REMARKS ON THE INFINITESIMAL RIGIDITY OF CLOSED CONVEX SURFACES

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1. Introduction. It is Efimov[2)] who pointed out that a closed convex surface is infinitesimally rigid outside its planar portions[3)] even if it contains some parts on which Gaussian curvature K vanishes identically. The proof[2)] given by Efimov depends upon the "Drehriss-method"[4)] which requires both the surface and the admissible infinitesimal deformation vectors to be, e. g., of class C'''. In this paper we shall give an alternative proof of the same theorem under less differentiability conditions to both the surface and the admissible infinitesimal deformation vectors, resorting to the methods developed in [2]. The following is the theorem to be proved here:

1.1. Theorem. *In the euclidean 3-space a closed convex surface S is infinitesimally rigid, outside its planar portions*[3)] *if it exist, if*

(*a*) *S is piecewise of class C'',*

(*b*) *the admissible infinitesimal* (*isometric*) *deformation vectors are of class C'.*

As we see in this theorem, no explicit restrictions are imposed upon the positiveness of Gaussian curvature of the surface S, but we only assume its non-negativeness or the convexity of the closed surface S in the weak sense that the surface S is a boundary of a convex body in 3-space and S may contain straight line segments. Next we assume that the surface S is piecewise of class C'', which means that the surface is of class C' as a whole and moreover it can be subdivided into finite number of pieces of surface, each of which is of class C'' including its piecewise smooth boundaries. That is to say, it is a surface of patchwork with respect to the second partial derivatives on S. Let $\mathfrak{x}$[5)] $= \mathfrak{x}(u, v)$ be a regular ($W > 0$) representation of the closed convex surface S, where u, v are local para-

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2) See [1]. The number in square brackets refers to the bibliography at the end of this paper.

3) For its exact meaning see 3.3 of this paper.

4) See [1] or [4].

5) German small letter respresents free or position vector.

meters. Let $\mathfrak{z}(u, v)$ be a vector function with the same parameters u, v and of class C'. If $\mathfrak{z}(u, v)$ satisfies the following partial differential equation for every line element everywhere on the surface S:

$$(1) \qquad d\mathfrak{z}\, d\mathfrak{x} = 0,$$

$\mathfrak{z}(u, v)$ is said to be an *infinitesimal* (*isometric*) *deformation vector of the surface* S. If any $\mathfrak{z}(u, v)$ which satisfies (1) is necessarily trivial, i. e., $\mathfrak{z}(u, v) = \mathfrak{a} \times \mathfrak{x} + \mathfrak{b}$, for two constant vectors $\mathfrak{a}$ and $\mathfrak{b}$, we say the surface $\mathfrak{x} = \mathfrak{x}(u, v)$ is *infinitesimally rigid*.

The proof of THEOREM 1.1 is divided into two parts, i. e., § 2 and § 3. In § 2 we recall the important integral formula which was developed in [2]. In fact this paper begins from where the proof of the similar theorem in [2] ends. In § 3 we leave [2] and go further to complete the proof.

1.2 Before going to § 2, we make some preparatory remarks. As is well-known, Gaussian curvature of a closed convex surface which is piecewise of class C'' cannot identically vanish on the whole of S. In fact the area of the spherical image of S is not zero but $\pm 2\pi$. Therefore we can assert that there exists a point O on S such that Gaussian curvature K of S is positive at the point O and is also positive at any near-by point of O.

Now let X be any point on the surface S different from the point O. Then the straight line through X and the point O intersects the surface S at no other points on S than X and O. For, if it were not the case, the whole straight line segment OX would lie on the closed convex surface S and consequently the Gaussian curvature of S at the point O could not be positive and this is a contradiction.

1.3 We take the point O as the origin of the orthogonal coordinates system of the 3-space, and take the tangential plane of S at the point O as its (x, y)-plane. Now we apply the projective transformation (2), (see pp. 246—248, [2]),

$$(2) \qquad x' = \frac{x}{z}, \qquad y' = \frac{y}{z}, \qquad z' = \frac{1}{z}$$

to the space, which maps the (x, y)-plane onto the plane at infinity. By the projective transformation (2) the surface S is mapped onto an open convex surface S' having similar shape as a paraboloid, and moreover the pencil of straight lines through the point O is mapped onto a family of straight lines which are parallel to z'-axis. Then we see that the open surface S' can be represented non-parametrically by x', y' and S' is a (1—1)-image of the whole (x', y')-plane i. e., any point P$'$ on S' can be represented by a position vector $(x', y', z'(x', y'))$.

