ON THE APPROXIMATION TO SOME LIMITING DISTRIBUTIONS

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It is well known that

a) binomlal distribution with mean np, variance npq approaches to Poisson distribution with $\lambda = np$ as mean and variance when $n \to \infty$,

b) let X_i $(i = 1, 2, \cdots, n)$ obey the uniform distribution :

$$\begin{aligned} P(X_i \leqq x) &= 1 & x &\geqq \frac{1}{2}, \\ &= x + \frac{1}{2} & |x| &\leqq \frac{1}{2}, \\ &= 0 & x &\leqq -\frac{1}{2}, \end{aligned}$$

then $\sum_{i=1}^{n} X_i/\sqrt{n/12} \to N(0, 1)$ as $n \to \infty$.

With respect to these facts, we shall deduce the approximation formulas for the above two distributions, and also evaluate the absolute error.

Concerning these problems,

a) the relative error in the Poisson approximation to binomial distribution was estimated by W. Feller [3, p. 114] and J. Uspensky [4, p. 137],

b) the distribution of the sum of uniformly distributed random variables was discussed by Uno [5] and Uspensky [4]. In [5] the different method to compute the exact value of this distribution and the table (for $n \leqq 10$) are given.

Our approximation formula is an improvement of Uspensky's. Same methods as in [1], [2] can also be used for our purposes, so we omit the detail of them here.

§ 1. Poisson approximation to the binomial distribution.

Let X_i be as follows:

$$P(X_i = 1) = p,$$
$$P(X_i = 0) = q,$$

then $\sum_{i=1}^{n} X_i$ is a random variable which obeys the binomial distribution with mean np, variance npq and has a c. f.:

$$f_n(t) = (q + pe^{it})^n = \{1 + p(e^{it} - 1)\}^n, \tag{1}$$

and corresponding Poisson distribution with mean, variance $\lambda = np$ has

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a c. f.:

$$\exp\{\lambda(e^{it}-1)\}.$$

1° *Taylor expansion of $f_n(t)$.* Taking logarithm of the both side in (1), we have, remembering that p is small, the following:

$$\begin{aligned}\log f_n(t) &= n\log\{1+p(e^{it}-1)\}\\ &= np(e^{it}-1)-\frac{np^2}{2}(e^{it}-1)^2+n\Theta,\end{aligned}\tag{2}$$

where

$$\Theta = \int_0^{p(e^{it}-1)} \frac{z^2}{1+z}\,dz,$$

$$|\Theta| \leqq \frac{1}{\min\limits_{|t|<\pi}|q+pe^{it}|}\,\frac{3}{p^3}\,|e^{it}-1|^3 \leqq \frac{1}{q-p}\,\frac{p^3}{3}\,|t|^3.$$

From (2), it follows easily that

$$\begin{aligned}f_n(t) &= e^{\log f_n(t)} = e^{\lambda(e^{it}-1)}e^{-\frac{\lambda^2}{2n}(e^{it}-1)^2}e^{n\Theta}\\ &= e^{\lambda(e^{it}-1)}\left\{1-\frac{\lambda^2}{2n}(e^{it}-1)^2+\frac{1}{2}\left(\frac{\lambda^2}{2n}\right)^2|e^{it}-1|\right.\\ &\qquad\left.\cdot\, e^{\frac{\lambda^2}{2n}|e^{it}-1|^2}\vartheta\right\}\left\{1+|n\Theta|\,e^{|n\Theta|}\vartheta\right\}\\ &= e^{\lambda(e^{it}-1)}-\frac{\lambda^2}{2n}\{e^{\lambda(e^{it}-1)+2it}-2e^{\lambda(e^{it}-1)+it}+e^{\lambda(e^{it}-1)}\}\\ &\quad+\frac{1}{2}\left(\frac{\lambda^2}{2n}\right)^2|e^{it}-1|^4e^{\frac{\lambda^2}{2n}|e^{it}-1|^2}\vartheta\, e^{\lambda(e^{it}-1)}\\ &\quad+|n\Theta|\,e^{|n\Theta|}\vartheta\, e^{-\frac{\lambda^2}{2n}(e^{it}-1)^2}e^{\lambda(e^{it}-1)},\end{aligned}\tag{3}$$

where ϑ unspecified complex-valued quantities such that $|\vartheta|<1$.

