# ON THE DISTRIBUTION OF COMPLETION TIMES FOR RANDOM COMMUNICATION IN THE TASK-ORIENTED GROUP WITH A SPECIAL STRUCTURE

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## § 1. Introduction.

Recently, A. Bavelas, L. S. Christie, R. D. Luce and J. Macy, Jr. have introduced the task-oriented group. According to them, the task-oriented group consists of a number of individuals and a communication network. And each individual has initially one piece of information which must be transmitted to all the others to complete the task. At every sending time each individual sends all the information he has required to one other individual chosen at random from the possibilities given by the communication network.

By introducing a Markov chain H. G. Landau [1] has shown how to calculate the distribution of the completion times. But this method needs the transition probabilities $a_{\alpha\beta}$ from the information state $c^{(\alpha)}$ to $c^{(\beta)}$ after one sending time. And it seems to be generally difficult to calculate $a_{\alpha\beta}$.

Now we shall denote by $T(l, m, n)$ the task-oriented group with the network indicated by Figure 1 where the numbers of links in $\overrightarrow{ecd}$, $\overrightarrow{dbe}$ and $\overrightarrow{dae}$ are respectively $l$, $m$ and $n$. $T(l, m, n)$ is the simplest case from a topological view-point, because it is shown by R. D. Luce's theorem [2] that $T(l, m, n)$ is of order 1 free from tree form. In our paper we shall give the distributions of completion times of $T(l, m, n)$ for the following exclusive cases.

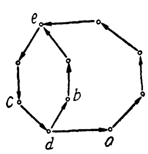

Figure 1.

Case (I) : $m = n \geqq 2$;
Case (II) : $m = 1,\ n \geqq 2$;
Case (III) : $m + 1 = n,\ m \geqq 2$;
Case (IV) : $m + 1 < n \leqq 2m,\ m \geqq 2$;
Case (V) : $2m < n,\ m \geqq 2$.

We can assume $m \leqq n$ without loss of generality owing to the symmetricity property on $m$ and $n$. And the discussion for case $m = n = 1$ is trivial. Hence we shall omit the case $m > n$ and $m = n = 1$ in this paper. All cases except these two are included in the above five cases.

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## § 2. An example of calculation of the distribution.

To simplify the explanation of our method we treat at first the task-oriented group $T(2, 2, 5)$ whose individuals are $a_1, a_2, \cdots, a_7$ and $b_1$. (See Figure 2.)

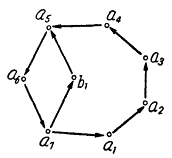
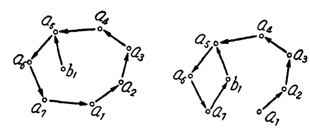


Figure 2.

Every individual except $a_7$ has then respectively a unique possible recipient while $a_7$ has two ones i. e. $a_1$ and $b_1$. These two choices give two operations of transmission. Let $A$ and $B$ be the sending operations by which $a_7$ takes $a_1$ and $b_1$ respectively, as his recipient and other individuals take their unique recipient. (See Figure 3.)

Then we can represent a system of trials in $N$ sending times as a permutation of $N$ elements which are respectively either $A$ or $B$. So we shall identify frequently each trial-system with its coresponding permutation.



Operation A. Operation B.
Figure 3.

We know easily that the completion times in this case are at least 7 (sending times), because $a_1$'s information must reach to $b_1$ in these times.

Now we study at first the system of 7 trials by which the task is completed. In order that all informations reach to $a_1$ it is necessary and sufficient that $a_2$'s information reaches to $a_1$. Hence this condition for $a_1$ becomes the one that at least one in the 6 th and 7 th trials is $A$ and can be represented by the diagram of Figure 4, where the five shaded parts show that the first five trials are arbitrary ones in $A$ and $B$ and the other two parts show that at least one in the 6 th and 7 th trials is $A$.

