ON THE PROLONGATION OF AN OPEN RIEMANN SURFACE OF FINITE GENUS

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In his paper [3], M. Heins has proved the following theorem: "If F is an open Riemann surface of genus one, the set of principal moduli of all tori W that are prolongations of F is bounded". Making use of the concept of Teichmüller space introcuced in [7] (see also [1]), we are able to express this fact, with a minor modification, as follows: "The W's form a bounded set in Teichmüller space for torus". A theorem which will be proved in the present report may be regarded as containing an extension of this theorem for any genus greater than one.

1. We begin with a topological preparation.

Let F be an open Riemann surface of a finite genus, W be a closed Riemann surface of the same genus as F, and $q = f(p)$ be a topological mapping of F onto a subdomain contained in W. Then, f induces a homomorphism η_f of $\mathfrak{G}_F$ (the fundamental group of F) into $\mathfrak{G}_W$ (the fundamental group of W) in the well-known manner.

LEMMA 1. (i) *η_f maps $\mathfrak{G}_F$ onto $\mathfrak{G}_W$.*

(ii) *The kernel of η_f depends only upon F; i.e., if there exists another W', f', then the kernels of η_f and $\eta_{f'}$ coincide.*

In fact, the kernel of η_f is the smallest normal subgroup of $\mathfrak{G}_F$ containing every such contour that divides F into two parts, one of which is of schlichtartig.

We omit the proof in details, since it is obtained rather easily.

2. Next, we introduce Teichmüller space according to [1]. For an integer $g > 1$, we consider all closed Riemann surfaces W of genus g. Fix one of them, say W_0. With respect to any W, let α be a homotopy-class of orientation-preserving topolpgical mappings of W_0 onto W, and consider the pair (W, α). (W, α) and (W', α') are said to be equivalent, if there exists a conformal mapping of W onto W' belonging to the homotopy-class $\alpha'\alpha^{-1}$. The equivalene class will be denoted by $\langle W, \alpha\rangle$ and the whole set of these classes is called *Teichmüller space*, which will be denoted by T_g.

In this space, we define the distance of any two points $\langle W, \alpha\rangle$, $\langle W', \alpha'\rangle$

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by a quantity $\log(\inf K_f)$, where K_f denotes the maximal dilatation of f, a quasiconformal mapping of W onto W', and the infimum is taken over all f belonging to the homotopy-class $\alpha'\alpha^{-1}$. It is not so difficult to see that T_g is a metric space by means of this distance.

As was proved in [1] (cf. [7]), T_g is homeorphic to $(6g-6)$-dimensional Euclidean space.

3. Let F be an open Riemann surface of genus $g\,(1<g<\infty)$. A Riemann surface W is said to be a *prolongation* of F if there exists a conformal mapping $q=f(p)$ of F onto a proper subdomain of W (we say, for simplicity's sake, that f maps F conformally *into* W). We consider W together with f, in other words, we consider them as a pair $\{W, f\}$. Among them, we collect all $\{W, f\}$ such that W are closed Riemann surfaces of genus g, and denote this set by $\mathfrak{P}(F)$. It is well-known that snch a $\{W, f\}$ actually exists.

We fix one $\{W_0, f_0\}\in\mathfrak{P}(F)$. For any $\{W, f\}\in\mathfrak{P}(F)$, the composed mapping $f\circ f_0^{-1}$ maps $f_0(F)\subset W_0$ conformally into W. Then, by LEMMA 1 it induces an isomorphism of $\mathfrak{G}_{W_0}$ onto $\mathfrak{G}_W$, therefore it determines a homotopy-class of orientation-preserving topological mappings of W_0 onto W, which will be denoted by α_f. We can easily see that, for $\{W, f\}$, $\{W, f'\}\in\mathfrak{P}(F)$, the mappings f, f' are homotopic to each other (as continuous mappings of F into W) if and only if $\alpha_f=\alpha_{f'}$.