2° THEOREM 1 (non-cumulative case).

Let

$$b(k;\,np) = \frac{n!}{k!\,(n-k)!}\,p^k q^{n-k},$$

$$p(k;\,\lambda) = \frac{\lambda^k e^{-\lambda}}{k!},$$

then we have

$$b(k;\,n,p) = p(k;\,\lambda)-\frac{\lambda^2}{2n}\Delta^2 p(k;\,\lambda)+R_1$$

where

$$\Delta^2 p(k;\,\lambda) = [\{p(k-2;\,\lambda)-p(k-1;\,\lambda)\}-\{p(k-1;\,\lambda)-p(k;\,\lambda)\}]$$

(that is $\Delta^2 p(k;\lambda)$ denotes the difference of second order) *and*

$$p(k';\lambda)=0 \qquad \textit{for} \qquad k'<0,$$

$$|R_1|\leqq \frac{1}{n^2\pi}\left(\frac{I_4(\lambda)}{8}+\frac{I_3(\lambda)}{3}\,\frac{e^{\frac{8\lambda^3}{3n^2(1-2\lambda/n)}}}{1-2\lambda/n}\right)e^{\frac{2\lambda^2}{n}}\ ^{1)}$$

and

$$I_j(\lambda)=\lambda^j\int_0^{\pi}e^{-2\lambda\sin^2\frac{t}{2}}\,t^j\,dt.$$

Proof. Since

$$(4)\qquad b(k;np)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_n(t)\,e^{-ikt}\,dt,$$

replacing $f_n(t)$ by the right hand side of (3), we have the following equality:

$$\begin{aligned}b(k;n,p)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{\lambda(e^{it}-1)}\,e^{-ikt}\,dt\\
(5)\qquad &-\frac{\lambda^2}{2n}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\{(e^{\lambda(e^{it}-1)+2it}-e^{\lambda(e^{it}-1)+it})\right.\\
&\left.-(e^{\lambda(e^{it}-1)+it}-e^{\lambda(e^{it}-1)})\}\,e^{-ikt}\,dt\right]+\frac{1}{2\pi}\int_{-\pi}^{\pi}R\,e^{-ikt}\,dt,\end{aligned}$$

where R is the sum of 3rd and 4th terms in (3). Hence

$$(6)\qquad b(k;n,p)=p(k;\lambda)+\frac{\lambda^2}{2n}\,\Delta^2 p(k;\lambda)+R_1$$

and

$$\begin{aligned}|R_1|&\leqq\frac{\lambda^4}{16\pi n^2}\int_{-\pi}^{\pi}t^4\,e^{\lambda(\cos t-1)}\,dt\cdot e^{\frac{2\lambda^2}{n}}\\
&+\frac{1}{q-p}\,\frac{\lambda^3}{6\pi n^2}\int_{-\pi}^{\pi}|t|^3\,e^{\lambda(\cos t-1)}\,dt\cdot e^{\frac{2\lambda^2}{n}+\frac{8\lambda^3}{3n^2}\frac{1}{q-p}}\\
&\leqq\frac{1}{n^2}\left[\frac{t^4}{8\pi}\int_0^{\pi}e^{-2\lambda\sin^2\frac{t}{2}}\,t^4\,dt\right.\\
&\left.+\frac{\lambda^3}{3\pi}\,\frac{1}{q-p}\int_0^{\pi}e^{-2\lambda\sin^2\frac{t}{2}}\,t^3\,dt\cdot e^{\frac{1}{q-p}\frac{8\lambda^3}{3n^2}}\right]e^{\frac{2\lambda^2}{n}}.\end{aligned}$$

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REMARK 1. In § 1, (2) putting

$$\Theta=\frac{np^3}{3}(e^{it}-1)^3-n\Theta' \qquad \text{and} \qquad \Theta'=\int_0^{p(e^{it}-1)}\frac{z^3}{1+z}\,dz,$$

we have the following

THEOREM 1'.