Let $\mathfrak{z} = (X, Y, Z)$ be any infinitesimal (isometric) deformation vector of the surface S. Then $\mathfrak{z}$ is a vector function of point P$: (x, y, z)$ on S.

The purpose of § 2 and § 3 is to prove $\mathfrak{z}$ is trivial. For this purpose we can assume without loss of generality that

(3) $\mathfrak{z}=0$ and $d\mathfrak{z}=0$ on S at the point O. (See p. 263, [2].)

According to Darboux [3], we define a new vector function $\mathfrak{z}'=(X', Y', Z')$ by

(4) $$X'=\frac{X}{z},\quad Y'=\frac{Y}{z},\quad Z'=-\frac{xX+yY+zZ}{z}$$

where $\mathfrak{z}=(X, Y, Z)$ is the infinitesimal isometric deformation vector of S at the point (x, y, z) on S. Then $\mathfrak{z}'$ is also an infinitesimal isometric deformation vector of S' at the point (x', y', z') which corresponds to the point (x, y, z) on S by the projective transformation (2). If this $\mathfrak{z}'$ is trivial with respect to the surface S', the $\mathfrak{z}$ is also trivial with respect to the surface S. (See p. 248 of [2].) We shall show in § 2 and § 3 that the infinitesimal isometric deformation vector $\mathfrak{z}'$ of S' vanishes identically on S' outside its planar portions.

2. (For the details of this section, see [2] on pp. 243—257.)

2.1 We hereafter deal exclusively with the open convex surface S' which we represent non-parametrically by position vector $(x, y, z(x, y))$. For the sake of simplicity we avoid to use the symbol $(x', y', z'(x', y'))$ which is in 1.3. Now the open surface S' is represented by a position vector function

(5) $$\mathfrak{x}=\mathfrak{x}(x, y)=(x, y, z(x, y))$$

where $\mathfrak{x}$ is piecewise of class C'' and also convex[6] in the weak sense, (see 1.1). We write simply $\mathfrak{z}$ for the $\mathfrak{z}'$ of 1.3. This $\mathfrak{z}$ allows the parametric representation with respect to the same x, y as in (5):

(6) $$\mathfrak{z}=\mathfrak{z}(x, y)=(X(x, y),\ Y(x, y),\ Z(x, y))$$

By assumption $\mathfrak{z}$ is of class C'.

Since $\mathfrak{z}(x,y)$ of (6) is an infinitesimal (isometric) deformation vector of the surface S' of (5), we have by definition

(7) $$d\mathfrak{z}\,d\mathfrak{x}=0$$

everywhere on S'. Here (7) is equivalent to the triple system of linear partial differential equations:

(8) $$\mathfrak{z}_x\,\mathfrak{x}_x=X_x+pZ_x=0,$$

(9) $$\mathfrak{z}_x\,\mathfrak{x}_y+\mathfrak{z}_y\,\mathfrak{x}_x=0,$$

6) Differentiability and convexity of S are preserved by (2).

$$\mathfrak{z}_y \mathfrak{x}_y = Y_y + qZ_y = 0, \tag{10}$$

where $p = z_x(x, y)$ and $q = z_y(x, y)$.

Next we define two auxiliary functions $a(x, y)$ and $b(x, y)$ by the following inner products:

$$a(x, y) = \mathfrak{z}\mathfrak{x}_x = X + pZ, \tag{11}$$

$$b(x, y) = \mathfrak{z}\mathfrak{x}_y = Y + qZ. \tag{12}$$

Since $a(x, y)$ and $b(x, y)$ are piecewise of class C', considering (8), (9) and (10), we see that

$$a_x = \mathfrak{z}\mathfrak{x}_{xx} = rZ, \tag{13}$$

$$\frac{1}{2}(a_y + b_x) = \mathfrak{z}\mathfrak{x}_{xy} = sZ, \tag{14}$$

$$b_y = \mathfrak{z}\mathfrak{x}_{yy} = tZ, \tag{15}$$

hold except at the points at which r, s or t is discontinuous, while $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$.

Multiplying (13) by (15) and then subtracting the square of (14) we have

$$a_x b_y - a_y b_x = (rt - s^2) Z^2 + \frac{1}{4}(a_y - b_x)^2. \tag{16}$$

We integrate (16) over a large circle C_ρ of radius ρ in the (x, y)-plane and with its centre at the origin. We have

$$\iint_{C_\rho} (a_x b_y - a_y b_x)\, dx\, dy = \iint_{C_\rho} (rt - s^2) Z^2\, dxdy + \frac{1}{4}\iint_{C_\rho} (a_y - b_x)^2\, dx\, dy. \tag{17}$$

Here the integration is carried out upon each patch of domains on which the integrand is of class C'' including boundary. Now the left-hand side of (17) can be converted into the contour integral on each patch of domains where S being of class C'' and summing them up, we have

$$\iint_{C_\rho} (a_x b_y - a_y b_x)\, dx\, dy = \frac{1}{2}\int_{\dot{C}_\rho} (adb - bda) \tag{18}$$

where $\dot{C}_\rho$ means the boundary of C_ρ. There is on pp. 249—257, [2] rather long calculation which gives the limit of (18) for $\rho \to +\infty$, i. e.,

$$\lim_{\rho\to+\infty}\int_{\dot{C}_\rho} (a\,db - b\,da) = 0. \tag{19}$$

3.

