# ON SOME LIMIT THEOREMS FOR THE SUMS OF IDENTICALLY DISTRIBUTED INDEPENDENT RANDOM VARIABLES

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The contents of this note contain two different parts. In § 1, we are concerned with the renewal theory, and in § 2 a limit theorem for probability densities.

## 1. Some extensions of the result of Lévy for the coin-tossing game.

In this section, we are concerned with the distributions of the number of zeros of the partial sums of the independent and identically lattice distributed random variables. Let $X_1, X_2,\cdots, X_n,\cdots$ be identically lattice distributed independent random variables. We assume, without loss of generality, that $X_1, X_2,\cdots, X_n,\cdots$ are integral valued random variables with span 1. In the coin-tossing game,

$$\Pr\{X_i = 1\} = \Pr\{X_i = -1\} = \frac{1}{2}.$$

Let $S_k = X_1 + X_2 \cdots + X_k$, $k = 1, 2, \cdots$, and let $N_n$ denote the number of $S_k$'s, $1 \leqq k \leqq n$, which are zero. In the coin-tossing game, the following result is known [1, p. 253].

$$\begin{aligned} \lim_{n\to\infty} \Pr\left\{\frac{N_n}{\sqrt{n}} \leqq x\right\} &= \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} dt \qquad &&\text{for} \quad x \geqq 0, \\ &= 0 &&\text{for} \quad x < 0. \end{aligned}$$

For a fixed integer $j$, even if we denote by $N_n$ the number of $S_k$'s, $1 \leqq k \leqq n$, $S_k = j$, the above result is also obviously true.

Now it seems to me that the following extension of this result was not yet given in references explicitly.

THEOREM 1. *Let $X_1, X_2, \cdots, X_n, \cdots$ be identically lattice distributed independent random variables taking only integral values, and let its span be* 1.[1] *We assume also that*

$$(*) \qquad EX_i = 0, \qquad D^2X_i = \sigma^2.$$

*Then*

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1) That is, the greatest common divisor of all differences $k-j$ for which $\Pr\{X_i=k\} > 0$, $\Pr\{X_i=j\}$ is equal to unity.

$$(1) \qquad \lim_{n\to\infty} \Pr\left\{\frac{N_n}{\sqrt{n}\,\sigma} \leqq x\right\} = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2/2}dt \qquad \text{for} \quad x \geqq 0,$$
$$= 0 \qquad \text{for} \quad x < 0.$$

*where* $N_n$ *is the number of* $\sum_{i=1}^{k} X_i = S_k$, $1 \leqq k \leqq n$, *such that* $S_k = j$, *for a fixed integer* $j$.

THEOREM 2. *If the assumptions of Theorem* 1 *are valid, except the condition* (*), *and if for some* $\alpha$, $1 < \alpha < 2$,

$$(2) \qquad \lim_{n\to\infty} \Pr\left\{\frac{S_n}{cn^{1/\alpha}} \leqq x\right\} = V_\alpha(x)$$

*where* $V_\alpha(x)$ *is a symmetric stable distribution with exponent* $\alpha$, *then*

(i) *for* $1 < \alpha < 2$, *we have*

$$(3) \qquad \lim_{n\to\infty} \Pr\left\{\frac{N_n}{c^{-1}\pi^{-1}\alpha^{-1}\Gamma(1/\alpha)\Gamma(1-1/\alpha)n^{1-1/\alpha}} \leqq x\right\}$$
$$= 1 - G_{1-1/\alpha}\left(\left(\frac{x}{\Gamma(1-1/\alpha)}\right)^{-\frac{1}{1-1/\alpha}}\right),$$

*where* $G_\beta(z)$ *is the stable distribution defined by the characteristic function*

$$\gamma_\beta(z) = \exp\left\{-|z|^\beta\left(\cos\frac{\pi\beta}{2} - i\sin\frac{\pi\beta}{2}\operatorname{sgn} z\right)\Gamma(1-\beta)\right\},$$

(ii) *for* $\alpha = 1$, *we have*

$$(4) \qquad \lim_{n\to\infty} \Pr\left\{\frac{N_n}{c^{-1}\pi^{-1}\log n} \leqq x\right\} = \int_0^x e^{-t}dt, \qquad \textit{for} \quad x \geqq 0,$$

*and*
$$= 0 \qquad \textit{for} \quad x < 0,$$

(iii) *for* $\alpha < 1$, $\{N_n\}$ *is bounded with probability* 1.

