ON APPROXIMATION BY TRIGONOMETRIC POLYNOMIALS

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In a paper [1], S. Bochner stated the following proposition:

Let $\{\sigma_n(x)\}$ *be a sequence of exponential polynomials, each* $\sigma_n(x)$ *being at most of order* n. *If*

$$\frac{1}{2\pi}\int_0^{2\pi} |\sigma_n(x)|\, dx \leqq M$$

and if there exist three constants α, ε, $C > 0$ *such that*

$$|\sigma_n(x)| \leqq C e^{-\varepsilon n}$$

for x *in*

$$-\alpha \leqq x \leqq \alpha,$$

then the sequence $\sigma_n(x)$ *converges to* 0 *uniformly in* $-\pi \leqq x \leqq \pi$.

However, this proposition is not true in general. In fact, M. E. Noble [2] proved that:

For any fixed $\delta > 0$ *and for every positive integer* n, *there exists a trigonometrical polynomial* $T_n(x)$ *at most of order* n *with constant term* 1 *satisfying the following conditions*:

(i) *there exists an absolute constant* A *such that*

$$|T_n(x)| \leqq A/\delta \qquad \text{for } -\pi \leqq x \leqq \pi,$$

(ii) *for* $\delta \leqq |x| \leqq \pi$

$$|T_n(x)| = O(\exp(-A(\delta)n)), \qquad A(\delta) > 0.$$

The purpose of the present paper is to prove a correct version of Bochner's proposition mentioned above.

THEOREM. *Let* $\{s_n(x)\}$ *be a sequence of exponential polynomials, each* $s_n(x)$ *being at most of order* n. *If*

$$\frac{1}{2\pi}\int_0^{2\pi} |s_n(x)|\, dx \leqq M, \tag{1}$$

and if there exist two constants ε, $C > 0$ *and an interval* (a, b) *of length* r *such that*

$$|s_n(x)| \leqq Ce^{-\varepsilon n} \qquad \text{in } (a, b) \tag{2}$$

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and if $\varepsilon r > 2\pi\rho$, *where* $\rho = \min_{0<t<\infty} 2t(e^t+1)/(e^t-1)$, *then* $s_n(x)$ *converges uniformly to zero in the interval* $-\pi \leqq x \leqq \pi$.

Proof. The proof is almost nothing but a repetition of Bochner's one. Let t_0 be the value of t at which the function $2t(e^t+1)/(e^t-1)$ $(t>0)$ takes on the minimum value, then $0 < t_0 < \infty$. We can suppose that the constant M in (1) satisfies the condition

$$M \cdot \frac{e^{t_0}}{e^{t_0}-1} < 1. \tag{3}$$

We can write $s_n(x)$ in the form

$$s_n(x) = \sum_{\nu=0}^{2n} a_{n\nu}\, e^{i(\nu-n)x},$$

and then it follows from (1) that

$$|a_{n\nu}| \leqq M. \tag{4}$$

Now let us set

$$F_n(z) = \sum_{\nu=0}^{2n} a_{n\nu}\, z^\nu, \qquad z = e^t e^{ix},$$
$$G_n(z) \doteq F_n(z)\, z^{-2n}$$

and

$$M_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \log|G_n(e^t e^{ix})|\, dx,$$

then as was shown by Bochner,

$$M_n(t) \leqq M_n(0) \qquad \text{for} \qquad t > 0. \tag{5}$$

(This can be seen alternatively from the fact that the function $\log|G_n(z)|$ is subharmonic in the domain obtained by excluding the origin). It follows by (2) that for sufficiently large n

$$M_n(0) = \frac{1}{2\pi}\int_0^{2\pi} \log|F_n(e^{ix})|\, dx \leqq -\frac{r\varepsilon n}{2\pi} + C_1, \tag{6}$$

and by (4) that

$$\begin{aligned} |G_n(e^t e^{ix})| &= \left|\sum_{\nu=0}^{2n} a_{n\nu}\, e^{(\nu-2n)t}\, e^{(\nu-2n)ix}\right| \\ &\leqq M \sum_{\nu=0}^{\infty} \frac{1}{e^{\nu t}} = M \cdot \frac{e^t}{e^t-1}. \end{aligned} \tag{7}$$

Therefore we have by (3)

$$|G_n(e^{t_0} e^{ix})| < 1. \tag{8}$$

Applying Poisson–Jensen's formula and remembering that the function $G_n(z)$ has a pole at most of order $2n$ at the origin, we have by (5)

$$\log|F_n(e^{ix})| = \log|G_n(e^{ix})|$$
$$\leqq \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{2t_0}-1}{e^{2t_0}-2e^{t_0}\cos(x-y)+1}\log|G_n(e^{t_0}e^{iy})|\,dy + 2nt_0.$$

From (7), (5) and (6) we obtain

$$\log|F_n(e^{ix})| \leqq \frac{1}{2\pi}\cdot\frac{e^{t_0}-1}{e^{t_0}+1}\cdot\int_0^{2\pi}\log|G_n(e^{t_0}e^{iy})|\,dy + 2nt_0$$
$$\leqq \frac{e^{t_0}-1}{e^{t_0}+1}\left\{-n\left(\frac{r\varepsilon}{2\pi}-\frac{2(e^{t_0}+1)\,t_0}{e^{t_0}-1}\right)+C_1\right\}.$$

Since

$$r\varepsilon > 2\pi\rho = 2\pi\,\frac{2t_0(e^{t_0}+1)}{e^{t_0}-1}$$

by the assumption of the theorem, it follows that

$$\log|s_n(x)| = \log|F_n(e^{ix})| \leqq \frac{e^{t_0}-1}{e^{t_0}+1}\{-n\alpha+C_1\},$$

where we set

$$\alpha = r\varepsilon - 2\pi\rho.$$

This is equivalent to

$$|s_n(x)| \leqq C_2 e^{-\beta n}, \qquad \beta = \frac{e^{t_0}-1}{e^{t_0}+1}\,\alpha > 0,$$

which proves the theorem.

References

[1] S. Bochner, Localization of best approximation, Contribution to Fourier Analysis. Annals of Mathematics Studies **25** (1950), 3—23.

[2] M. E. Noble, Coefficient properties of Fourier series with a gap condition. Math. Annalen **128** (1954), 55—62.

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