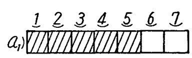



Figure 4.

By the similar considerations for $a_2, a_3, \cdots, a_7$ and $b_1$ we get Figure 5 where the 7 th part of the last line coresponding to $b_1$ shows that the 7 th trial must be $B$. But the 5 th∼7 th lines are unnecessary and the condition for $a_3$ is included in the one for $a_4$. So the diagram of Figure 5 is reduced to the one of Figure 6.

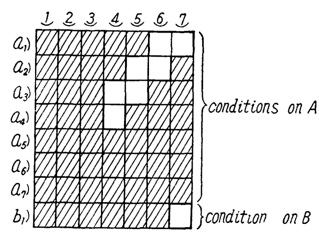



Figure 5.

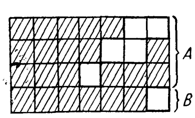



Figure 6.

The 3 rd and 4 th lines in this diagram inform that the 4 th and 7 th trials must be respectively $A$ and $B$, and consequently we know that the 6 th trial is also $A$.

Hence the type of the trial-systems completing the task must be $(*, *, *, A, *, A, B)$ where $*$ is an arbitrary trial in $A$ and $B$. Conversely it is obvious that the task is completed by the system of this type. Hence the number of the systems of 7 trials by which the task is completed is $2^4 = 16$ whose exponent 4 is the number of $*$-marks in the above type of trial-systems.

We refer to the number of the systems of $N$ trials by which the task is completed with $R'_N$ and also to the number of the systems of $N$ trials whose $N$th trial just completes the task with $R_N$. Evidently $R_N = R'_N - 2R'_{N-1}$, ($N \geqq 8$) and $R_7 = R'_7 = 16$ by the above discussion.

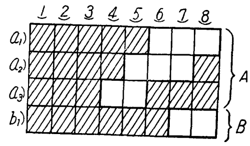


Figure 7.

Next to find $R'_8$ and $R_8$ we treat the systems of 8 trials. Discussing the condition under which the task is completed in this case, we get easily Figure 7 as the reduced diagram corresponding to Figure 6 in the preceding case.

Thereupon we consider the following two cases.

Case (1): The 7th trial is $A$;

Case (2): The 7th trial is $B$.

In the case (1) the conditions on $A$ indicated by the 1st and 2nd lines in Figure 7 are satisfied automatically and the 8th trial must be $B$ by the last line. So the 7th and 8th trials are determined. On the other hand we know from the 3rd line in Figure 7 that the 1st$\sim$3rd and 6th trials are arbitrary and at least one in the 4th and 5th trials must be $A$. Hence the number of the completing trial-systems in this case (1) is $(2^2 - 1)\cdot 2^4 = 6 \times 2^3$. Similarly in the case (2) the condition on $B$ indicated by the last line is satisfied and it is obvious also that the 1st$\sim$3rd trials are arbitrary. Illustrating the remained conditions on the 4th$\sim$6th and 8th trials, we get Figure 8. The number of the permutations which consist of $A$ and $B$ and satisfy the condition indicated by Figure 8 is

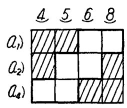


Figure 8.

$$H(3, 2, 1) = 2^{3+2+1} - 2^{3+1-2} - (3-1)\cdot 2^{3-2} = 8$$

by Lemma 1 of §3. Accordingly the number of the trial-systems in this case (2) is $8 \times 2^3$. Summing up, we get $R'_8 = 6 \times 2^3 + 8 \times 2^3 = 112$ and $R_8 = 112 - 2 \times 16 = 80$.

The similar considerations will give $R'_9 = 360$ and $R_9 = 136$.

Finally we shall deal with the systems of $N$ trials ($N \geqq 10$) which complete the task. The reduced diagram in this case coresponding to Figure 6 in the case $N = 7$ is the following Figure 9. This case is divided to the following three mutually exclusive cases.