Putting this W_0 on center, we construct Teichumüller space T_g as in the last section. Then the above consideration gives us a correspondence

$$\mathfrak{P}(F)\ni\{W, f\} \longrightarrow \langle W, \alpha_f\rangle\in T_g.$$

The image of $\mathfrak{P}(F)$ in T_g will be denoted by $P(F)$.

Now, what we should like to prove is the following

THEOREM. *$P(F)$ is compact and connected in T_g.*

We shall show the compactness in §§ 4, 5 and connectedness in §§ 6, 7.

4. We know that the family $\mathfrak{P}(F)$ is compact in a certain sense (see [2]). As we shall show, this compactness implies the compactness of $P(F)$ in T_g under a condition that $P(F)$ is bounded in T_g.

In order to show the boundedness of $P(F)$, it suffices to prove the following proposition:

"*There exists a constant $K_0<\infty$ with the following property: For any $\{W, f\}\in\mathfrak{P}(F)$, there is a quasiconformal mapping of W_0 onto W, belonging to the homotopy-class α_f, and whose maximal dilatation is majorated by K_0*".

To prove this, the following lemma will be required.

LEMMA 2. *Let B be a ring-domain in z-plane, one complementary continuum of which is $1\leqq|z|\leqq\infty$ and the other contains $z=0$. Let $\mathfrak{F}$ be the family of* *functions* $w(z)$, *which are regular, univalent,* $0<|w(z)|<1$ *in* B *and* $|w(z)|=1$ *on* $|z|=1$. *Then, there exist constants* c, c' *such that* $0<c\leqq c'<\infty$ *and*

$$c\leqq\left|\frac{dw}{dz}\right|\leqq c' \qquad on\ |z|=1$$

for all $w(z)\in\mathfrak{F}$.

Proof of LEMMA 2. We know that $\mathfrak{F}$ is a normal family and any converging subsequence of $\mathfrak{F}$ converges uniformly on a sufficiently narrow annulus containing $|z|=1$, and furthermore, limit function is not a constant (see [6], p. 636). So that, if there exists no number c satisfyihg the above condition, we obtain $w_n(z)\in\mathfrak{F}$ and $z_n\in(|z|=1)(n=1,2,\cdots)$ such that $\lim_{n\to\infty}\frac{dw_n}{dz}(z_n)=0$. For a suitably chosen subsequence, we get $\lim w_{n_j}(z)=w_0(z)\in\mathfrak{F}$ and $\lim z_{n_j}=z_0\in(|z|=1)$, which implies $\frac{dw_0}{dz}(z_0)=0$, a contradiction. Similarly, we can see the existence of number c', q.e.d.

Now, let us prove the above proposition. First of all, we can easily find a compact subdomain D of F (it means that D is a subdomain of F and the closure of D is compact in F), which is of genus g and the relative boundary ∂D of which consists of only one closed Jordan curve. Moreover, we can take a ring-domain A in $F-D$, one of the boundary components of which coincides with ∂D. On W_0, the subdomain $W_0-\overline{f_0(D)}$ is simply connected. So that, it can be mapped conformally onto the unit disc $|z|<1$ in z-plane by a function $z=\varphi_0(p)$, where it may be assumed that $z=0$ corresponds to $W_0-\overline{f_0(F)}$. On the ring-domain $\varphi_0\circ f_0(A)$, numbers c and c' are determined by LEMMA 2. We show in the sequel that a number

$$K_0=\max\left(c',\ \frac{1}{c}\right)$$

is the desired.

Take an arbitrary $\{W,f\}\in\mathfrak{P}(F)$. A simply connected domain $W-\overline{f(D)}$ can be mapped conformally onto a unit disc $|w|<1$ by a function $w=\varphi(p)$, where $w=0$ corresponds to a point in $W-\overline{f(F)}$. A composed function

$$w=\varphi\circ f\circ f_0^{-1}\circ\varphi_0^{-1}\equiv\Phi(z)$$

belongs to the family $\mathfrak{F}$ of LEMMA 2 with respect to the ring-domain $B=\varphi_0\circ f_0(A)$, so that we get $c\leqq|d\Phi/dz|\leqq c'$ on $|z|=1$.

