\mathfrak{R}_q = \{\Phi(z)\}$ be the class of analytic functions which are single-valued, regular and of positive real part in the annulus $(0<)\ q<|z|<1$ and normalized by the conditions

$$\Re\Phi(z)=1 \quad \text{along } |z|=q \quad \text{and} \quad \int_{-\pi}^{\pi} \Im\Phi(qe^{i\theta})\,d\theta = 0.$$

Let $\hat{\mathfrak{R}}_q = \{\Phi(z)\}$ be the class of analytic functions which are single-valued, regular and of positive real part in the annulus $(0<)\ q<|z|<1$ and normalized by the condition

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(re^{i\theta})\,d\theta = 1 \qquad (q<r<1).$$

The class $\mathfrak{R}_0$ is, of course, regarded as the limit case of $\mathfrak{R}_q$ as $q\to 0$. On the other hand, the equation of the normalizing condition for $\hat{\mathfrak{R}}_q$ involves $r$ merely apparently, since its left-hand member has the value independent of $r$. In fact, it asserts only that the constant term in the Laurent expansion of $\Phi(z)\in\hat{\mathfrak{R}}_q$ is equal to unity. Consequently, $\mathfrak{R}_q$ is a subclass of $\hat{\mathfrak{R}}_q$. However, this subclass plays a distinguished role within the extended class. In fact, as shown in theorem 1 below, the class $\hat{\mathfrak{R}}_q$ is constructed in terms of this subclass in a simple manner.

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In general, let $\mathfrak{F}=\{\Phi(z)\}$ be a class of functions defined in a domain $D$. For any fixed $z\in D$, the value of $\Phi(z)\in\mathfrak{F}$ may be regarded as a functional of argument function $\Phi$ defined in $\mathfrak{F}$. From this point of view, we denote it especially by $W_z[\Phi]$, $z$ being a parameter point. The *range-set* $\Omega_z(\mathfrak{F})$ for the class $\mathfrak{F}$ is then defined as the set consisting of all possible values $W_z[\Phi]$ when $\Phi(z)$ extends over the class $\mathfrak{F}$; in symbol,

$$\Omega_z(\mathfrak{F})=\bigcup_{\Phi(z)\in\mathfrak{F}} W_z[\Phi].$$

The purpose of the present paper is to determine the range-sets precisely for the classes enumerated above.

## 2. Representation formulas.

The tools of attack in the present paper are, as in the previous ones [3; 4], also the integral representations of Stieltjes type valid for any function of the respective classes. The representations for $\mathfrak{R}_0$ and $\mathfrak{R}_q$ have been stated in the previous papers [3; 4] and will be re-formulated in the following two lemmas.

LEMMA 1. *It is necessary and sufficient for $\Phi(z)\in\mathfrak{R}_0$ that $\Phi(z)$ is representable by means of Herglotz integral*

$$\Phi(z)=\int_{-\pi}^{\pi}\frac{e^{i\varphi}+z}{e^{i\varphi}-z}d\rho(\varphi)$$

*with a real-valued function $\rho(\varphi)=\rho_\Phi(\varphi)$ defined for $-\pi<\varphi\leqq\pi$ which is increasing and has the total variation equal to unity.*

LEMMA 2. *It is necessary and sufficient for $\Phi(z)\in\mathfrak{R}_q$ that $\Phi(z)$ is representable by means of integral*

$$\Phi(z)=\int_{-\pi}^{\pi}\Phi^*(ze^{-i\varphi})\,d\rho(\varphi)$$

*with a function $\rho(\varphi)=\rho_\Phi(\varphi)$ of the same nature as in lemma 1. The function $\Phi^*(z)$ is defined by*

$$\Phi^*(z)=\frac{2}{i}\left(\zeta(i\lg z)-\frac{\eta_1}{\pi}i\lg z\right)$$

*where the elliptic zeta-function concerns (here and also throughout the paper) the Weierstrassian theory with primitive periods*

$$2\omega_1=2\pi \quad and \quad 2\omega_3=-2i\lg q.$$

The representation formula in lemma 2 for the class $\mathfrak{R}_q$ will be now generalized to that for $\hat{\mathfrak{R}}_q$ as stated in the lemma below. It will be sufficient to give an outline of the proof since its substantial part has been established in a previous paper [1].

LEMMA 3. *It is necessary and sufficient for $\Phi(z)\in\hat{\mathfrak{R}}_q$ that $\Phi(z)$ is representable by means of a formula*

$$\Phi(z)=\int_{-\pi}^{\pi}\Phi^*(ze^{-i\varphi})\,d\rho(\varphi)+\int_{-\pi}^{\pi}\Phi^*\Big(\frac{q}{ze^{-i\varphi}}\Big)\,d\tau(\varphi)-1$$

*where $\rho(\varphi)=\rho_\Phi(\varphi)$ and $\tau(\varphi)=\tau_\Phi(\varphi)$ are functions of the same nature as $\rho(\varphi)$ in lemma 1 and $\Phi^*(z)$ denotes the function introduced in lemma 2.*

*Proof.* Based on the characterizing properties of $\hat{\mathfrak{R}}_q$, the function defined by

$$\rho(r,\varphi)=\frac{1}{2\pi}\int_{-\pi}^{\varphi}\Re\Phi(re^{i\theta})\,d\theta\quad(-\pi<\varphi\leqq\pi,\ q<r<1)$$

is, qua function of $r$, uniformly bounded with respect to $\varphi$ and is, qua function of $\varphi$, of bounded variation for every value of $\varphi$. Moreover, for any $r$, it is an increasing function of $\varphi$ satisfying $\rho(r,-\pi)=0$ and $\rho(r,\pi)=1$. Hence, we can apply the result obtained in the previous paper [1] yielding

$$\Phi(z)=\int_{-\pi}^{\pi}\frac{2}{i}\zeta(i\lg z+\varphi)\,d\rho(\varphi)-\int_{-\pi}^{\pi}\frac{2}{i}\zeta_3(i\lg z+\varphi)\,d\tau(\varphi)+ic,$$