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1) See REMARK 3.

$$b(k;\, n,\, p) = p(k;\,\lambda) - \frac{\lambda^2}{2n}\, \Delta^2\, p(k;\,\lambda)$$

(6′) $$+\frac{1}{n^2}\left(\frac{\lambda^4\, \Delta^4\, p(k;\,\lambda)}{8} + \frac{\lambda^3\, \Delta^3\, p(k;\,\lambda)}{3}\right) + R_2$$

where

$$\Delta^3\, p(k;\,\lambda) = \Delta^2\, p(k-1;\,\lambda) - \Delta^2\, p(k;\,\lambda),$$

$$\Delta^4\, p(k;\,\lambda) = \Delta^3\, p(k-1;\,\lambda) - \Delta^3\, p(k;\,\lambda)$$

and

$$|R_2| \leqq \frac{e^{\frac{2\lambda^2}{n}}}{2n^3\pi}\left[\frac{1}{24}\, I_6 + \frac{1}{3}\, I_5 + \frac{1}{2(1-2\lambda/n)}\, I_4\, e^{\frac{8\lambda^3}{3n^2} + \frac{4\lambda^4}{n^3(1-2\lambda/n)}}\right]$$

$$+\frac{1}{18\, n^4\pi}\, I_6\, e^{\frac{2\lambda^2}{n} + \frac{8\lambda^3}{3n^2}}.$$

(6′) gives a good approximation for comparably small λ.

3° THEOREM 2 (cumulative case).

(7) $$\sum_{k=l+1}^{l'} b(k;\, n,\, p) = \sum_{k=l+1}^{l'} p(k;\,\lambda) - \frac{\lambda^2}{2n}\, [\Delta^2 P(l';\,\lambda) - \Delta^2 P(l;\,\lambda)] + S_1,$$

where

$$P(l;\,\lambda) = \sum_{k=0}^{l} p(k;\,\lambda)$$

and

$$\Delta^2 P(l;\,\lambda) = [P(l;\,\lambda) - P(l-1;\,\lambda)] - [P(l-1;\,\lambda) - P(l-2;\,\lambda)],$$

$$|S_1| \leqq \frac{\lambda}{n^2}\left(\frac{I_3(\lambda)}{8} + \frac{I_2(\lambda)}{3}\, \frac{e^{\frac{8\lambda^3}{3n^2(1-2\lambda/n)}}}{1-2\lambda/n}\right) e^{\frac{2\lambda^2}{n}}.$$

Proof. Summing up (4) from $l+1$ to l', we have

(8) $$\begin{aligned}\sum_{k=l+1}^{l'} b(k;\, n,\, p) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\left\{\sum_{k=l+1}^{l'} e^{-ikt}\right\} dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\, \frac{e^{-i(l+1)t} - e^{-i(l'+1)t}}{1-e^{-it}}\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_n(t)\, e^{-i\frac{l+l'+1}{2}t}\, \frac{\sin\left(\frac{l'-l}{2}\right)t}{\sin\frac{t}{2}}\, dt.\end{aligned}$$

Substituting (3) into (8)

$$\sum_{k=l+1}^{l'} b(k;\, n,\, p) = \sum_{k=l+1}^{l'} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda(e^{it}-1)}\, e^{-ikt}\, dt$$

$$-\frac{\lambda^2}{2n}\left[\sum_{k=l+l}^{l'} \frac{1}{2\pi}\int_{-\pi}^{\pi}\{e^{\lambda(e^{it}-1)+2it} - 2e^{\lambda(e^{it}-1)+it} + e^{\lambda(e^{it}-1)}\}\, e^{-ikt}\, dt\right]$$

$$+\frac{1}{2\pi}\int_{-\pi}^{\pi} R e^{-i\frac{l+l'+1}{2}t}\frac{\sin\left(\frac{l'-l}{2}\right)t}{\sin\frac{1}{2}}dt$$

$$=\sum_{k=l+1}^{l'} p(k;\lambda)+\frac{\lambda^2}{2n}(\Delta^2 P(l;\lambda)-\Delta^2 P(l';\lambda))+S_1.$$

Furthermore, S_1 can be estimated as follows:

$$|S_1|\leqq\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2}\left(\frac{\lambda^2}{2n}\right)^2|e^{it}-1|^4 e^{\frac{\lambda^2}{2n}|e^{it}-1|^2}\vartheta e^{\lambda(e^{it}-1)}\right.$$

$$\left.\cdot e^{-i\frac{l+l'+1}{2}t}\frac{\sin\frac{l'-l}{2}t}{\sin\frac{t}{2}}dt\right|$$

$$+\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} n\,\Theta\, e^{|n|\vartheta}e^{-\frac{\lambda^2}{2n}(e^{it}-1)^2}e^{\lambda(e^{it}-1)}e^{-i\frac{l+l'+1}{2}t}\frac{\sin\frac{l'-l}{2}t}{\sin\frac{t}{2}}dt\right|.$$