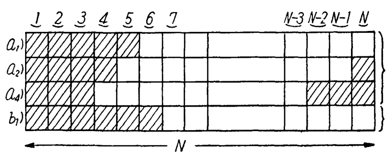


Figure 9.

Case (1): At least one $A$ and at least one $B$ exist respectively in the 7th$\sim$($N-3$)th trials;

Case (2): All of the 7th$\sim$($N-3$)th trials are $A$;

Case (3): All of the 7th$\sim$($N-3$)th trials are $B$.

In the case (1) the conditions on $A$ and $B$ indicated by Figure 9 has been satisfied completely. Hence the number of the completing systems in this case (1) is $(2^{N-9} - 2)\cdot 2^9 = 2^N - 2^{10}$. In the case (2) only the condition on $B$

indicated by the last line of Figure 9 remains, because the conditions on $A$ are wholly satisfied. So the number of the completing systems in the case (2) is $(2^3-1)\times 2^6 = 56\times 2^3$. In the case (3) the conditions of Figure 9 are reduced to those on the 4th$\sim$6th and $(N-2)$th$\sim N$ th trials which are represented as Figure 10. Of course the 1st$\sim$3rd trials are arbitrary. Hence the corresponding diagrams have been omitted. However, the number of the permutations which consist of $A$ and $B$ and satisfy the conditions of Figure 10 is

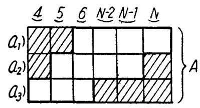


Figure 10.

$$H(3,3,2) = 2^{3+3+2-2} - 3^{3+2-2} - (3-1)\cdot 2^{3-2} = 52.$$

Hence the number of the completing systems in this case (3) is $52\times 2^3$. Summing up, we know that

$$R'_N = 2^N - 2^{10} + 56\times 2^3 + 52\times 2^3$$

$$= 2^N - 160 \qquad \text{for} \quad N \geqq 10,$$

$$R_{10} = 144$$

and

$$R_N = 2^N - 160 - 2\cdot(2^{N-1} - 160)$$

$$= 160 \qquad \text{for} \quad N \geqq 11.$$

If the result of the $N$th trial is denoted by $S_N$, then

$$P(S_N = A) = P(S_N = B) = 1/2$$

and all trials are mutually independent. (See Landau [1].) Hence the probability that the completion time $T$ is equal to $N$ is $R_N/2^N$. So the distribution of completion time $T$ in this task-oriented group is as follows:

$$P(T=7) = 16/2^7, \qquad P(T=8) = 80/2^8, \qquad P(T=9) = 136/2^9,$$

$$P(T=10) = 144/2^{10} \quad \text{and} \quad P(T=N) = 160/2^N \quad \text{for} \quad N \geqq 11.$$

Moreover, the mean completion time is

$$E(T) = \frac{7\times 160}{2^7} + \frac{8\times 80}{2^8} + \frac{9\times 136}{2^9} + \frac{10\times 144}{2^{10}} + \sum_{N=11}^{\infty} \frac{160N}{2^N}$$

$$= 9.0468\cdots$$

by Lemma 3 of §3.

We shall retain the notations $T$, $R_N$ and $R'_N$ used in this section also in the subsequent sections for treating the general cases.

## 3. Some lemmas for general cases.

DEFINITION 1. We denote by $H(\alpha,\beta,\gamma)$ the number of the permutations of $(\alpha+\beta+\gamma-2)$ elements which consist of $A$ and $B$ and satisfy the following conditions:

Condition 1: There is at least one $A$ among the 1st$\sim\beta$th positions;
Condition 2: There is at least one $A$ among tne 2nd$\sim(\beta+\gamma)$th positions;
Condition 3: There is at least one $A$ among the 3rd$\sim(\beta+\gamma+1)$th positions;
.......................................;
Condition $\alpha$: There is at least one $A$ among $\alpha$th$\sim(\alpha+\beta+\gamma-2)$th positions.