$c$ being a real constant. Here $\rho(\varphi)$ and $\tau(\varphi)$ are obtained as the limits of $\rho(r,\varphi)$ as $r$ tends to 1 and $q$ along respective suitable sequences, so that they are both increasing and have the total variation equal to unity. In order to determine the value of $c$, we observe the integral of $\Phi(z)/z$ along $|z|=r$ $(q<r<1)$ which, after divided by $2\pi i$, becomes

$$\begin{aligned}1&=\frac{1}{2\pi i}\int_{|z|=r}\frac{\Phi(z)}{z}\,dz=\frac{1}{2\pi i}\int_{|z|=r}\Phi(z)\,d\lg z\\&=\frac{1}{\pi i}\int_{-\pi}^{\pi}\lg\frac{\sigma(i\lg z+\pi+\varphi)}{\sigma(i\lg z-\pi+\varphi)}\,d\rho(\varphi)-\frac{1}{\pi i}\int_{-\pi}^{\pi}\lg\frac{\sigma_3(i\lg z+\pi+\varphi)}{\sigma_3(i\lg z-\pi+\varphi)}\,d\tau(\varphi)+ic\\&=\frac{1}{\pi i}\int_{-\pi}^{\pi}(\pi i+2\eta_1(i\lg z+\varphi))\,d\rho(\varphi)-\frac{1}{\pi i}\int_{-\pi}^{\pi}2\eta_1(i\lg z+\varphi)\,d\tau(\varphi)+ic\\&=1+\frac{2\eta_1}{\pi}\int_{-\pi}^{\pi}\varphi\,d\rho(\varphi)-\frac{2\eta_1}{\pi}\int_{-\pi}^{\pi}\varphi\,d\tau(\varphi)+ic.\end{aligned}$$

This determines the value of $c$. The same result will follow alternatively by observing directly the constant term in the Laurent expansion of $\Phi(z)$ which is obtained by means of the expansions of zeta-functions followed by termwise integration. Substituting the value of $c$ thus determined, we get

$$\begin{aligned}\Phi(z)=&\int_{-\pi}^{\pi}\frac{2}{i}\Big(\zeta(i\lg z+\varphi)-\frac{\eta_1}{\pi}(i\lg z+\varphi)\Big)d\rho(\varphi)\\&-\int_{-\pi}^{\pi}\frac{2}{i}\Big(\zeta_3(i\lg z+\varphi)-\frac{\eta_1}{\pi}(i\lg z+\varphi)\Big)d\tau(\varphi).\end{aligned}$$

Finally, in order to replace the $\zeta_3$-function in the last integral by the $\zeta$-

function, we use the defining equation $\zeta_3(u)=\zeta(u-i\lg q)-\eta_3$ together with the Legendre relation $\eta_1\cdot(-i\lg q)-\eta_3\pi=\pi i/2$. Then, remembering that $\zeta(u)$ is an odd function, we readily get

$$-\int_{-\pi}^{\pi}\frac{2}{i}\Big(\zeta_3(i\lg z+\varphi)-\frac{\eta_1}{\pi}(i\lg z+\varphi)\Big)d\tau(\varphi)$$

$$=\int_{-\pi}^{\pi}\frac{2}{i}\Big(\zeta\Big(i\lg\frac{q}{z}-\varphi\Big)-\frac{\eta_1}{\pi}\Big(i\lg\frac{q}{z}-\varphi\Big)\Big)d\tau(\varphi)-1.$$

Thus we obtain the desired representation

$$\Phi(z)=R(z)+T(z)-1;$$

$$R(z)=\int_{-\pi}^{\pi}\Phi^*(ze^{-i\varphi})\,d\rho(\varphi),\qquad T(z)=\int_{-\pi}^{\pi}\Phi^*\Big(\frac{q}{ze^{-i\varphi}}\Big)d\tau(\varphi).$$

Conversely, consider a function $\Phi(z)$ of the last form constructed by means of any $\rho(\varphi)$ and $\tau(\varphi)$ of the assigned nature. Then, by lemma 2, the function $R(z)$ belongs to $\mathfrak{R}_q$. The expression of $T(z)$ can be transformed, by change of integration variable, into

$$T(z)=\int_{-\pi}^{\pi}\Phi^*\Big(\frac{q}{z}e^{-i\varphi}\Big)d(-\tau(-\varphi))$$

in which $-\tau(-\varphi)$ is of the same nature as $\tau(\varphi)$, so that the function $T(q/z)$ belongs also to $\mathfrak{R}_q$. Hence, in particular, the real part of $\Phi(z)$ is bounded below. Further, in view of the boundary behaviors of functions of $\mathfrak{R}_q$, we have

$$\varliminf_{z\to z_1}\Re R(z)\geqq 0,\qquad \lim_{z\to z_1}\Re T(z)=1,$$

$$\lim_{z\to z_q}\Re R(z)=1,\qquad \varliminf_{z\to z_q}\Re T(z)\geqq 0$$

as $z$ in $q<|z|<1$ tends to any point $z_1$ on $|z|=1$ or any point $z_q$ on $|z|=q$. Consequently, $\Re\Phi(z)$ remains positive throughout the annulus $q<|z|<1$. Since the constant terms of $R(z)$ as well as $T(z)$ in their Laurent expansions are both equal to unity, that of $\Phi(z)$ is also equal to unity. Hence $\Phi(z)$ belongs surely to $\hat{\mathfrak{R}}_q$.

It would be noted that the representation for $\hat{\mathfrak{R}}_q$ given in lemma 3 may be expressed more briefly by

$$\Phi(z)=\frac{2}{i}\int_{-\pi}^{\pi}\Big(\zeta(i\lg z+\varphi)-\frac{\eta_1}{\pi}\varphi\Big)d\rho(\varphi)-\frac{2}{i}\int_{-\pi}^{\pi}\Big(\zeta_3(i\lg z+\varphi)-\frac{\eta_1}{\pi}\varphi\Big)d\tau(\varphi).$$

However, we have intentionally retained a term involving $\lg z$ in every integral in order to make the integrand to be single-valued, while this term has naturally appeared in the representation for $\mathfrak{R}_q$ given in lemma 2. The representation for $\hat{\mathfrak{R}}_q$ reduces to that for $\mathfrak{R}_q$ when we put $\tau(\varphi)=(\varphi+\pi)/2\pi$ (or simply $\tau(\varphi)=\varphi/2\pi$). In fact, we then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi^*\left(\frac{q}{ze^{-i\varphi}}\right)d\varphi=1.$$

Now, from the argument employed in the second half of the proof of lemma 3, we can derive a consequence showing the structure of the class $\hat{\mathfrak{R}}_q$ in terms of the restricted class $\mathfrak{R}_q$.

THEOREM 1. *Any function* $\Phi(z)\in\hat{\mathfrak{R}}_q$ *is expressed in the form*

$$\Phi(z)=R(z)+T(z)-1;\qquad R(z)\in\mathfrak{R}_q,\qquad T(z)\in\mathfrak{R}_q'$$

*where* $\mathfrak{R}_q'$ *designates the class consisting of all functions* $T(z)$ *such that* $T(q/z)$ *belongs to* $\mathfrak{R}_q$, *and the decomposition is unique. Conversely, any function of this form belongs to* $\hat{\mathfrak{R}}_q$. *In other words, the class* $\{\Phi(z)-1\}_{\Phi(z)\in\hat{\mathfrak{R}}_q}$ *is the direct sum of two classes* $\{R(z)-1\}_{R(z)\in\mathfrak{R}_q}$ *and* $\{T(z)-1\}_{T(z)\in\mathfrak{R}_q'}$.