A mapping $w=H(z)$ of $|z|\leqq1$ onto $|w|\leqq1$ defined by

$$H(re^{i\theta})=r\Phi(e^{i\theta}) \qquad (0\leqq r\leqq1,\ 0\leqq\theta<2\pi)$$

is evidently a topological mapping of $|z|\leqq1$ onto $|w|\leqq1$ and it coincides with Φ on $|z|=1$. Furthermore, it is differentiable in $0<|z|<1$ and its dilatation at a point $z=re^{i\theta}$ is equal to

$$\max\left(\left|\frac{d\Phi(e^{i\theta})}{dz}\right|,\ 1\Big/\left|\frac{d\Phi(e^{i\theta})}{dz}\right|\right),$$

which is majorated by $\max(c', 1/c) = K_0$.

By the aid of this mapping, we define a mapping $q = h(p)$ of W_0 onto W by

$$h(p) = \begin{cases} f \circ f_0^{-1}(p) & p \in f_0(D), \\ \varphi \circ H \circ \varphi_0^{-1}(p) & p \in W_0 - f_0(D). \end{cases}$$

It is evidently a topological mapping of W_0 onto W, and has a maximal dilatation not greater than K_0. It is not difficult to see that it belongs to our homotopy-class α_f, which completes the proof of the boundness of $P(F)$.

5. Now, we show the compactness of $P(F)$. Since $P(F)$ is bounded, any sequence $\langle W_n, \alpha_{f_n}\rangle \in P(F)$ $(n = 1, 2, \cdots)$ contains a subsequence (we denote it again by $\{n\}$) such that $\langle W_n, \alpha_{f_n}\rangle \to \langle W^*, \alpha^*\rangle \in T_g$. It suffices to show that $\langle W^*, \alpha^*\rangle \in P(F)$, i.e. there exists a $\{W^*, f^*\}$ in $\mathfrak{P}(F)$ such that $\langle W^*, \alpha_f{}^*\rangle = \langle W^*, \alpha_f{}^*\rangle$. We shall construct a mapping function f^* by the aid of uniformization. In what follows, we write α_n instead of α_{f_n}, for the sake of simplicity.

Let us denote by $q = h_n(p)$ the extremal quasiconformal mapping of W_0 onto W_n of the homotopy-class $\alpha_n (n = 1, 2, \cdots)$[1], and by $q = h^*(p)$ that of W_0 onto W^* of α^*. Let $|z| < 1$ be a universal covering surface of W_0, and $|w| < 1$ be that of W_n, W^* such that $w = 0$ lies above $h_n(p_0)$, $h^*(p_0)$ respectively, where p_0 is the projection of $z = 0$ on W_0. Mappings h_n, h^* are interpreted as mappings $w = h_n(z)$, $w = h^*(z)$ of $|z| < 1$ onto $|w| < 1$, satisfying a condition that $h_n(0) = h^*(0) = 0$. Let $\mathfrak{G}_0$, $\mathfrak{G}_n$, $\mathfrak{G}^*$ be the groups of covering transformations of W_0, W_n, W^*, respectively $(n = 1, 2, \cdots)$. Then, $h_n(z)$ and $h^*(z)$ induce isomorphisms

$$\mathfrak{G}_0 \ni S \to S^{\alpha_n} \in \mathfrak{G}_n \qquad (n = 1, 2, \cdots),$$
$$\mathfrak{G}_0 \ni S \to S^{\alpha^*} \in \mathfrak{G}^*,$$

such that

$$h_n(Sz) = S^{\alpha_n} h_n(z) \qquad (n = 1, 2, \cdots), \tag{1}$$
$$h^*(Sz) = S^{\alpha^*} h^*(z),$$

respectively. We can see that the assumption $\lim_{n\to\infty}\langle W_n, \alpha_n\rangle = \langle W^*, \alpha^*\rangle$ implies that

$$\lim_{n\to\infty} h_n(z) = h^*(z) \tag{2}$$

holds uniformly in $|z| < 1$ (see [1], p. 56). So that, from (1) and (2), we get

$$\lim_{n\to\infty} S^{\alpha_n}(w) = S^{\alpha^*}(w) \tag{3}$$

in $|w| < 1$, for any $S \in \mathfrak{G}_0$.