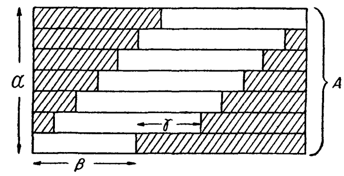


Figure 11.

(See Figure 11.)

LEMMA 1.

$$H(\alpha,\beta,\gamma) = 2^{\alpha+\beta+\gamma-2} - 2^{\alpha+\gamma-2} - (\alpha-1)\cdot 2^{\alpha-2} \qquad \textit{for} \quad \alpha \leqq \beta+1,$$
$$= 2^{\alpha+\beta+\gamma-2} - 2^{\alpha+\gamma-2} - \beta 2^{\alpha-2} - (\alpha-\beta-1)(2^{\beta}-1)\cdot 2^{\alpha-\beta-1}$$
$$\textit{for} \quad \beta+1 < \alpha \leqq \beta+\gamma+1.$$

*Proof.* We can easily see that

$$H(1,\beta,*) = 2^{\beta}-1,$$
$$H(2,\beta,\gamma) = 2^{\gamma}H(1,\beta,*)-1,$$

and

$$H(\alpha,\beta,\gamma) = 2H(\alpha-1,\beta,\gamma) - 2^{\alpha-2} \qquad \text{for} \quad 2 \leqq \alpha \leqq \beta+1$$
$$= 2H(\alpha-1,\beta,\gamma) - (2^{\beta}-1)\cdot 2^{\alpha-\beta-1}$$
$$\text{for} \quad \beta+2 \leqq \alpha \leqq \beta+\gamma+1.$$

The conclusion of the lemma is then obtained from these recurrence formulas for $\alpha$.

LEMMA 2. *If* $\beta+\gamma+1<\alpha$, *then*

$$H(\alpha,\beta,\gamma) = 2H(\alpha-1,\beta,\gamma) - H(\alpha-\beta-\gamma,\beta,\gamma).$$

LEMMA 3.

$$\sum_{N=r}^{\infty} \frac{N}{2^N} = \frac{r+1}{2^{r-1}}.$$

DEFINITION 2. We denote by $K(\alpha,\beta,\gamma;\delta,\varepsilon)$ the number of permutations of $(\alpha+\beta+\gamma+\delta-2)$ elements which consist of $A$ and $B$ and satisfy the following conditions:

Condition 1: There is at least one $A$ among the 1st$\sim\beta$th positions;
Condition 2: There is at least one $A$ among the 2nd$\sim(\beta+\gamma)$th positions;
..................................;

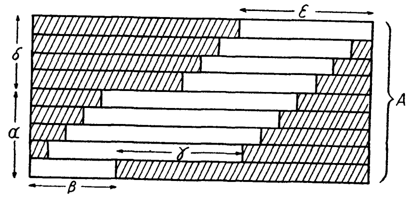


Figure 12.

Condition $\alpha$: There is at least one $A$ among $\alpha$ th$\sim(\alpha+\beta+\gamma-2)$ th positions;

Condition $(\alpha+1)$: There is at least one $A$ among $(\alpha+\beta+\gamma-\varepsilon)$ th $\sim(\alpha+\beta+\gamma-1)$ th positions;

.........................................;

Condition $(\alpha+\delta)$: There is at least one $A$ among $(\alpha+\beta+\gamma+\delta-\varepsilon-1)$ th$\sim(\alpha+\beta+\gamma+\delta-2)$ th positions.

(See Figure 12.)