*Proof.* It is only necessary to prove the uniqueness of the decomposition since the remaining part of the theorem has been verified in the proof of lemma 3. For that purpose, let any decomposition of $\Phi(z)$ be

$$\Phi(z)=R_1(z)+T_1(z)-1;\qquad R_1(z)\in\mathfrak{R}_q,\qquad T_1(z)\in\mathfrak{R}_q'.$$

Equating two decompositions of $\Phi(z)$, we have

$$R_1(z)-R(z)=T(z)-T_1(z).$$

The left-hand member can be prolonged analytically beyond $|z|=q$ while the right-hand member can be prolonged analytically beyond $|z|=1$. Hence, every member remains regular throughout the closed annulus $q\leqq|z|\leqq1$. The boundary value of its real part vanishes everywhere. Consequently, it must reduce to a purely imaginary constant (or zero). But the constant term in its Laurent expansion also vanishes so that this constant is equal to zero.

A remark is stated on theorem 1. The decomposition of $\Phi(z)\in\hat{\mathfrak{R}}_q$ given in the theorem is not of such nature that every component is of positive real part. By adjusting the kernels in the representation of lemma 3 by means of harmonic measures, we would obtain a representation

$$\Phi(z)=\int_{-\pi}^{\pi}\left(\Phi^*(ze^{-i\varphi})-\frac{\lg(ze^{-i\varphi})}{\lg q}\right)d\rho(\varphi)$$

$$+\int_{-\pi}^{\pi}\left(\Phi^*\left(\frac{q}{ze^{-i\varphi}}\right)-\frac{\lg\dfrac{q}{ze^{-i\varphi}}}{\lg q}\right)d\tau(\varphi)$$

valid for $\Phi(z)\in\hat{\mathfrak{R}}_q$. Two integrals in the last expression are both of positive real part and moreover their real parts are single-valued throughout the annulus $q<|z|<1$. However, they are not single-valued and really have the purely imaginary periodicity moduli equal to $-2\pi i/\lg q$ and $2\pi i/\lg q$, respectively, which cancel each other.

### 3. Range for $\mathfrak{R}_0$.

We first observe the range-set $\Omega_z(\mathfrak{R}_0)$. Several properties of this set will be previously evident. For instance, since with any pair $\Phi_0(z)$ and $\Phi_1(z)$ from $\mathfrak{R}_0$ the function $(1-\lambda)\Phi_0(z)+\lambda\Phi_1(z)$ belongs to $\mathfrak{R}_0$ for any constant $\lambda$ with $0<\lambda<1$, the set is convex; with $\Phi(z)\in\mathfrak{R}_0$ the function $\Phi(\varepsilon z)$ belongs to $\mathfrak{R}_0$ for any constant $\varepsilon$ with $|\varepsilon|=1$ so that the set depends on $z$ through its absolute value alone, namely $\Omega_z(\mathfrak{R}_0)=\Omega_{|z|}(\mathfrak{R}_0)$; with $\Phi(z)\in\mathfrak{R}_0$ the function $1/\Phi(z)$ belongs to $\mathfrak{R}_0$ so that the set remains invariant under the transformation $w\,|\,1/w$; etc. But theorem 2 below for explicit determination of $\Omega_z(\mathfrak{R}_0)$ is really a well-known classical result. In fact, it is nothing but a transform of Schwarz lemma by means of a linear transformation from the unit circle onto the right half-plane. Nevertheless, we shall re-state the result and give a proof based on lemma 1. The proof given here will serve as a model in dealing with other classes by a similar method.

THEOREM 2. *The range-set $\Omega_z(\mathfrak{R}_0)$ with $|z|=r$ $(0\leqq r<1)$ laid on the $w$-plane is the closed circular disc defined by*

$$\left|\frac{w-1}{w+1}\right|\leqq r.$$

*Let $\omega$ be any boundary point of $\Omega_r(\mathfrak{R}_0)$ and $z_0$ any assigned point with $|z_0|=r<1$. Then the equation $\Phi(z_0)=\omega$ holds for $\Phi(z)\in\mathfrak{R}_0$ if and only if $\Phi(z)$ is the linear function given by*

$$\Phi(z)=\frac{(\omega+1)z_0+(\omega-1)z}{(\omega+1)z_0-(\omega-1)z}.$$

*Proof.* The representation for $\Phi(z)\in\mathfrak{R}_0$ given in lemma 1 implies by virtue of the properties of $\rho(\varphi)$ that the set $\Omega_z(\mathfrak{R}_0)$ is a subset of the closed circular disc with the circumference described by $w=(e^{i\varphi}+z)/(e^{i\varphi}-z)$ as $\varphi$ varies from $-\pi$ to $\pi$. In fact, any half-plane $\Re(e^{-i\psi}w)\leqq d$ containing this circumference surely contains the point $\Phi(z)$. This disc is expressed by $|(w-1)/(w+1)|\leqq|z|$. As noticed above, the range-set is convex. Hence, in order to show that this set coincides with the whole disc, it is sufficient to verify that the circumference of the disc belongs to the set. Now, for any $\omega$ with $|(\omega-1)/(\omega+1)|=r$ and any assigned $z_0$ with $|z_0|=r$, define the constant $\varepsilon$ by

$$\frac{\varepsilon+z_0}{\varepsilon-z_0}=\omega \quad \text{i. e.} \quad \varepsilon=\frac{\omega+1}{\omega-1}z_0.$$

Then we have $|\varepsilon|=1$. Again by virtue of the properties of $\rho(\varphi)$ associated to $\Phi(z)$, the relation

$$\frac{\varepsilon + z_0}{\varepsilon - z_0} = \omega = \Phi(z_0) = \int_{-\pi}^{\pi} \frac{e^{i\varphi} + z_0}{e^{i\varphi} - z_0} d\rho(\varphi)$$

holds if and only if $d\rho(\varphi)$ vanishes everywhere except at $\varphi = \arg \varepsilon$ where $\rho(\varphi)$ has the jump with the height equal to unity. The function corresponding to this $\rho(\varphi)$ is given by

$$\Phi(z) = \frac{\varepsilon + z}{\varepsilon - z} = \frac{(\omega + 1)z_0 + (\omega - 1)z}{(\omega + 1)z_0 - (\omega - 1)z}.$$

Thus the range-set coincides with the whole disc and the remaining part of the theorem is simultaneously established.