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1) It is a quasiconformal mapping belonging to α_n, whose maximal dilatation is smaller than that of any other mapping of α_n; see [1, 7].

Next, let $|\zeta|<1$ be a universal covering surface of F and $\Gamma=\{\sigma\}$ be the covering transformations. To $\{W_0, f_0\}$ and $\{W_n, f_n\}(n=1, 2, \cdots)$, functions $z=f_0(\zeta)$ and $w=f_n(\zeta)$ correspond, which map $|\zeta|<1$ into $|z|<1$ and $|w|<1$, respectively. Representing the homomorphism η_{f_0} (introduced in § 1) as

$$\Gamma \ni \sigma \rightarrow S \in \mathfrak{G}_0,$$

we obtain

$$(4)\qquad \begin{aligned} f_0(\sigma\zeta) &= Sf_0(\zeta), \\ f_n(\sigma\zeta) &= S_n S^{\alpha_n} S_n^{-1} f_n(\zeta) \qquad (n=1,2,\cdots), \end{aligned}$$

where $S_n \in \mathfrak{G}_n$ is chosen independent of σ so that

$$(5)\qquad |f_n(0)| \leqq 1-\delta < 1 \qquad (n=1,2,\cdots)$$

may hold. (It is possible because of (2).) We know that $\{f_n(\zeta)\}_{n=1}^{\infty}$ is a normal family and, by (5), the limit function of any converging subsequence is not a constant (see [2]). Furthermore, by (4), a suitably chosen subsequence of $\{S_n(w)\}_{n=1}^{\infty}$ canverges to a linear transformation ($\not\equiv$ const.). So that, denoting a subsequence by $\{n\}$ again, we obtain

$$\lim f_n(\zeta) = \hat{f}(\zeta), \qquad \lim S_n(w) = \hat{S}(w).$$

Consequently,

$$\lim S_n f_n(\zeta) = \hat{S}\hat{f}(\zeta) \equiv f^*(\zeta) \not\equiv \text{const.},$$

and by (3) and (4),

$$f^*(\sigma\zeta) = S^{\alpha^*} f^*(\zeta) \qquad \text{for any } \sigma \in \Gamma,$$

which determines a conformal mapping f^* of F into W^* such that $\alpha_{f^*}=\alpha^*$.

6. To prove the connectedness of $P(F)$, a preparation is required.

Let $F_1 \subset F_2 \subset \cdots \uparrow F$ be an exhaustion of F, each member of which is a compact subdomain of genus g and has a relative boundary consists of a finite number of closed analytic curves. With respect to any $\{W, f\} \in \mathfrak{P}(F)$, if we restrict f on F_k, then $\{W, f\}$ can be seen as a prolongation of F_k. In this sense $\mathfrak{P}(F) \subset \mathfrak{P}(F_k)$, so that we obtain $P(F) \subset \bigcap_{k=1}^{\infty} P(F_k)$.

LEMMA 3. $$P(F) = \bigcap_{k=1}^{\infty} P(F_k).$$

Proof. Let $\langle W, \alpha\rangle$ be an element of $\bigcap_{k=1}^{\infty} P(F_k)$. There exist conformal mappings f_k of F_k into $W\,(k=1, 2, \cdots)$, which are homotopic to f_1. The consideration in the last section shows that a subsequence $\{f_{k_j}\}$ converges uniformly in the wider sense on F to a conformal mapping f^* ($\not\equiv$ const.) of F into W, which is homotopic to f_1. So that $\{W, f^*\}$ belongs to $\mathfrak{P}(F)$, to which $\langle W, \alpha\rangle$ corresponds, and it implies $\bigcap_{k=1}^{\infty} P(F_k) \subset P(F)$, q. e. d.