LEMMA 4.

$$K(\alpha,\beta,\gamma;1,\varepsilon)=2H(\alpha,\beta,\gamma)-(2^{\beta+\gamma-\varepsilon}-1)\cdot 2^{\alpha-1} \quad \textit{for} \quad \alpha+\gamma-1\leqq\varepsilon,$$

$$=2H(\alpha,\beta,\gamma)-(2^{\beta}-1)(2^{\beta+\gamma-\varepsilon}-1)\cdot 2^{\alpha-\beta-1}$$

$$\text{for} \quad \alpha-1\leqq\varepsilon\leqq\alpha+\gamma-2 \quad \textit{and} \quad \beta\leqq\alpha-1,$$

$$=2H(\alpha,\beta,\gamma)-2^{\alpha+\beta+\gamma-\varepsilon-1}+2^{\alpha+\gamma-\varepsilon-1}+1$$

$$\textit{for} \quad \alpha-1\leqq\varepsilon\leqq\alpha+\gamma-2 \quad \textit{and} \quad \alpha\leqq\beta.$$

*And if* $\varepsilon\leqq\alpha-2$, *then*

$$K(\alpha,\beta,\gamma;1,\varepsilon)=2H(\alpha,\beta,\gamma)-K(\alpha-\varepsilon,\beta,\gamma;1,\beta+\gamma-\varepsilon).$$

LEMMA 5.

$$K(\alpha,\beta,\gamma;\delta,\varepsilon)=2K(\alpha,\beta,\gamma;\delta-1,\varepsilon)-K(\alpha,\beta,\gamma;\delta-\varepsilon-1,\varepsilon)$$

$$\textit{for} \quad \delta\geqq\varepsilon+2,$$

$$=2K(\alpha,\beta,\gamma;\delta-1,\varepsilon)-H(\alpha+\delta-\varepsilon-1,\beta,\gamma)$$

$$\textit{for} \quad 2\leqq\delta\leqq\varepsilon+1 \quad \textit{and} \quad \alpha+\delta\geqq\varepsilon+3,$$

$$=2K(\alpha,\beta,\gamma;\delta-1,\varepsilon)-H(2,\beta,\alpha+\gamma+\delta-\varepsilon-3)$$

$$\textit{for} \quad 2\leqq\delta\leqq\varepsilon+1 \quad \textit{and} \quad \alpha+\gamma+\delta\geqq\varepsilon+3>\alpha+\delta,$$

$$=2K(\alpha,\beta,\gamma;\delta-1,\varepsilon)-2^{\alpha+\beta+\gamma+\delta-\varepsilon-3}$$

$$\textit{for} \quad 2\leqq\delta\leqq\varepsilon+1 \quad \textit{and} \quad \varepsilon+3>\alpha+\gamma+\delta.$$

## § 4. The case where $m=n\geqq 2$.

In this case we get Figure 14 as the diagram corresponding to the condition that the system of $N$ trials completes the task. From it we know that these conditions are equivalent to that $A$ and $B$ exist respectively at least one by one in $(l+m)$ th$\sim(N-m+2)$ th trials. (See Figure 15.)

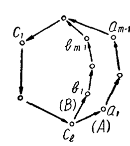



Figure 13.

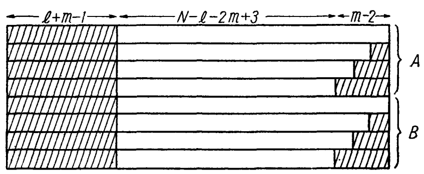



Figure 14.

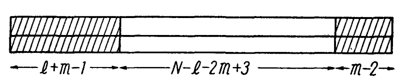



Figure 15.

Hence

$$R'_N = (2^{N-l-2m+3} - 2)\,2^{l+m-1+m-2}$$
$$= 2^N - 2^{f-1} \qquad \text{for } N \geqq f,$$

where $f = l + 2m - 1$ is the minimum completion time. So we have

$$R_N = R'_N - 2R'_{N-1} = 2^{f-1} \qquad \text{for } N > f$$

and

$$R_f = 2^{f-1}.$$

Consequently we get the following theorem.