In theorem 2 just proved, for a fixed $z_0$ with $z_0 \neq 0$, when $\omega$ describes the whole boundary of $\Omega_{z_0}(\Re_0)$, then the quantity $\varepsilon$ defined by $(\varepsilon + z_0)/(\varepsilon - z_0) = \omega$ describes the whole unit circumference monotonously. Hence, for any fixed $z_0$ with $|z_0| = r$, the boundary of $\Omega_{z_0}(\Re_0)$, i.e. $|(w-1)/(w+1)| = r$, and the family of rational functions $\{(\varepsilon + z)/(\varepsilon - z)\}$ with $|\varepsilon| = 1$ correspond in one-to-one way. While for any point $\omega_0$ interior to $\Omega_r(\Re_0)$ there correspond infinitely many functions $\Phi(z)$ of $\Re_0$ such that $\Phi(z_0) = \omega_0$ for an assigned $z_0$ with $|z_0| = r$, there is a unique linear function of the form

$$\Phi(z) = 1 - \lambda + \lambda \frac{\varepsilon + z}{\varepsilon - z} \qquad (0 \leqq \lambda < 1,\ |\varepsilon| = 1)$$

satisfying this condition. Actual values of $\lambda$ and $\varepsilon$ are given by

$$2\lambda = \sqrt{(\Re\omega_0 - 1)^2 + \frac{1 - r^2}{r^2}|\omega_0 - 1|^2} - (\Re\omega_0 - 1), \qquad \varepsilon = \left(1 + \frac{2\lambda}{\omega_0 - 1}\right) z_0;$$

for $\omega_0 = 1$ we put simply $\lambda = 0$, $\varepsilon$ being arbitrary.

### 4. Range for $\Re_q$.

The class $\Re_q$ may be regarded as a straightforward generalization of the class $\Re_0$, the latter being the limit case of the former as $q \to 0$. The range-set $\Omega_z(\Re_q)$ can be determined also after the model of the case $\Re_0$. By similar reasons as above, we see that this set is convex and depends substantially on $|z|$ alone.

THEOREM 3. *The range-set $\Omega_z(\Re_q)$ with $|z| = r$ $(q \leqq r < 1)$ laid on the $w$-plane is the closed convex set bounded by the image-curve of $|z| = r$ by the mapping*

$$w = \Phi^*(z) \equiv \frac{2}{i}\left(\zeta(i \lg z) - \frac{\eta_1}{\pi} i \lg z\right).$$

*Let $\omega$ be any boundary point of $\Omega_r(\Re_q)$ and $z_0$ any assigned point with $|z_0| = r > q$. Then the equation $\Phi(z_0) = \omega$ holds for $\Phi(z) \in \Re_q$ if and only if $\Phi(z)$ is the function given by*

$$\Phi(z)=\Phi^*(ze^{-i\alpha})$$

*where the real value $\alpha$ is defined uniquely (within modulo $2\pi$) by the condition $\Phi(z_0)=\omega$.*

*Proof.* The present theorem can be proved quite similarly as theorem 2 by making use of lemma 2 instead of lemma 1. We have only to notice that the curve described by $w=\Phi^*(ze^{-i\varphi})$, for a fixed $z$ with $|z|=r$ $(q<r<1)$, as $\varphi$ varies from $-\pi$ to $\pi$ is an oval which is strictly convex everywhere. Now, this curve may be regarded also as the image of $|z|=r$ by the mapping $w=\Phi^*(z)$ which transforms the annulus $q<|z|<1$ univalently onto the right half-plane cut along a vertical rectilinear segment; cf. [4, §3]. Hence, by virtue of a result previously established, it is strictly convex; cf. [2, corollary 2 of theorem 5.1].

For any point $\omega_0$ interior to $\Omega_r(\mathfrak{R}_q)$ there correspond infinitely many functions $\Phi(z)\in\mathfrak{R}_q$ such that $\Phi(z_0)=\omega_0$ for an assigned $z_0$ with $|z_0|=r$. However, among them there exists a unique function of the form

$$\Phi(z)=1-\lambda+\lambda\Phi^*(\bar{\varepsilon}z) \qquad (0\leqq\lambda<1,\ |\varepsilon|=1)$$

satisfying this condition. Actual values of $\lambda$ and $\varepsilon$ are determined as follow: There exists a unique boundary point of $\Omega_r(\mathfrak{R}_q)$ lying on the half-line $\arg(w-1)=\arg(\omega_0-1)$ which is denoted by $\omega$. Then the values of $\lambda$ and $\varepsilon$ are obtained by the equations

$$\lambda(\omega-1)=\omega_0-1, \qquad \Phi^*(\bar{\varepsilon}z_0)=\omega.$$

## 5. Range for $\hat{\mathfrak{R}}_q$.

We now proceed to consider the general class $\hat{\mathfrak{R}}_q$. Its connection to the restricted class $\mathfrak{R}_q$ has been established in theorem 1. Accordingly, its range-set $\Omega_z(\hat{\mathfrak{R}}_q)$ will be readily determined by means of theorem 3. Its convexity as well as apparent dependence on $\arg z$ follows obviously also by a similar reasoning as before. For brief statement of the theorem, we understand that the *cross sum* of two point sets $A$ and $B$ means the set consisting of all the points which are of the form $a+b$ with $a\in A$ and $b\in B$.

THEOREM 4. *The range-set $\Omega_z(\hat{\mathfrak{R}}_q)$ with $|z|=r$ $(q<r<1)$ is the closed set given by the cross sum of $\Omega_z(\mathfrak{R}_q)$ and $\Omega_{q/z}(\mathfrak{R}_q)\equiv\Omega_z(\mathfrak{R}_q')$ followed by the leftward translation by unity. Let $\omega$ be any boundary point of $\Omega_r(\hat{\mathfrak{R}}_q)$ and $z_0$ any assigned point with $|z_0|=r$. Then the equation $\Phi(z_0)=\omega$ holds for $\Phi(z)\in\hat{\mathfrak{R}}_q$ if and only if $\Phi(z)$ is the function given by*

$$\Phi(z)=\Phi^*(ze^{-i\alpha})+\Phi^*\left(\frac{q}{ze^{-i\beta}}\right)-1$$

*where the real values $\alpha$ and $\beta$ are defined uniquely (within modulo $2\pi$) by* *the condition* $\Phi(z_0)=\omega$. *More precisely, let* $\omega(t)$ $(q<t<1)$ *denote the point of contact of the tangent drawn to the boundary of* $\Omega_t(\Re_q)$ *which is parallel to the tangent of* $\Omega_z(\hat{\Re}_q)$ *at* $\omega$. *Then* $\alpha$ *and* $\beta$ *are determined by the equations*

$$\Phi^*(z_0e^{-i\alpha})=\omega(r) \quad \text{and} \quad \Phi^*\left(\frac{q}{z_0e^{-i\beta}}\right)=\omega\left(\frac{q}{r}\right),$$