Consequently, to prove the connectedness of $P(F)$, it suffices to show that of $P(F_k)$.

7. Now, we show the connectedness of $P(F_k)$; in other words, we show the connectedness of $P(F)$ under a supposition that F is a compact subdomain of a Riemann surface, relative boundary ∂F of which consists of a finite number of closed analytic curves. As will be seen in the proof below, we may assume that ∂F consists of only one curve.

We try to connect any $\langle W, \alpha\rangle$, $\langle W', \alpha'\rangle \in P(F)$ by a curve $\langle W_t, \alpha_t\rangle$ $(0 \leqq t \leqq 1)$ in $P(F)$.

Let $\{W, f\}$ and $\{W', f'\}$ be elements of $\mathfrak{P}(F)$, to which $\langle W, \alpha\rangle$ and $\langle W', \alpha'\rangle$ correspond respectively. Take a ring-domain A in F which is bounded by two closed analytic curves, one of which coincides with ∂F. Since a domain $W - \overline{f(F-A)}$ is simply connected, we can map it conformally by $z = \varphi(p)$ onto a disc $|z| < 1$ in such a way that $z = 0$ corresponds to a point contained in $W - \overline{f(F)}$. Similarly we map $W' - \overline{f'(F-A)}$ onto $|w| < 1$ by $w = \varphi'(p)$. A composed function

$$w = \varphi_0' \circ f' \circ f_0^{-1} \circ \varphi^{-1}(z) \equiv \Phi(z)$$

maps the ring-domain $\varphi \circ f(A)$ conformally onto $\varphi' \circ f'(A)$, where $|z| = 1$ and $|w| = 1$ correspond mutually.

Now, we need the following lemma concerning the function family in LEMMA 2.

LEMMA 4. *Let B be a ring-domain in z-plane, one complementary continuum of which is $1 \leqq |z| \leqq \infty$ and the other contains $z = 0$. Let $\mathfrak{F}$ be the family of functions $w(z)$, which are regular, univalent, $0 < |w(z)| < 1$ in B, $|w(z)| = 1$ on $|z| = 1$. Then, for any $\Phi(z) \in \mathfrak{F}$, there exists a one-parameter family of functions $\Phi(z)$ $(0 \leqq t \leqq 1)$, satisfying the following conditions*:

i) *$\Phi_t(z) \in \mathfrak{F}$ for any fixed t;*

ii) *$\Phi_t(z)$ is continuous with respect to t for any fixed z;*[2]

iii) *$\Phi_0(z) = z$, $\Phi_1(z) = \Phi(z)$.*

Proof of LEMMA 4. Though we may prove this lemma from a general theory (see [5]), Prof. Y. Komatu has kindly taught the author a method of explicit construction of $\Phi_t(z)$ as follows.

First of all, we may assume without loss of generality that B is a concentric annulus $\rho < |z| < 1$ and $\Phi(1) = 1$. Let $U(w)$ be a function which is harmonic in $B' = \Phi(B)$, $U(w) = 0$ on $|w| = 1$ and $U(w) = 0$ on the other boundary component of B'. For every $t\,(0 < t < 1)$, denoting by B_t' a ring-domain in B', bounded by $|w| = 1$ and the niveau curve $U(w) = t$, we can find a number $r_t > 1$ such that

$$\mathrm{mod}\,[B_t' \cup (1 \leqq |w| < r_t)] = \log \frac{1}{p}.$$

So that, there exists a function $w = \Psi_t(z)$ which maps B conformally onto $B_t' \cup (1 \leqq |w| < r_t)$, where $\Psi_t(1) = r_t$. Then, the function-family defined by

2) We do not require here the continuity with respect to $p \times t$.

$$\Phi_t(z) = \begin{cases} z & \text{for } t = 0, \\ \dfrac{1}{r_t} & \text{for } 0 < t < 1, \\ \Phi(z) & \text{for } t = 1 \end{cases}$$

is the required, q. e. d.