To prove Theorem 1 and Theorem 2, we shall use the following lemmas.

LEMMA 1. *Under the assumptions of Theorem* 1, *we have*

$$\Pr\{S_k = k\} = \frac{1}{\sqrt{2\pi}\sigma} n^{-1/2} + c_{n,k},$$

*for any fixed integer* $k$, *where* $c_{n,k} = o(n^{-1/2})$.

LEMMA 2. *Under the assumptions of Theorem* 2, *we have*

$$\Pr\{S_k = k\} = c^{-1}\pi^{-1}\alpha^{-1}\Gamma(1/\alpha)n^{-1/\alpha} + c_{n,k},$$

*for any fixed integer* $k$, *where* $c_{n,k} = o(n^{-1/\alpha})$.

These lemmas are easy consequences of the local limit theorems of Gnedenko–Kolmogorov [3, p. 233, 236].

Now the proofs of Theorem 1 and Theorem 2, are the same as in G. Kallianpur and H. Robbins's proof [4, Theorem 3.1]. For example, the proof of (i) in Theorem 2 is as follows.

Let

$$\xi_j = 1 \quad \text{if} \quad S_j = k,$$
$$= 0 \quad \text{otherwise}.$$

Then

$$N_n = \sum_{j=1}^{n} \xi_j.$$

We have

$$EN_n = \sum_{j=1}^{n} E\xi_j = \sum_{j=1}^{n} \Pr\{S_j = k\}$$

$$= c^{-1}\pi^{-1}\alpha^{-1}\Gamma(1/\alpha) \sum_{j=1}^{n} j^{-1/\alpha} + \sum_{j=1}^{n} c_{j,n}.$$

Since

$$\sum_{j=1}^{n} j^{-1/\alpha} \sim \frac{n^{1-1/\alpha}}{1-1/\alpha}, \qquad \sum_{j=1}^{n} c_{j,k} = o(n^{1-1/\alpha})$$

by Lemma 2, we have

$$EN_n \sim \frac{n^{1-1/\alpha}}{1-1/\alpha}. \tag{5}$$

For any positive integer $r$ $(r \geqq 2)$,

$$\begin{aligned} EN_n^{\,r} &= \sum_{j_1=1}^{n} \cdots \sum_{j_r=1}^{n} E\{\xi_{j_1}\xi_{j_2}\cdots\xi_{j_r}\} \\ &= \sum_{j_1=1}^{n} E\xi_{j_1}^{r} + r\sum_{1\leq j_1<j_2\leq n} E\xi_{j_1}^{r}\xi_{j_2} + \cdots + r!\sum_{1\leq j_1<\cdots<j_r\leq n} E\xi_{j_1}\xi_{j_2}\cdots\xi_{j_r}. \end{aligned} \tag{6}$$

Now

$$\begin{aligned} E\xi_{j_1}\xi_{j_2}\cdots\xi_{j_r} &= \Pr\{S_{j_1} = k,\ S_{j_2} = k,\ \cdots,\ S_{j_r} = k\} \\ &= \Pr\{S_{j_1} = k,\ S_{j_2} - S_{j_1} = 0,\ \cdots,\ S_{j_r} - S_{j_{r-1}} = 0\} \\ &= \Pr\{S_{j_1} = k\}\Pr\{S_{j_2} - S_{j_1} = 0\}\cdots\Pr\{S_{j_r} - S_{j_{r-1}} = 0\} \\ &= (c\pi\alpha)^{-r}\Gamma(1/\alpha)^r\,[j_1(j_2 - j_1)\cdots(j_r - j_{r-1})]^{-1/\alpha} + A_{j_1 j_2\cdots j_r}, \end{aligned}$$