*respectively.*

*Proof*. The first part of the theorem follows readily from theorem 1 combined with theorem 3. To verify the second part, we notice that the point of contact $\omega(t)$ is uniquely determined for any assigned $\omega$ and $t$ since the set $\Omega_t(\Re_q)$ is strictly convex. Further the same reason implies that the decomposition of the boundary point $\omega$ into the form $a+b-1$ with $a\in\Omega_z(\Re_q)$ and $b\in\Omega_{q/z}(\Re_q)$ is unique and is really given by

$$\omega=\omega(r)+\omega\left(\frac{q}{r}\right)-1$$

(while the decomposition of an interior point is not unique). Consequently, the second part is also established.

A similar remark as stated subsequently to theorem 3 holds here also. In fact, remembering theorem 1, it is readily verified that for any interior point $\omega_0\in\Omega_r(\hat{\Re}_q)$ there corresponds a unique function $\Phi(z)\in\hat{\Re}_q$ of the form

$$\Phi(z)=1-\lambda+\lambda\left(\Phi^*(\bar{\varepsilon}z)+\Phi^*\left(\frac{q\varepsilon'}{z}\right)-1\right) \quad (0\leqq\lambda\leqq1,\ |\varepsilon|=|\varepsilon'|=1)$$

which satisfies $\Phi(z_0)=\omega_0$ for an assigned $z_0$ with $|z_0|=r$. Actual values of $\lambda$, $\varepsilon$ and $\varepsilon'$ may be determined similarly as before. Of course, there exist infinitely many functions $\Phi(z)\in\hat{\Re}_q$ which is subject merely to the condition $\Phi(z_0)=\omega_0$ for an interior point $\omega_0\in\Omega_r(\hat{\Re}_q)$ and an assigned point $z_0$ with $|z_0|=r$.

Finally we state a supplementary remark. According to the circumstances, it will be convenient to introduce a tangential polar coordinate system on the $w$-plane with the pole at $w=1$ and the initial ray lying on the positive real axis. With respect to this system, let $p_t(\psi)$ denote the supporting function of the oval $\Omega_t(\Re_q)$ $(q<t<1)$ which is bounded by the curve

$$w=\Phi^*(te^{i\theta}), \qquad -\pi<\theta\leqq\pi.$$

Analytically stated, let the function $p_t(\psi)$ be defined by

$$p_t(\psi)=\operatorname*{Max}_{-\pi<\theta\leqq\pi}\Re(e^{-i\psi}(\Phi^*(te^{i\theta})-1)).$$

Let further $\theta_t(\omega)$ be the real value uniquely (within modulo $2\pi$) determined by the equation $\Phi^*(te^{i\theta_t(\omega)})=\omega(t)$ which is equivalent to

$$p_t(\psi_t(\omega))=\Re(e^{-i\psi_t(\omega)}(\Phi^*(te^{i\theta_t(\omega)})-1)),$$

$\psi_t(\omega)$ being the angular coordinate of the supporting line of $\Omega_t(\hat{\mathfrak{R}}_q)$ at $\omega$, i.e. $\arg(w-1)=\psi_t(\omega)$ being the equation of the perpendicular drawn from $w=1$ to the supporting line (tangent) of $\Omega_t(\hat{\mathfrak{R}}_q)$ at $\omega$. Then theorem 4 may be re-stated in terms of supporting functions also as follows.

COROLLARY. *Let $p_t(\psi)$ denote the supporting function of $\Omega_t(\mathfrak{R}_q)$ with respect to a tangential polar coordinate system with the pole at $w=1$. Then the range-set $\Omega_z(\hat{\mathfrak{R}}_q)$ is the closed convex set with the supporting function*

$$p_r(\psi)+p_{q/r}(\psi) \qquad (r=|z|)$$

*with respect to the same system. For any boundary point $\omega$ of $\Omega_z(\hat{\mathfrak{R}}_q)$ and any assigned point $z_0=re^{i\theta_0}$, the equation $\Phi(z_0)=\omega$ holds for $\Phi(z)\in\hat{\mathfrak{R}}_q$ if and only if $\Phi(z)$ is the function given by*

$$\Phi(z)=\Phi^*(ze^{-i\alpha})+\Phi^*\left(\frac{q}{ze^{-i\beta}}\right)-1$$

*where the real constants $\alpha$ and $\beta$ are given by*

$$\alpha=\theta_0-\theta_r(\omega) \quad \text{and} \quad \beta=\theta_0+\theta_{q/r}(\omega)$$

*and $\theta_t(\omega)$ is defined by $\Phi^*(te^{i\theta_t(\omega)})=\omega(t)$ with $\omega(t)$ defined in theorem 4.*

Finally, we observe here as a supplement the *mapping character* of a function defined by

$$\Phi(z)=\Phi(z;\alpha,\beta)\equiv\Phi^*(ze^{-i\alpha})+\Phi^*\left(\frac{q}{ze^{-i\beta}}\right)-1,$$

$\alpha$ and $\beta$ being any real constants. While for extremal function for theorem 4, there exists a certain relation between $\alpha$ and $\beta$, such a relation is indifferent in the following discussions. As shown previously [2, §3] and referred to above, the function $w=\Phi^*(z)$ maps the annulus $q<|z|<1$ univalently onto the right half-plane cut along a vertical rectilinear segment laid on $\Re w=1$ which is bisected by the real axis. Based on the boundary behavior of $\Phi^*(z)$, it is prolongable into a function meromorphic throughout the punctured plane $0<|z|<\infty$. In view of the inversion principle, we get the functional relations

$$\Phi^*\left(\frac{1}{\bar{z}}\right)=-\overline{\Phi^*(z)}, \qquad \Phi^*\left(\frac{q^2}{\bar{z}}\right)=2-\overline{\Phi^*(z)}.$$