We note that the normality of $\mathfrak{F}$ shown in the proof of LEMMA 2 implies the fact that

$$\lim_{t' \to t} \frac{d\Phi_{t'}}{dz} = \frac{d\Phi_t}{dz} \tag{1}$$

holds uniformly on $|z| = 1$.

To continue, we apply LEMMA 4 to $B = \varphi \circ f(A)$ and obtain $\Phi_t(z)$ $(0 \leqq t \leqq 1)$. By the aid of it, a family of Riemann surfaces $W_t (0 \leqq t \leqq 1)$ can be constructed as follows: W_t is the union of sets $f(F - A) \subset W$ and $|w| \leqq 1$, where points $p \in \alpha\,(f(F - A))$ and $w = \Psi_t \circ \varphi(p) \in (|w| = 1)$ are identified; as local parameters, the original are chosen in $f(F - A)$ and w in $|w| < 1$, which are connected across $\partial(f(F - A)) = (|w| = 1)$ in the usual manner. Then, a mapping defined by

$$f_t(p) = \begin{cases} f(p) & \text{for } p \in F - A, \\ \Phi_t \circ \varphi \circ f(p) & \text{for } p \in \bar{A} \end{cases}$$

maps F conformally into W_t. Consequently, we obtain

$$\{W_t, f_t\} \in \mathfrak{P}(F) \qquad (0 \leqq t \leqq 1),$$

to which corresponds

$$\langle W_t, \alpha_t \rangle \in P(F) \qquad (0 \leqq t \leqq 1). \tag{2}$$

Evidently $\langle W_0, \alpha \rangle$[3] $= \langle W, \alpha \rangle$, $\langle W_1, \alpha_1 \rangle = \langle W', \alpha' \rangle$.

Finally, we show that (2) is a continuous curve in T_g, i. e.

$$\lim_{t' \to t} \langle W_{t'}, \alpha_{t'} \rangle = \langle W_t, \alpha_t \rangle$$

holds for any $t\,(0 \leqq t \leqq 1)$. For this purpose, we define a topological mapping $w = H_t(z)$ of $|z| \leqq 1$ onto $|w| \leqq 1$ by

$$H_t(re^{i\theta}) = r\Phi_t(e^{i\theta}) \qquad (0 \leqq r \leqq 1,\ 0 \leqq \theta < 2\pi;\ 0 \leqq t \leqq 1)$$

and, by making use of it, construct a topological mapping $q = h(p)$ of W_t onto W_t':

$$h(p) \begin{cases} p & \text{for } p \in f(W - A), \\ H_{t'} \circ H_t^{-1} & \text{for } p \longleftrightarrow w \in (|w| \leqq 1). \end{cases}$$

We can immediately see that h belongs to the homotopy-class $\alpha_{t'}\alpha_t^{-1}$. So that

3) W_0 means $W_{t=0}$; it is *not* W_0 of § 3 put on the center of T_g.

$$\begin{aligned}&\text{dist.}\ (\langle W_t, \alpha_t\rangle, \langle W_{t'}, \alpha_{t'}\rangle)\\ &\quad \leqq \log(\text{maximal dilatation of } h)\\ &\quad = \log \max_{|z|=1} \max\left(\left|\frac{d\Phi_t}{dz}\right| \Big/ \left|\frac{d\Phi_{t'}}{dz}\right|, \left|\frac{d\Phi_{t'}}{dz}\right| \Big/ \left|\frac{d\Phi_t}{dz}\right|\right)\end{aligned}$$

which converges to 0 for $t' \to t$, by (1).

Consequently, we see that $P(F)$ is a connected set in T_g.

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