say. Then, as in the proof of Kallianpur and Robbins, since

$$\sum_{1\leq j_1<j_2<\cdots<j_r\leq n} (c\pi\alpha)^{-r}\Gamma(1/\alpha)[j_1(j_2-j_1)\cdots(j_r-j_{r-1})]^{-1/\alpha}$$
$$\sim \frac{[\Gamma(1/\alpha)\Gamma(1-1/\alpha)]^r}{(c\pi\alpha)^r\Gamma(1+r(1-1/\alpha))} n^{r(1-1/\alpha)},$$
$$\sum_{1\leq j_1<j_2<\cdots<j_r\leq n} A_{j_1 j_2\cdots j_r} \sim o(n^{r(1-1/\alpha)}),$$

we have

$$r!\sum_{1\leq j_1<j_2<\cdots<j_r\leq n} E\xi_{j_1}\xi_{j_2}\cdots\xi_{j_r} \sim \frac{[\Gamma(1/\alpha)\Gamma(1-1/\alpha)]^r}{(c\pi\alpha)^r\Gamma(1+r(1-1/\alpha))} r!\,n^{r(1-1/\alpha)}. \tag{7}$$

Since

$$\sum_{1\leq j_1<j_2<\cdots<j_l\leq n} E\xi_{j_1}^{r_1}\xi_{j_2}^{r_2}\cdots\xi_{j_l}^{r_l} = \sum_{1\leq j_1<j_2<\cdots<j_l\leq n} E\xi_{j_1}\xi_{j_2}\cdots\xi_{j_l}$$

the other terms of (6) except the last one are, by (7),

$$o(n^{r(1-1/\alpha)}).$$

Thus we have

$$EN_n^r \sim \frac{1}{\Gamma[1+r(1-1/\alpha)]}\left[\frac{\Gamma(1/\alpha)\Gamma(1-1/\alpha)}{c\pi\alpha}\right]^r r!\, n^{r(1-1/\alpha)},$$

from which, by the same arguments as in the Kallianpur and Robbins's, we can complete the proof of (i) in Theorem 2.

Theorem 1 and (ii) of Theorem 2 can be proved in the similar manner. The result of (iii) of Theorem 2 is an easy consequence of Borel–Catelli's lemma.

**2. A frequency function from of central limit theorem.**

W. L. Smith [5] proved the following theorem:

Let $X_1, X_2, \cdots, X_n, \cdots$ be identically distributed independent random variables with a distribution function $F(x)$ and let its characteristic function be $\phi(t)$. If

(A) $$EX_i = 0, \qquad D^2X_i = 1$$

(B) $$|\phi(t)| \leqq A/|t|^\alpha, \qquad \text{for} \quad |t| \geqq R,$$

for some positive $A$, $R$, $\alpha$, then, for sufficiently large $n$, the random variables $S_n/\sqrt{n}$ have always probability densities $h_n(x)$ and it holds

$$\lim_{n\to\infty} |x|^l h_n(x) = \frac{|x|^l}{\sqrt{2\pi}} e^{-x^2/2} \qquad \text{for} \quad 0 \leqq l \leqq 2,$$

uniformly with respect to $x$ in the interval $(-\infty < x < \infty)$.

On the other hand, Gnedenko–Kolmogorov [3] proved the following theorem: If

(A) $$EX_i = 0, \qquad D^2X_i = 1,$$

(B′) if the probability density $p_m(x)$ of the sum $S_m$ exists for some $m \geqq 1$ and $p_m(x)$ belongs to the class $L^r(-\infty, \infty)$ for some $r$, $1 < r \leqq 2$, then

$$\lim_{n\to\infty} h_n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

uniformly with respect to $x$ in the interval $(-\infty < x < \infty)$.

Obviously the Gnedenko–Kolmogorov's conclusion is implied in Smith's. But for their assumptions, Smith's are contained in Gnedenko–Kolmogorov's. Because, under the assumptions of Smith's theorem, $\phi^m(t) \in L(-\infty, \infty)$ for $m > 1/\alpha$, and so by the inversion formula, the density function $p_m(x)$ of $S_m$ exists, and

$$2\pi p_m(x) = \int_{-\infty}^{\infty} e^{-itx}\phi^m(t)\,dt.$$

Thus $p_m(x)$ is bounded in the whole interval $(-\infty < x < \infty)$, from which, with $p_m(x) \in L$, it holds that $p_m(x)$ belongs to $L^r$ for all $r \geqq 1$.