On the other hand, based on the unicity of the mapping under the normalization $\Phi^*(1)=\infty$, we have

$$\overline{\Phi^*(\bar{z})}=\Phi^*(z).$$

Consequently, $\Phi^*(z)$ satisfies the identities

$$\Phi^*\left(\frac{1}{z}\right)=-\Phi^*(z), \qquad \Phi^*\left(\frac{q^2}{z}\right)=2-\Phi^*(z),$$

whence follow $\Phi^*(z/q^2) = -2 + \Phi^*(z)$ and $\Phi^*(q^3/z) = 2 + \Phi^*(q/z)$. By means of these relations we see that the function $\Phi(z) = \Phi(z;\, \alpha, \beta)$ satisfies

$$\Phi\Big(\frac{z}{q^2}\Big) = \Phi^*\Big(\frac{ze^{-i\alpha}}{q^2}\Big) + \Phi^*\Big(\frac{q^3}{ze^{-i\beta}}\Big) - 1$$

$$= -2 + \Phi^*(ze^{-i\alpha}) + 2 + \Phi^*\Big(\frac{q}{ze^{-i\beta}}\Big) - 1 = \Phi(z).$$

For convenience, we introduce a new variable $u$ and its function $\Psi(u)$ by

$$u = i \lg z, \qquad \Psi(u) = \Phi(z).$$

Then, the last equation shows that $\Psi(u)$ possesses a period $-2i \lg q$. On the other hand, since $\Phi(z)$ is single-valued with respect to $z$, $\Psi(u)$ possesses another period $2\pi$. Hence, the function $\Psi(u)$ meromorphic in $u$ is an elliptic function, its primitive periods being $2\pi$ and $-2i \lg q$. Its irreducible poles lying at $-\alpha$ and $-\beta + i \lg q$ are both of order 1 and hence the order of $\Psi(u)$ is equal to 2. Now, the basic annulus $q < |z| < 1$ corresponds in the $u$-plane to a half of the fundamental period parallelogram (rectangle). All these properties of $\Psi(u) = \Phi(z)$ can be also deduced by making use of its explicit expression in terms of zeta-functions. On the other hand, $\Re\Phi(z)$ remains positive in the basic annulus and its boundary value vanishes everywhere except at singularities $e^{i\alpha}$ and $qe^{i\beta}$. Consequently, we conclude that the image of $q < |z| < 1$ by the mapping $w = \Phi(z;\, \alpha, \beta)$ is the two-sheeted right half-planes connected crosswise along a cut between two branch-points. The points at infinity lying on the boundary components originate from $z = e^{i\alpha}$ and $z = qe^{i\beta}$.

## 6. Generalization.

We have observed in previous papers [3; 4] a linear operator $\mathcal{L}$ which has $\mathfrak{R}_0$ or $\mathfrak{R}_q$ as its domain of argument function and produces by applying to any $\Phi(z)$ of its domain a single-valued regular analytic function $\mathcal{L}[\Phi(z)]$, and which is homogeneous of degree zero, i. e., for any constant $\gamma$, the function $\mathcal{L}[\Phi(z)]$ is transferred after substitution $z|\gamma z$ into $\mathcal{L}[\Phi(\gamma z)]$. We suppose here also as before that the operator $\mathcal{L}$ is commutative with the integration with respect to $\rho(\varphi)$ in the representation for $\Phi(z)$ stated in lemma 1 or 2. By virtue of the last supposition, it would have been substantially only necessary to define merely the $\mathcal{L}$-operation applied to the kernel of the representation. In fact, the operation in the whole domain would naturally follow by means of the supposition. Now, we shall show that theorems 2 and 3 admit straightforward generalizations with reference to such an operator. Let, in general, the class of functions $\{\mathcal{L}[\Phi(z)]\}$ in which $\Phi(z)$ extends over a class $\mathfrak{F}$ be denoted by $\mathcal{L}[\mathfrak{F}]$. We begin with a generalization of theorem 2.

THEOREM 5. *Let $\mathcal{L}$ be a linear operator defined for $\mathfrak{R}_0$ and of the* *nature stated above. Then, for any fixed $z$ with $|z|=r<1$, the range-set $\Omega_z(\mathcal{L}[\mathfrak{R}_0])$ is the smallest convex hull which contains the image-curve $C_r$ of $|z|=r$ by the (not necessarily univalent) mapping $w=\mathcal{L}[(1+z)/(1-z)]$. Any boundary point $\omega$ of the hull is attained by $\mathcal{L}[\Phi(z)]$ with $\Phi(z)\in\mathfrak{R}_0$ at an assigned point $z_0=re^{i\theta_0}$ if and only if $\Phi(z)$ is a rational function of the form*

$$\Phi(z)=\sum_{h=1}^{m}\rho_h\frac{e^{i(\theta_0+\alpha_h)}+z}{e^{i(\theta_0+\alpha_h)}-z}.$$

*Here $\alpha_h$ and $\rho_h$ $(h=1,\cdots,m)$ are subject to the conditions*

$$\mathcal{L}\left[\frac{1+re^{-i\alpha_h}}{1-re^{-i\alpha_h}}\right]=\omega_h \qquad (h=1,\cdots,m),$$

$$\rho_h\geqq 0\ (h=1,\cdots,m),\quad \sum_{h=1}^{m}\rho_h=1,\quad \sum_{h=1}^{m}\rho_h\omega_h=\omega,$$

*and $\omega_h$ $(h=1,\cdots,m)$ denote the points on the boundary of the convex hull where the supporting line at $\omega$ touches the curve $C_r$, the multiplicity of contact being taken into account; the integer $m$ may depend on $\omega$.*

***Proof.*** Based on the representation formula for $\mathfrak{R}_0$ given in lemma 1 together with the assumption imposed on $\mathcal{L}$, we get, for any $\Phi(z)\in\mathfrak{R}_0$,

$$\mathcal{L}[\Phi(z)]=\int_{-\pi}^{\pi}\mathcal{L}\left[\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right]d\rho(\varphi).$$

Since $\rho(\varphi)$ is an increasing function with the total variation equal to unity, this relation implies readily that the value of $\mathcal{L}[\Phi(z)]$ for an assigned $z$ with $|z|=r$ belongs to the smallest convex hull which contains the curve described by $w=\mathcal{L}[(e^{i\varphi}+z)/(e^{i\varphi}-z)]$ as $\varphi$ varies from $-\pi$ to $\pi$. By virtue of the homogeneity of $\mathcal{L}$, this curve coincides with $C_r$ as the configuration. Hence the range-set is a subset of the convex hull stated in the theorem. Now, since the analytic curve $C_r$ is regular, it has then a finite number of points common with any straight line. Introducing a tangential polar coordinate system on the $w$-plane with the pole at an arbitrary point $w_0$ interior to the hull and the initial ray parallel to the positive real axis, let the angular coordinate of the supporting line at a boundary point $\omega$ of the hull be denoted by $\psi(\omega)$. The quantity defined by

$$S_\omega(\theta)=\Re\left(e^{-i\psi(\omega)}\left(\mathcal{L}\left[\frac{1+re^{i\theta}}{1-re^{i\theta}}\right]-w_0\right)\right),$$