3.1. Since the sign of Gaussian curvature is projective invariant by (2) and the original closed convex surface S is piecewise of class C'', we have $rt-s^2 \geqq 0$, while r, s, t could have discontinuity across the piecewise smooth curves. As is mentioned before, the integration is carried out on each patch of domains, on which S' is of class C'' respectively. So we can conclude from (17), (18) and (19) that

$$\iint_{C_\rho} (a_y - b_x)^2 \, dx \, dy = 0 \tag{20}$$

and

$$\iint_{C_\rho} (rt - s^2) Z^2 \, dx \, dy = 0 \tag{21}$$

for any ρ, because both of the integrands are non-negative. From (20) and (14) we have

$$a_y = b_x = sZ \tag{22}$$

that holds except at the points at which r, s or t is discontinuous. Recalling (11) and (12) we have by calculation

$$X_y + pZ_y = 0 \tag{23}$$

and

$$Y_x + qZ_x = 0 \tag{24}$$

which hold *everywhere* on the (x, y)-plane, because by assumption p, q are continuous and X, Y, Z are of class C' and the set of the points at which r, s or t is discontinuous is non-dense in the (x, y)-plane.

Since in a sufficiently small vicinity of the point O on S Gaussian curvature K of S is positive, we see that the $rt-s^2$ of the integrand in (21) is also positive outside a circle C_{ρ_0} for sufficiently large ρ_0. So we have from (21)

$$Z = 0 \quad \text{outside } C_{\rho_0} \text{ for sufficiently large } \rho_0. \tag{25}$$

3.2. Next we eliminate p from (8) and (23) and have

$$X_x Z_y - X_y Z_x = 0 \tag{26}$$

which holds everywhere on (x, y)-plane. Integrating (26) over a domain D defined by $x^2+y^2 \leq \rho^2$, $\rho > \rho_0$ and $y \geqq y_0$, (D is a circular segment), we have

$$\iint_D (X_x Z_y - X_y Z_x) \, dx \, dy = 0. \tag{27}$$

Since $X(x, y)$ and $Z(x, y)$ are of class C' by assumption, it can be converted into the contour integral

$$\int_{\dot{D}} Z dX = 0, \tag{28}$$

where $\dot{D}$ is the boundary of D. In (28) the curvilinear part of integral along the circle $\dot{C}_\rho$ is zero because $Z = 0$ on $\dot{C}_\rho$ by (25). So we have

$$\int_{-\rho'}^{\rho'} Z(x, y_0)\, X_x(x, y_0)\, dx = 0 \tag{29}$$

where $\rho' = (\rho^2 - y_0^2)^{1/2}$. Putting the relation (8), $X_x = -pZ_x$ into (29) we have

$$\begin{aligned} &\int_{-\rho'}^{\rho'} Z(x, y_0)\, p(x, y_0)\, Z_x(x, y_0)\, dx \\ &= \frac{1}{2} \int_{-\rho'}^{\rho'} p(x, y_0)\, d(Z^2(x, y_0)) = 0 \end{aligned} \tag{30}$$

where Z is of class C' and $p(x, y_0)$ is a monotone[7] function of x, with fixed y_0. Then by the formula of integration by parts of Riemann-Stieltjes integral, we have

$$\begin{aligned} \int_{-\rho'}^{\rho'} p(x, y_0)\, d(Z^2(x, y_0)) &= p(x, y_0)\, Z^2(x, y_0) \Big|_{x=-\rho'}^{x=\rho'} \\ &- \int_{-\rho'}^{\rho'} Z^2(x, y_0)\, dp(x, y_0) = 0 \end{aligned} \tag{31}$$

where the first term of the left-hand side is zero by (25). Finally we have

$$\int_{-\rho'}^{\rho'} Z^2(x, y_0)\, dp(x, y_0) = 0. \tag{32}$$

Now eliminating q from (24) and (10) we have

$$Y_x Z_y - Y_y Z_x = 0 \tag{33}$$

everywhere on the (x, y)-plane. We integrate (33) over the domain E defined by $x^2 + y^2 \leq \rho^2$, $\rho > \rho_0$ and $x \leq x_0$, $|x_0| < \rho$. Then by the similar calculations to those from (27) to (32), we have

$$\int_{-\rho'}^{\rho'} Z^2(x_0, y)\, dq(x_0, y) = 0. \tag{34}$$

3.3 In this paragraph we shall show that $Z(x_0, y_0) = 0$ at any point (x_0, y_0), $x_0^2 + y_0^2 \leq \rho^2$, if $z(x, y)$ is of class C'' and $r + t > 0$[8] in some neigh-