Now we shall prove that the conclusion of Smith is also true under the assumptions of Gnedenko–Kolmogorov. That is:

THEOREM 3. *Under the assumptions of* (A), (B′),

$$\lim_{n\to\infty} |x|^l h_n(x) = \frac{|x|^l}{\sqrt{2\pi}} e^{-x^2/2} \qquad \textit{for } 0 \leqq l \leqq 2,$$

*uniformiy with respect to x in the interval* $(-\infty < x < \infty)$.

*Proof.* Since $p_m(x)$ belongs to the class $L^r$, by a theorem of Titchmarsh, we have

$$\phi^m(t) = \int_{-\infty}^{\infty} e^{itx} p_m(x)dx \in L^{r'} \qquad \text{for} \quad r' = \frac{r}{r-1}.$$

Thus

$$(1) \qquad p^n(s) \in L$$

for all $n \geqq mr/(r-1)$.

Let

$$\theta_n(t) \equiv \left\{\phi\left(\frac{t}{\sqrt{n}}\right)\right\}^n, \qquad n = 1, 2, \cdots,$$

then we have

$$(2) \quad \theta_n''(t) = (n-1)\left\{\phi'\left(\frac{t}{\sqrt{n}}\right)\right\}^2 \left\{\phi\left(\frac{t}{\sqrt{n}}\right)\right\}^{n-2} + \phi''\left(\frac{t}{\sqrt{n}}\right)\left\{\phi\left(\frac{t}{\sqrt{n}}\right)\right\}^{n-1}.$$

Since, by assumptions

$$\left|\phi'\left(\frac{t}{\sqrt{n}}\right)\right| \leqq E(|X_i|) \leqq 1, \qquad \left|\phi''\left(\frac{t}{\sqrt{n}}\right)\right| \leqq 1,$$

by (1),

$$(3) \qquad \theta_n''(t) \in L.$$

Clearly, we have

$$\theta_n(t) = \int_{-\infty}^{\infty} e^{itx} h_n(x)dx, \qquad \theta_n''(t) = -\int_{-\infty}^{\infty} e^{itx} x^2 h_n(x)dx,$$

and hence from a theorem on Fourier transform, using (3),

$$(4) \qquad 2\pi h_n(x) = \int_{-\infty}^{\infty} e^{itx}\theta_n(t)dt, \qquad 2\pi x^2 h_n(x) = -\int_{-\infty}^{\infty} e^{itx}\theta_n''(t)dt.$$

Thus we have

$$(5) \qquad x^2 h_n(x) - x^2 e^{-x^2/2} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}(\theta_n''(t) - (t^2-1)e^{-t^2/2})dt$$

To prove the theorem, it is sufficient to show that

$$R_n = \int_{-\infty}^{\infty} e^{-itx}(\theta_n''(t) - (t^2-1)e^{-t^2/2})dt \to 0, \qquad \text{as} \quad n \to \infty$$

uniformly with respect to $x$ $(-\infty < x < \infty)$.

Following after Kolmogorov–Gnedenko's arguments, we represent $R_n$ as the sum of four integrals:

$$I_1 = \int_{-A}^{A} e^{-itx}(\theta''_n(t) - (t^2-1)e^{-t^2/2})dt, \qquad I_2 = \int_{|t| \geqq A} e^{-itx}(t^2-1)e^{-t^2/2}dt,$$

$$I_3 = \int_{A \leqq |t| \leqq \varepsilon\sqrt{n}} e^{-itx}\theta_n''(t)dt, \qquad I_4 = \int_{|t| \geqq \varepsilon\sqrt{n}} e^{-itx}\theta_n''(t)dt,$$

where the number $A > 0$ depends on $\varepsilon$ arbitrarily given and will be chosen later.