qua function of $\theta$, then attains its maximum, $M_\omega$ say, for $\theta=-\alpha_h$ $(h=1,\cdots,m)$ and for these values of $\theta$ alone (within modulo $2\pi$). Hence we have

$$\Re(e^{-i\psi(\omega)}(\mathcal{L}[\Phi(re^{i\theta_0})]-w_0))=\int_{-\pi}^{\pi}S_\omega(\theta_0-\varphi)\,d\rho(\varphi)\leqq M_\omega$$

and the equality sign in the last inequality holds if and only if the quantity $d\rho(\varphi)$ associated to $\Phi(z)$ vanishes everywhere except for

$$\varphi \equiv \theta_0 + \alpha_h \qquad (\mathrm{mod}\, 2\pi) \qquad\qquad (h = 1, \cdots, m)$$

and hence $\Phi(z)$ is a rational function of the form

$$\Phi(z) = \sum_{h=1}^{m} \rho_h \frac{e^{i(\theta_0+\alpha_h)} + z}{e^{i(\theta_0+\alpha_h)} - z};$$

$$\rho_h \geqq 0 \ (h = 1, \cdots, m), \qquad \sum_{h=1}^{m} \rho_h = 1,$$

$\rho_h$ denoting here the height of jump of $\rho(\varphi)$ at $\theta_0 + \alpha_h$, i. e. $\rho_h = \rho(\theta_0 + \alpha_h + 0) - \rho(\theta_0 + \alpha_h - 0)$. In order to assure the equation $\mathcal{L}[\Phi(z_0)] = \omega$, it is only necessary to supplement a condition

$$\omega = \sum_{h=1}^{m} \rho_h \mathcal{L}\left[\frac{e^{i(\theta_0+\alpha_h)} + z_0}{e^{i(\theta_0+\alpha_h)} - z_0}\right] = \sum_{h=1}^{m} \rho_h \omega_h.$$

Thus the theorem has been established in its whole scope.

Now, several sorts of choice are possible for $\mathcal{L}$, among which some have been illustrated in [3]. Here we mention an illustrating example of theorem 5 corresponding to a particular choice of $\mathcal{L}$. Let $z$ be any fixed point in $|z| < 1$ and $W_z[z\Phi'(z)]$ denote the functional defined for $\mathfrak{R}_0$ by which the value $z\Phi'(z)$ at $z$ corresponds to each argument function $\Phi(z) \in \mathfrak{R}_0$.

COROLLARY. *The range-set defined by*

$$\Omega_z\left(z \frac{d}{dz} [\mathfrak{R}_0]\right) = \bigcup_{\Phi(z) \in \mathfrak{R}_0} W_z[z\Phi'(z)]$$

*is the smallest closed convex hull which contains the image-curve $C_r$ of $|z| = r$ by the mapping $w = 2z/(1-z)^2$. The boundary of the hull consists of the whole curve $C_r$ for $r \leqq 2 - \sqrt{3}$ but it consists of an arc of $C_r$ given by $w = 2re^{i\theta}/(1 - re^{i\theta})^2$, $-\alpha_1 \leqq \theta \leqq \alpha_1$ and a rectilinear segment connecting its end-points for $2 - \sqrt{3} < r < 1$, $\alpha_1 = \alpha_1(r)$ being determined by $\cos\alpha_1 = -(1 - 6r^2 + r^4)/(2r(1+r^2))$, $0 < \alpha_1 < \pi$. Any boundary point $\omega$ of the hull is attained by $z\Phi'(z)$ with $\Phi(z) \in \mathfrak{R}_0$ if and only if $\Phi(z)$ is a rational function of the form*

$$\Phi(z) = \frac{e^{i(\theta_0+\alpha)} + z}{e^{i(\theta_0+\alpha)} - z}$$

*or*

$$\Phi(z) = \lambda \frac{e^{i(\theta_0+\alpha_1)} + z}{e^{i(\theta_0+\alpha_1)} - z} + (1-\lambda) \frac{e^{i(\theta_0-\alpha)} + z}{e^{i(\theta_0-\alpha)} - z}$$

*provided that $\omega$ lies on the* (*closed*) *curvilinear part or the* (*open*) *rectilinear part of the boundary, respectively. Here $\alpha$ is determined by $\omega = 2re^{-i\alpha}/(1 - re^{-i\alpha})^2$ and $\lambda$ with $0 < \lambda < 1$ is determined by $\Phi(z_0) = \omega$.*

*Proof.* By taking especially

$$\mathcal{L} = z\frac{d}{dz},$$

we see that the corollary is an immediate consequence of theorem 5 except an explicit statement on the shape of the smallest convex hull containing $C_r$. Now, for the function

$$f(z) = \frac{2z}{(1-z)^2} \equiv z\frac{d}{dz}\frac{1+z}{1-z} \qquad (|z|<1)$$

we get

$$1 + \Re\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})} = \frac{(1-r^2)(1+4r\cos\theta+r^2)}{1-2r^2\cos 2\theta+r^4}$$

which is non-negative for all values of $\theta$ if and only if $r \leqq 2-\sqrt{3}$. Hence, the curve $C_r$ is (strictly) convex for such a value of $r$ while otherwise it possesses a dented arc centred at $-2r/(1+r)^2$; cf. [5]. For the latter case, in order to construct the boundary of the convex hull, we have to replace the arc by the rectilinear segment connecting $f(re^{\pm i\alpha_1})$, where $\alpha_1$ is the value of $\theta$ given in the corollary which is obtained by the condition that $\Re f(re^{i\theta})$ attains its minimum.

Actual calculation shows that explicit values concerned are given by

$$re^{\pm i\alpha_1} = \frac{1}{2(1+r^2)}(-1+6r^2-r^4 \pm i(1-r^2)\sqrt{-1+14r^2-r^4}),$$

$$f(re^{\pm i\alpha_1}) = \frac{1+r^2}{4(1-r^2)^2}(-(1+r^2) \pm i\sqrt{-1+14r^2-r^4}).$$

A result on $\Re_q$ analogous to theorem 5 may be stated as follows.