THEOREM 1. *In the case of $T(l, m, n)$ where $m = n \geqq 2$, we have that*

$$P(T = N) = \frac{2^{f-1}}{2^N} \qquad \textit{for } N \geqq f,$$

*where $f = l + 2m - 1$. And the mean completion time is*

$$E(T) = f + 1.$$

## § 5. The case where $m = 1$ and $n \geqq 2$.

When a system of trials in this case completes the task, then it satisfies the conditions on $A$ indicated by Figure 17.
($N \geqq$ min. compl. time $f = l + n - 1$.)

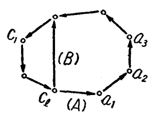



Figure 16.

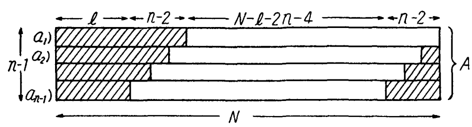



Figure 17.

Hence $R'_N = 2^l H(n-1,\ N-l-n+2,\ 1)$. From it we can get the distribution of the completion time.

THEOREM 2. *In the case of $T(l, 1, n)$ where $n \geqq 2$, we have*

$$P(T = N) = \frac{R'_f}{2^f}, \qquad \textit{for } N = f,$$
$$= \frac{R'_N - 2R'_{N-1}}{2^N} \qquad \textit{for } N > f$$

*where $f \equiv$ min. compl. time $= l + n - 1$ and*

$$R'_N = 2^l H(n-1,\ N-l-n+2,\ 1).$$

REMARK. Especially $T(l, 1, 2)$ gives us, as the distribution of its completion time $T$,

$$P(T = N) = \frac{2^{f-1}}{2^N} \qquad \textit{for } N \geqq f,$$

because

$$R'_N = 2^l(2^{N-l} - 1) = 2^N - 2^{f-1}$$

and

$$R_N = R'_N - 2R'_{N-1} = 2^{f-1}$$

for $N \geqq f$. And its mean completion time is

$$E(T) = \sum_{N=f}^{\infty} N \cdot \frac{2^{f-1}}{2^N} = f + 1,$$

## § 6. The case of $T(l, m, m+1)$ where $m \geqq 2$.

The conditions that a system of $N$ trials completes the task are shown by the Figure 18.

($N \geqq$ min. compl. time $f = l + 2m - 1$.)

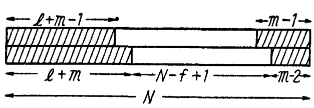



Figure 19.

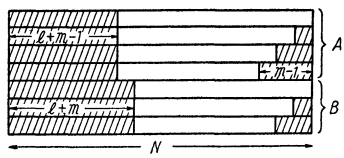



Figure 18.

From it we get Figure 19 as the reduced diagram. Hence

$$R'_f = R_f = 2^{l+2m-3} = 2^{f-2}$$

and

$$R'_N = (2^{N-f} - 2) \cdot 2^f + 2^{l+2m-2} + 2^{l+2m-2}$$
$$= 2^N - 2^f \qquad \text{for } N > f,$$

from which we have

$$R_f = 2^{f-2}, \qquad R_{f+1} = 2^{f+1} - 2^f - 2 \cdot 2^{f-2} = 2^{f-1}$$

and

$$R_N = 2^N - 2^f - 2^N + 2^{f+1} = 2^f \qquad \text{for } N > f + 1.$$

And the mean completion time is

$$E(T) = \frac{f}{4} + \frac{f+1}{4} + \sum_{N=f+2}^{\infty} \frac{N \cdot 2^f}{2^N} = f + \frac{7}{4} = l + 2m + \frac{3}{4}.$$

THEOREM 3. *In the case of $T(l, m, m+1)$ where $m \geqq 2$, we have*

$$P(T = f) = P(T = f + 1) = \frac{1}{4}$$

*and*

$$P(T = N) = 2^{f-N} \qquad \textit{for } N \geqq f + 2,$$

*where $f = l + 2m - 1$.*

*The mean completion time is*

$$E(T) = f + \frac{7}{4} = l + 2m + \frac{3}{4}.$$

## § 7. The case of $T(l, m, n)$ where $m + 1 < n \leqq 2m$ and $m \geqq 2$.

We shall obtain at once Figures 20 and 21 as the reduced diagrams.