7) The surface S' is convex and piecewise of class C'' and so is its non-parametric representation, $z = z(x, y)$.

8) Without loss of generality we can assume $p(x, y_0)$ and $q(x_0, y)$ are non-decreasing functions with respective to x and y respectively. In this case r and t are both non-negative.

bourhood of the point (x_0, y_0). Suppose $z(x, y)$ satisfies the conditions mentioned above and also those assumed about the surface S'. Then Z^2dp in (32) and Z^2dq in (34) are both non-negative[8] because $z(x, y)$ is convex and smooth enough. Therefore we have

$$\int_\alpha^\beta Z^2(x, y_0)\, dp(x, y_0) = 0 \tag{35}$$

and

$$\int_\alpha^\beta Z^2(x_0, y)\, dq(x_0, y) = 0 \tag{36}$$

for any α and β, $-\rho' \leq \alpha < \beta \leq \rho'$. As $z(x, y)$ is of class C'' in a neighbourhood U of the point (x_0, y_0), so the differential $dp(x, y_0)$ with respect to x and $dq(x_0, y)$ with respect to y can be written by

$$dp(x, y_0) = r(x, y_0)\, dx \tag{37}$$

and

$$dq(x_0, y) = t(x_0, y)\, dy \tag{38}$$

in the neighbourhood U of the point (x_0, y_0). If $r + t > 0$, then either $r > 0$ or $t > 0$. Suppose $r > 0$ in U. Since by assumption r and Z are both continuous in U, it follows from (35) and (37) that $Z(x, y_0)$ vanishes at any point (x, y_0) which belongs to U. If $t > 0$, then by the similar argument it follows from (36) and (38) that $Z(x_0, y)$ vanishes at any point (x_0, y) which belongs to U. That is to say, if $r + t > 0$ in U, then $Z(x_0, y_0) = 0$.

Recalling that $Z(x, y)$ is of class C' and $z(x, y)$ is piecewise of class C'', we conclude that $Z(x, y)$ vanishes identically outside the open set of points at which both r and t of the surface S' vanish simultaneously. This exceptional open set of points may consists of any number of domains and on each of them the surface S' is planar. Now we shall show that a and b vanish identically everywhere on the (x, y)-plane. In fact, on each of the patches of domain on which r, s and t are continuous, all a_x, a_y, b_x and b_y vanish identically, because we have (13), (15) and (22), and we have $Z(x, y) = 0$ outside the planar portions and $r = s = t = 0$ on the planar portions of S'. Consequently both a and b are constant on each of the patches of domain mentioned above. But a, b are continuous everywhere in (x, y)-plane, which means that a, b are constant on the whole (x, y)-plane respectively. Moreover we find, on p. 250 of [2], that $a(x, y) \to 0$ and $b(x, y) \to 0$ for $x^2 + y^2 \to \infty$. Therefore we finally say that a and b vanish identically on the whole (x, y)-plane. By the definitions $a = X + pZ$ and $b = Y + qZ$ we have

$$X = -pZ \tag{39}$$

and

$$Y = -qZ, \tag{40}$$

which mean that X, Y (and Z) vanish identically outside the planar portions of S'. But on the planar portions of S', $Z(x, y)$ may be an arbitrary function of class C' vanishing on each of the boundaries, while X and Y are defined by $X = -p_0 Z$ and $Y = -q_0 Z$ where p_0 and q_0 are constant.

By way of the inverse transformations of (2) and (4), we can set the above-mentioned results concerning the open surface S' back to the original closed surface S. That is to say, as the infinitesimal (isometric) deformation vector $\mathfrak{z}'$ of S' vanishes identically outside the planar portions of S', so does the image $\mathfrak{z}$ of $\mathfrak{z}'$ by the inverse transformation of (4). Considering the condition (3), $\mathfrak{z}$ is necessarily a trivial infinitesimal (isometric) deformation vector of the original surface S, while we have assumed $\mathfrak{z}$ is *any* vector satisfying (1). Thus the proof of THEOREM in 1.1 is complete.

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