THEOREM 6. *Let $\mathcal{L}$ be a linear operator defined for $\Re_q$ and of the nature stated at the beginning of the present section. Then, for any fixed $z$ with $|z|=r$ $(q<r<1)$, the range-set $\Omega_z(\mathcal{L}[\Re_q])$ is the smallest closed convex hull which contains the image-curve $C_r$ of $|z|=r$ by the (not necessarily univalent) mapping $w=\mathcal{L}[\Phi^*(z)]$, $\Phi^*(z)=(2/i)(\zeta(i\lg z)-(\eta_1/\pi)i\lg z)$. Any boundary point $\omega$ of the hull is attained by $\mathcal{L}[\Phi(z)]$ with $\Phi(z)\in\Re_q$ at an assigned point $z_0=re^{i\theta_0}$ if and only if $\Phi(z)$ is of the form*

$$\Phi(z) = \sum_{h=1}^{m}\rho_h\,\Phi^*(ze^{-i(\theta_0+\alpha_h)}).$$

*Here $\alpha_h$ and $\rho_h$ $(h=1,\cdots,m)$ are subject to the conditions*

$$\mathcal{L}[\Phi^*(re^{-i\alpha_h})] = \omega_h \qquad (h=1,\cdots,m),$$

$$\rho_h \geqq 0 \quad (h=1,\cdots,m), \qquad \sum_{h=1}^{m}\rho_h = 1, \qquad \sum_{h=1}^{m}\rho_h\omega_h = \omega,$$

*and $\omega_h=\omega_h(r)$ $(h=1,\cdots,m)$ denote the points on the boundary of the convex hull where the supporting line at $\omega$ touches the curve $C_r$, the multiplicity of contact being taken into account; the integer $m$ may depend on $\omega$.*

*Proof.* The present theorem is proved quite similarly as theorem 5. In fact, we have only to replace the Poisson kernel $(e^{i\varphi}+z)/(e^{i\varphi}-z)$ in lemma 1 by the kernel $\Phi^*(ze^{-i\varphi})$ in lemma 2.

A corollary analogous to that of theorem 5 may be formulated here also. Then the function $2z/(1-z)^2=(d/d\lg z)((1+z)/(1-z))$ will have to be replaced by $-2(\wp(i\lg z)+\eta_1/\pi)=(d/d\lg z)\Phi^*(z)$.

In order to establish a generalization of theorem 4 correspondingly to theorems 5 and 6, let $\mathcal{L}$ be a linear operator defined for $\hat{\mathfrak{R}}_q$ and of a similar nature as before. By virtue of the decomposition theorem 1, it suffices to suppose first that the operator is applicable to every function of the classes $\mathfrak{R}_q$ as well as $\mathfrak{R}_q'$ or merely to their kernels in respective representations. In fact, according to theorem 1, the $\mathcal{L}$-operator applied to any function $\Phi(z)\in\hat{\mathfrak{R}}_q$ is then naturally defined by

$$\mathcal{L}[\Phi(z)]=\mathcal{L}[R(z)]+\mathcal{L}[T(z)]-\mathcal{L}[1].$$

THEOREM 7. *Let $\mathcal{L}$ be a linear operator defined for $\hat{\mathfrak{R}}_q$ as above. Then, for any fixed $z$ with $|z|=r$ $(q<r<1)$, the range-set $\Omega_z(\mathcal{L}[\hat{\mathfrak{R}}_q])$ is the cross sum of $\Omega_z(\mathcal{L}[\mathfrak{R}_q])$ and $\Omega_z(\mathcal{L}[\mathfrak{R}_q'])\equiv\Omega_{q/z}(\mathcal{L}[\mathfrak{R}_q])$ followed by the leftward translation by $\mathcal{L}[1]$. Any boundary point $\omega$ of the range-set is attained by $\mathcal{L}[\Phi(z)]$ with $\Phi(z)\in\hat{\mathfrak{R}}_q$ at an assigned point $z_0=re^{i\theta_0}$ if and only if $\Phi(z)$ is of the form*

$$\Phi(z)=\sum_{h=1}^{m}\rho_h\,\Phi^*(ze^{-i\alpha_h})+\sum_{k=1}^{n}\tau_k\,\Phi^*\left(\frac{q}{ze^{-i\beta_k}}\right)-1.$$

*Here $\alpha_h$, $\beta_k$, $\rho_h$ and $\tau_k$ $(h=1,\cdots,m;\ k=1,\cdots,n)$ are subject to the conditions*

$$\mathcal{L}[\Phi^*(re^{-i\alpha_h})]=\omega_h(r),\qquad \mathcal{L}\left[\Phi^*\left(\frac{q}{re^{-i\beta_k}}\right)\right]=\omega_k\left(\frac{q}{r}\right),$$

$$\rho_h\geqq 0,\ \tau_k\geqq 0\quad (h=1,\cdots,m;\ k=1,\cdots,n),\qquad \sum_{h=1}^{m}\rho_h=\sum_{k=1}^{n}\tau_k=1,$$

$$\sum_{h=1}^{m}\rho_h\,\omega_h(r)+\sum_{k=1}^{n}\tau_k\,\omega_k\left(\frac{q}{r}\right)-1=\omega,$$

*and $\omega_h(r)$ $(h=1,\cdots,m)$ and $\omega_k(q/r)$ $(k=1,\cdots,n)$ denote the points on the boundaries of $\Omega_r(\mathcal{L}[\mathfrak{R}_q])$ and $\Omega_{q/r}(\mathcal{L}[\mathfrak{R}_q])=\Omega_r(\mathcal{L}[\mathfrak{R}_q'])$, respectively, where the supporting lines parallel to that of $\Omega_z(\mathcal{L}[\hat{\mathfrak{R}}_q])$ at $\omega$ touch the image-curves $C_r$ and $C_{q/r}$ of $|z|=r$ and $|z|=q/r$, respectively, by the mapping $w=\mathcal{L}[\Phi^*(z)]$, the multiplicity of contact being taken into account.*

*Proof.* The proof of the present theorem proceeds similarly as in theorem 4 by modifying after the model of the proof of theorem 5.

## REFERENCES

[1] KOMATU, Y., Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung. Jap. Journ. Math. **19** (1945), 203–215.
[2] KOMATU, Y., On conformal mapping of a domain with convex or star-like boundary. Kōdai Math. Sem. Rep. **9** (1957), 105–139.
[3] KOMATU, Y., On analytic functions with positive real part in a circle. Kōdai Math. Sem. Rep. **10** (1958), 64–83.
[4] KOMATU, Y., On analytic functions with positive real part in an annulus. Kōdai Math. Sem. Rep. **10** (1958), 84–100.
[5] NEVANLINNA, R., Über die schlichten Abbildungen des Einheitskreises. Översikt af Finska Vetenskaps-Soc. Förh. (A) **62** (1919/20), 1–14.
[6] ROGOSINSKI, W., Über positive harmonische Entwicklungen und typisch-reelle Potenzreihen. Math. Zeitschr. **35** (1932), 93–121.

DEPARTMENT OF MATHEMATICS,
TOKYO INSTITUTE OF TECHNOLOGY.