Now

$$\sin\frac{t}{2}>\frac{t}{\pi}\qquad\text{for}\qquad 0<t<\pi,$$

hence

$$|S_1|\leqq\frac{1}{\pi}\int_0^{\pi}\frac{\lambda^4}{8n^2}|t|^4 e^{\frac{2\lambda^2}{n}}e^{\lambda(\cos t-1)}\frac{1}{|t|/\pi}dt$$

$$+\frac{1}{\pi}\int_0^{\pi}\frac{1}{q-p}\frac{\lambda^3}{3n^2}|t|^3 e^{\frac{1}{q-p}\frac{8\lambda^3}{3n^2}+\frac{2\lambda^2}{n}+\lambda(\cos t-1)}\frac{1}{|t|/\pi}dt$$

$$\leqq\frac{1}{n^2}\left[\frac{\lambda^4}{8}\int_0^{\pi}e^{-2\lambda\sin^2\frac{t}{2}}t^3\,dt+\frac{\lambda^3}{3}\frac{1}{q-p}\int_0^{\pi}e^{-2\lambda\sin^2\frac{t}{2}}t^2\,dt\cdot e^{\frac{1}{q-p}\frac{8\lambda^3}{3n^2}}\right]e^{\frac{2\lambda^2}{n}}.$$

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REMARK 2. We can take a slight modification of THEOREM 2 as follows:

THEOREM 2′.

$$\sum_{k=l+1}^{l'} b(k;n,p)=\sum_{k=l+1}^{l'} p(k;\lambda)-\frac{\lambda^2}{2n}[\Delta^2 P(l';\lambda)-\Delta^2 P(l;\lambda)]$$

$$+\frac{1}{n^2}\left[\frac{\lambda^3}{3}(\Delta^3 P(l';\lambda)-\Delta^3 P(l;\lambda))\right.$$

$$\left.+\frac{\lambda^4}{8}(\Delta^4 P(l';\lambda)-\Delta^4 P(l;\lambda))\right]+S_2$$

where

$$|S_2|\leqq\frac{e^{\frac{2\lambda^2}{n}}\lambda}{2n^3}\left[\frac{I_5}{24}+\frac{I_4}{3}+\frac{I_3}{2(1-2\lambda/n)}e^{\frac{8\lambda^3}{3n^2}+\frac{4\lambda^4}{n^3(1-2\lambda/n)}}\right]+\frac{\lambda}{18n^4}I_5 e^{\frac{2\lambda^2}{n}+\frac{8\lambda^3}{3n^2}}.$$

REMARK 3. For the sake of numerical computation of $|R_1|$ and $|S_1|$, we shall give the numerical constants A, B, C, D in the next TABLE 1, where

$$A = I_4(\lambda)/8\pi, \quad B = I_3(\lambda)/3\pi,$$
$$C = I_3(\lambda)/8, \quad D = I_2(\lambda)/3.$$

We shall plot these values on the Fig. 1.

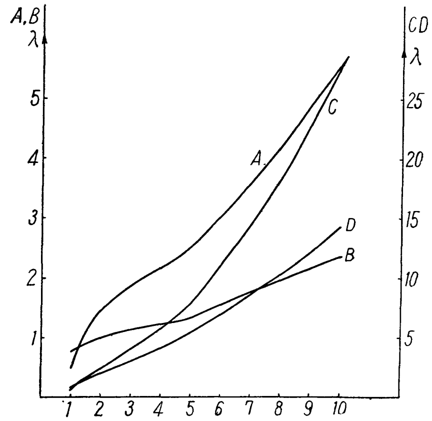

Fig. 1

REMARK 4. In order to see the degree of approximation roughly, we shall denote some numerical values in the following TABLE 2~5. In the TABLE 2~4, "1 st appr." denote the estimating values due to the TOHEREM 1 and "2 nd appr." denote the ones due to the THEOREM 1′. In the TABLE 5 "1 st appr." denotes the estimating values due to the THEOREM 2. And the velues of $b(k; n, p)$, $p(k; \lambda)$, $P_B(k)$ and $P_P(k)$ were taken from [3, p. 112] and [6, p. 688].