By Lemma 2 of [5], it follows that

$$\lim_{n\to\infty} \theta_n''(t) = (t^2-1)e^{-t^2/2}$$

uniformly with respect to $t$ in every finite interval and hence for any constant $A$

$$I_1 \to 0 \qquad \text{as} \qquad n \to \infty$$

uniformly with respect to $x$ $(-\infty < x < \infty)$. Choosing $A$ sufficiently large, we have, obviously $I_2 < \varepsilon$.

$$I_3 = \int_{A \leqq |t| \leqq \varepsilon\sqrt{n}} e^{-itx}\theta_n''(t)dt = \int_{A \leqq |t| \leqq \varepsilon\sqrt{n}} (n-1)\left\{\phi'\left(\frac{t}{\sqrt{n}}\right)\right\}^2\left\{\phi\left(\frac{t}{\sqrt{n}}\right)\right\}^{n-2}dt$$
$$+ \int_{A \leqq |t| \leqq \varepsilon\sqrt{n}} \phi''\left(\frac{t}{\sqrt{n}}\right)\left\{\phi\left(\frac{t}{\sqrt{n}}\right)\right\}^{n-1}dt \equiv J_1 + J_2,$$

say.

Since, in the neighbourhood of the point $t = 0$,

$$\phi(t) = 1 - \frac{t^2}{2} + o(t^2), \qquad \phi'(t) = -t + o(t),$$

we have

$$|\phi(t)| \leqq 1 - \frac{t^2}{4} \leqq e^{-t^2/4} \tag{6}$$

and

$$\sqrt{n}\,\phi'\left(\frac{t}{\sqrt{n}}\right) = -t + \varepsilon_n t, \qquad \varepsilon_n \to 0 \qquad \text{as} \quad n \to \infty,$$

that is,

$$n\left|\phi'\left(\frac{t}{\sqrt{n}}\right)\right|^2 < t^2 + \varepsilon t^2 \tag{7}$$

for large $n$. Thus we have

$$|J_1| \leqq \int_{A \leqq |t| \leqq \varepsilon\sqrt{n}} (n-1) \left|\phi'\left(\frac{t}{\sqrt{n}}\right)\right|^2 \left|\phi\left(\frac{t}{\sqrt{n}}\right)\right|^{n-2} dt$$

$$\leqq 2\int_A^\infty (t^2 + \varepsilon t^2) e^{-\frac{n-2}{4n}t^2} dt < \frac{\varepsilon}{2},$$

$$|J_2| \leqq \int_{A \leqq |t| \leqq \varepsilon\sqrt{n}} \left|\phi\left(\frac{t}{\sqrt{n}}\right)\right|^{n-1} dt \leqq 2\int_A^\infty e^{-\frac{n-1}{4n}t^2} dt < \frac{\varepsilon}{2}$$

for sufficiently large $A > 0$. Thus we have

$$|I_3| < \varepsilon.$$

Since $p_m(t) \in L$, $\phi^m(t) \to 0$ as $|t| \to \infty$, by the theorem of Riemann-Lebesgue, that is $\phi(t) \to 0$ as $|t| \to \infty$. Hence there exists a constant $c > 0$ such that

$$|\phi(t)| < e^{-c} \qquad \text{for all } |t| \geqq \varepsilon.$$

Let $\beta > mr/(r-1)$ be a constant. Then

$$|I_4| \leqq \int_{|t| \geqq \varepsilon\sqrt{n}} |\theta_n''(t)| dt$$

$$\leqq \int_{|t| \geqq \varepsilon\sqrt{n}} (n-1) \left|\phi\left(\frac{t}{\sqrt{n}}\right)\right|^{n-2} dt + \int_{|t| \geqq \varepsilon\sqrt{n}} \left|\phi\left(\frac{t}{\sqrt{n}}\right)\right|^{n-1} dt$$

$$\leqq 2(n-1)e^{-(n-2-\beta)c} \int_{c\sqrt{n}}^\infty \left|\phi\left(\frac{t}{\sqrt{n}}\right)\right|^\beta dt + 2e^{-(n-1-\beta)c} \int_{\varepsilon\sqrt{n}}^\infty \left|\phi\left(\frac{t}{\sqrt{n}}\right)\right|^\beta dt$$

$$\leqq 2\sqrt{n}\{(n-1)e^{-(n-2-\beta)c} + e^{-(n-1-\beta)c}\} \int_\varepsilon^\infty |\phi(t)|^\beta dt \to 0 \quad \text{as } n \to \infty.$$

The above estimations complete the proof.

## References


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