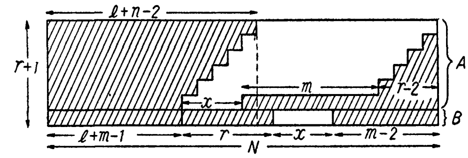


Figure 20. $(1 \leqq x \leqq r)$

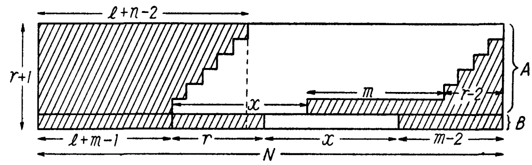


Figure 21. $(r < x)$

The former corresponds to the case $1 \leqq x \leqq r$ i.e. $N \leqq l + 2n - 3$ and the latter to the case $r < x$ i.e. $N > l + 2n - 3$, where $x = N - f + 1$, $r = n - m$ and the min. compl. time $f = l + m + n - 2$. From them we get

$$R'_N = (2^x - 2)(2^x - 1)\cdot 2^{N-2x} + 2^{l+m-1}\cdot H(r, x, m - x) \quad \text{for} \quad 1 \leqq x \leqq r,$$

$$= (2^{x-r} - 2)\cdot 2^{N-x+r} + (2^r - 1)\cdot 2^{N-x} + 2^{l+m-1}\cdot H(r, r, m)$$

$$\text{for} \quad r + 1 \leqq x.$$

Since we have by Lemma 1 of § 3

$$H(r, x, m - x) = 2^{r+m-2} - 2^{r+m-x-2} - x\cdot 2^{r-2} - (r - x - 1)(2^x - 1)\cdot 2^{r-x-1}$$

$$\text{for} \quad 1 \leqq x \leqq r - 2,$$

$$H(r, r - 1, m - r + 1) = 2^{n-2} - 2^{m-1} - (n - m - 1)\cdot 2^{n-m-2},$$

$$H(r, r, m - r) = 2^{n-2} - 2^{m-2} - (n - m - 2)\, 2^{n-m-2}$$

and

$$H(r, r, m) = 2^{n+r-2} - 2^{n-2} - (r - 1)\, 2^{r-2},$$

we can calculate $R'_N$, the distribution and the mean of the completion times.

THEOREM 4. *In the case of $T(l, m, n)$ where $m + 1 < n \leqq 2m$ and $m \geqq 2$, $f \equiv$ min. compl. time $= l + m + n - 2$. And we have*

$$P(T = f) = \frac{R'_f}{2^f}$$

*and*

$$P(T=N) = \frac{R'_N - 2R'_{N-1}}{2^N} \qquad \textit{for} \quad N > f,$$

*where*

$$R'_N = 2^N - 2^f + 2^{2f-N-2} - (N-f+1)2^{l+n-3}$$
$$- (l+2n-N-4)(2^{N-f+1}-1)\cdot 2^{2l+m+2n-N-5}$$
$$\textit{for} \quad f \leqq N \leqq l+2n-5,$$

$$R'_{l+2n-4} = 2^{l+2n-4} - 2^f + 2^{l+2m-2} - (n-m-1)\cdot 2^{l+n-3},$$

$$R'_{l+2n-3} = 2^{l+2n-3} - 2^f + 2^{l+2m-3} - (n-m-2)\cdot 2^{l+n-3}$$

*and*

$$R'_N = 2^N - 2^f - (n-m-1)\cdot 2^{l+n-3} \qquad \textit{for} \quad N \geqq l+2n-2.$$

REMARK. When we require only the mean completion time, it is convenient to use the following lemma.