TABLE 1

λ	A	B	C	D
1	.4584	.5270	.6208	8087
2	1.469	1.008	2.375	2.004
3	1.889	1.134	4.008	3.027
4	2.192	1.236	5.822	4.143
5	2.543	1.337	7.874	5.464
6	3.026	1.562	11.044	6.939
7	3.572	1.756	14.476	8.576
8	4.195	1.956	18.432	10.327
9	4.916	2.158	22.884	12.191
10	5.575	2.362	27.827	14.326

TABLE 2 ($n = 100$, $\lambda = 1$)

	$b(k; n, p)$	$p(k; \lambda)$	1 st appr.	2 nd appr.
0	.366032	.367879	.366040	.366032
1	.369730	.367879	.369718	.369729
2	.184865	.183940	.184860	.184865
4	.014942	.015328	.014945	.014941
7	.000063	.000073	.000062	.000063

TABLE 3 ($n = 10$, $\lambda = 1$)

	$b(k; n, p)$	$p(k; \lambda)$	1 st appr.	2 nd appr.
0	.3487	.3679	.3495	.3487
1	.3874	.3679	.3863	.3874
2	.1937	.1839	.1931	.1936
4	.0112	.0153	.0115	.0112
7	.0000	.0001	.0000	.0000

TABLE 4 ($n = 5,\ \lambda = 1$)

	$b(k;\, n, p)$	$p(k;\, \lambda)$	1 st appr.	2 nd appr.
0	.3277	.3679	.3311	.3284
1	.4096	.3679	.4047	.4090
2	.2048	.1839	.2030	.2044
4	.0064	.0153	.0076	.0063

TABLE 5 ($n = 10,\ \lambda = 1$)

k	$P_B(k)$	$P_P(k)$	1 st appr.
0	.3487	.3679	.3495
1	.7361	.7358	.7358
2	.9298	.9197	.9289
3	.9872	.9810	.9871
4	.9984	.9963	.9986
5	.9999	.9994	1.0000
6	1.0000	.9999	1.0000

Notation: $P_B(k) = \sum_{i=0}^{k} b(i;\, n, p)$, $P_P(k) = \sum_{i=0}^{k} p\,(i;\, \lambda)$.

§ 2. Normal approximation to the distribution of sum of uniformly distributed romdom variables.

Our normalized summand $(X_1 + X_2 + \cdots + X_n)\,/\sqrt{n/12}$ has the c. f.:

$$f_n(t) = \left(\frac{\sin t\sqrt{\dfrac{3}{n}}}{t\sqrt{\dfrac{3}{n}}} \right)^n . \tag{9}$$

1° *Taylor expansion of $f_n(t)$.* Now put for simplicity

$$z = t\sqrt{\frac{3}{n}}\,,$$

we have

$$\begin{aligned} \log f_n(t) &= n \log \frac{\sin z}{z} \\ &= \sum_{k=1}^{\infty} \log\left(1 - \frac{k^2\pi^2}{z^2}\right) \qquad \text{for} \qquad 0 \leqq z \leqq \pi. \end{aligned} \tag{10}$$

Then expand $\log(1 - z^2/k^2\pi^2)$, we have the following absolutely convergent series

$$\begin{aligned} \log \frac{\sin z}{z} = &- \frac{z^2}{\pi^2} - \frac{1}{2}\left(\frac{z^2}{\pi^2}\right)^2 - \frac{1}{3}\left(\frac{z^2}{\pi^2}\right)^3 - \int_0^{z^2/\pi^2} \frac{t^3}{1-t}\,dt \\ &- \frac{z^2}{2^2\pi^2} - \frac{1}{2}\left(\frac{z^2}{2^2\pi^2}\right)^2 - \frac{1}{3}\left(\frac{z^2}{2^2\pi^2}\right)^3 - \int_0^{z^2/2^2\pi^2} \frac{t^3}{1-t}\,dt \\ &- \frac{z^2}{3^2\pi^2} - \frac{1}{2}\left(\frac{z^2}{3^2\pi^2}\right)^2 - \frac{1}{3}\left(\frac{z^2}{3^2\pi^2}\right)^3 - \int_0^{z^2/3^2\pi^2} \frac{t^3}{1-t}\,dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ = &- \frac{z^2}{\pi^2}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right) - \frac{z^4}{2\pi^4}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right) - \frac{z^6}{3\pi^6}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right) - \left[\sum_{k=1}^{\infty}\int_0^{z^2/k^2\pi^2} \frac{t^3}{1-t}\,dt\right]. \end{aligned}$$