LEMMA 6.

$$E(T) = \lim_{N\to\infty}\left\{\frac{NR'_N}{2^N} - \sum_{\nu=f}^{N-1}\frac{R'_\nu}{2^\nu}\right\}.$$

*Proof.*

$$E(T) = \lim_{N\to\infty}\sum_{\nu=\infty}^{N}\frac{\nu R_\nu}{2^\nu} = \lim_{N\to\infty}\sum_{\nu=f}^{N}\frac{\nu(R'_\nu - 2R'_{\nu-1})}{2^\nu}$$
$$= \lim_{N\to\infty}\left\{\frac{NR'_N}{2^N} - \sum_{\nu=f}^{N-1}\frac{R'_\nu}{2^\nu}\right\}.$$

## § 8. The case of $T(l, m, n)$ where $2m < n$ and $m \geqq 2$.

In this case we also get

$$R'_N = 2^N - 2^f - (r-1)2^{f-m-1} \qquad \textit{for} \quad x > r \quad \text{i. e.} \quad N \geqq l+2n-2$$

in the similar manner as in the preceding section where $x = N-f+1$, $r = n-m$ and $f = l+m+n-2$. Next we consider the case $1 \leqq x \leqq r$.

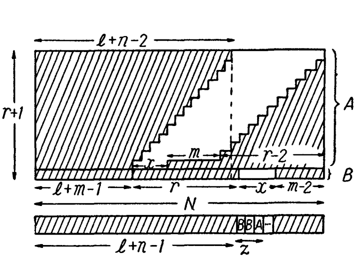


Figure 22. $(1 \leqq x \leqq r-m)$

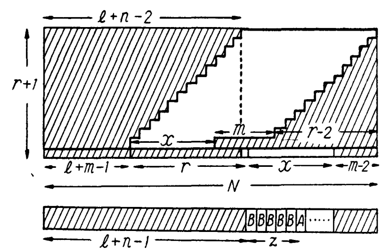


Figure 23. $(r-m+1 \leqq x \leqq r)$

Turning our attention to the $(l+n-1+z)$th trial which is the first $A$ after $(l+n-1)$th trial, we can express $R'_N$ in terms of $H(\alpha, \beta, \gamma)$ and $K(\alpha, \beta, \gamma; \delta, \varepsilon)$.

THEOREM 5. *In the case of $T(l, m, m)$ where $2m < n$ and $m \geqq 2$ we have*

$$P(T=f) = \frac{R'_f}{2^f},$$

$$P(T=N) = \frac{R'_N - 2R'_{N-1}}{2^N} \qquad \textit{for} \quad N > f,$$

*and*

$$R'_N = 2^{l+m-1}\Big\{\sum_{z=1}^{x} K(r-m+2-x, x, m; 1, x+m-z)\cdot 2^{x-z+m-2}$$

$$+ K(r-m+2-x, x, m; m-1, m-1)\Big\}$$

$$\textit{for} \quad 1 \leqq x \leqq r-m,$$

$$= 2^{l+m-1}\Big\{(2^{x-1}-1)(2^x-1)2^{r+m-x-2} + \sum_{z=2}^{x-r+m}(2^x-1)\cdot 2^{r+m-z-2}$$

$$+ \sum_{z=x-r+m+1}^{x}(2^r - 2^{z-x+r-m} - 2^{r-x}+1)\cdot 2^{x-z+m-2}$$

$$+ 2^{r-m+1}H(m-1, m-1, 1) - H(r-x, r-x, m-r+x)\Big\}$$

$$\textit{for} \quad r-m+1 \leqq x \leqq r,$$

$$= 2^N - 2^f - (r-1)2^{f-m-1} \qquad \textit{for} \quad x > r,$$

*where* $x = N-f+1$, $r = n-m$, $H(1, 1, m-1) = 2^{m-2}$, $H(0, 0, m) = 2^{m-3}$ *for* $m \geqq 3$ *and* $H(0, 0, 2) = 0$.

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