Hence, we can put

$$\log \frac{\sin z}{z} = -\frac{z^2}{6} - \frac{z^4}{180} - \frac{z^6}{2835} - \Theta_8. \tag{11}$$

And since (11) and (10) give

$$\left(\frac{\sin t \sqrt{\frac{3}{n}}}{t \sqrt{\frac{3}{n}}} \right)^n = e^{-\frac{n}{6}\left(t\sqrt{\frac{3}{n}}\right)^2 - \frac{n}{180}\left(t\sqrt{\frac{3}{n}}\right)^4 - \Theta_6}$$

or

$$= e^{-\frac{n}{6}\left(t\sqrt{\frac{3}{n}}\right)^2 - \frac{n}{180}\left(t\sqrt{\frac{3}{n}}\right)^4 - \frac{n}{2835}\left(t\sqrt{\frac{3}{n}}\right)^6 - n\Theta_8},$$

we have

$$\left(\frac{\sin t \sqrt{\frac{3}{n}}}{t \sqrt{\frac{3}{n}}} \right)^n = e^{-\frac{t^2}{2}} \Bigg[1 - \frac{t^4}{20\,n} + \left\{ \frac{t^8}{800\,n^2} - \frac{t^6}{105\,n^2} - \frac{t^{12}}{48000\,n^3}\, e^{-\frac{t^4}{20n}\theta} \right. \tag{12}$$

$$\left. + \frac{t^{10}}{20 \times 105\,n^3}\, e^{-\frac{t^4}{20n}\theta - \frac{t^6}{105n^2}\theta} + \frac{t^{12}}{2 \times (105)^2}\, e^{-\frac{t^6}{105n^2}\theta} \right\} - n\,\Theta_8 e^{-n\Theta_8\theta - \frac{t^4}{20n} - \frac{t^6}{105n^2}} \Bigg]$$

where

$$0 < \theta < 1,$$

$$\Theta_8 = \sum_{k=1}^{\infty} \int_0^{z^2/k^2\pi^2} \frac{t^3}{1-t}\, dt$$

and

$$\Theta_8 \leqq \frac{1}{1 - \frac{\alpha^2}{\pi^2}} \frac{1}{4} \frac{z^8}{\pi^8} \left(\sum_{k=1}^{\infty} \frac{1}{k^8} \right) = \frac{1}{1 - \frac{\alpha^2}{\pi^2}} \frac{t^8}{1400\,n^4},$$

where

$$z \leqq \alpha \qquad \text{or} \qquad t \leqq \sqrt{\frac{n}{3}}\,\alpha.$$

2° From Levy's inverse formula

$$\mathrm{P}\left\{ x < \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{\frac{n}{12}}} < x + h \right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-ith}}{it}\, e^{-itx} \left(\frac{\sin t \sqrt{\frac{3}{n}}}{t \sqrt{\frac{3}{n}}} \right)^n dt$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left(\frac{\sin t \sqrt{\frac{3}{n}}}{t \sqrt{\frac{3}{n}}} \right)^n \frac{\sin\,[t\,(x+h)] - \sin tx}{t}\, dt$$

$$= \int_0^{\sqrt{2n}} {}'' + \frac{1}{\pi}\int_{\sqrt{2n}}^{\infty} {}'' = I_1 + I_2,$$

where, using the equality (12)

$$I_1 = \frac{1}{\pi}\int_0^{\infty}\frac{\sin[t(x+h)] - \sin tx}{t} e^{-\frac{t^2}{2}} dt - \frac{1}{\pi}\int_{\sqrt{2n}}^{\infty}$$

$$- \frac{1}{\pi}\int_0^{\infty}\frac{\sin[t(x+h)] - \sin tx}{t} \frac{t^4}{20\,n} e^{-\frac{t^2}{2}} dt + \frac{1}{\pi}\int_{\sqrt{2n}}^{\infty} {}'' + R_2$$

$$= \Phi(x+h) - \Phi(x) - \frac{1}{20\sqrt{2\pi}\cdot n}[\{3(x+h) - (x+h)^3\} e^{-\frac{(x+h)^2}{2}}$$

$$- \{3x - x^3\} e^{-\frac{x^2}{2}}] + R_1 + R_2.$$

(i) Evaluation of R_1.

$$|R_1| \leqq \frac{2}{\pi}\int_{\sqrt{2n}}^{\infty}\frac{1}{t} e^{-\frac{t^2}{2}} dt + \frac{2}{\pi}\int_{\sqrt{2n}}^{\infty}\frac{t^3}{20\,n} e^{-\frac{t^2}{2}} dt = e^{-n}\Big[0.064 + \frac{0.383}{n}\Big].$$

(ii) Evaluation of R_2.

$$|R_2| \leqq \frac{2}{n^2\pi}\int_0^{\sqrt{2n}}\left|\frac{t^7}{800} - \frac{t^5}{105}\right| e^{-\frac{t^2}{2}} dt$$

$$+ \frac{2}{n^3\pi}\Big[\frac{1}{48000}\int_0^{\sqrt{2n}} t^{11} e^{-\frac{t^2}{2}} dt + \frac{1}{2100}\int_0^{\sqrt{2n}} t^9 e^{-\frac{t^2}{2}} dt$$

$$+ \frac{3}{1400}\int_0^{\sqrt{2n}}\frac{t^7}{1 - \frac{3t^2}{n\pi^2}} e^{-\frac{t^2}{2}} dt\Big] + \frac{2}{n^4\pi}\frac{1}{22050}\int_0^{\sqrt{2n}} t^{11} e^{-\frac{t^2}{2}} dt.$$

Now letting $\sqrt{2n} \to \infty$, we get

$$|R_2| \leqq \frac{1}{n^2}\Big[\frac{2}{\pi}\Big\{\frac{\mu_5(\alpha)}{105} - \frac{\mu_7(\alpha)}{800} + \frac{\mu_7 - \mu_7(\alpha)}{800} - \frac{\mu_5 - \mu_5(\alpha)}{105}\Big\}\Big]$$

$$+ \frac{1}{n^3}\Big[\frac{2}{\pi}\Big\{\frac{\mu_{11}}{48000} + \frac{\mu_9}{2100} + \frac{3\mu_7}{1400}\frac{1}{1 - \frac{3\beta^2}{n\pi^2}}\Big\}\Big] + \frac{1}{n^4}\Big[\frac{\mu_{11}}{22050}\frac{2}{\pi}\Big],$$

where

$$\alpha = \sqrt{\frac{800}{105}} = 2.76, \qquad \beta = \sqrt{2n}$$

and

$$\mu_n(x) = \int_0^x t^n e^{-\frac{t^2}{2}} dt, \qquad \mu_n \equiv \mu_n(\infty).$$

(iii) Evaluation of I_2.

Since our integrand can be variable in $\sqrt{2n} < t < \infty$, we have

$$\left|\frac{\sin t\sqrt{\frac{3}{n}}}{t\sqrt{\frac{3}{n}}}\right|^n \cdot \left|\frac{\sin[t(x+h)]-\sin tx}{t}\right| \leqq 2\left(\frac{3}{n}\right)^{-\frac{n}{2}} t^{-(n+1)},$$

hence

$$\left|\frac{1}{\pi}\int_{\sqrt{2n}}^{\infty}\left(\frac{\sin t\sqrt{\frac{3}{n}}}{t\sqrt{\frac{3}{n}}}\right)^n \frac{\sin[t(x+h)]-\sin tx}{t}\,dt\right|$$

$$\leqq \frac{2}{\pi}\left(\frac{3}{n}\right)^{-\frac{n}{2}}\int_{\sqrt{2n}}^{\infty} t^{-(n+1)}\,dt \leqq \frac{2}{\pi}\left(\frac{3}{n}\right)^{-\frac{n}{2}}\frac{1}{n}(\sqrt{2n})^{-n} = \frac{0.64}{n}e^{-1.8n}.$$

Summing up the above discussions we get the following

THEOREM 3. *With the same notations as in the above, we have*

$$\begin{aligned} &F(x+h)-F(x) \\ &= \Phi(x+h)-\Phi(x)-\frac{1}{20\sqrt{2\pi}\,n}\left[\{3(x+h)-(x+h)^3\}e^{-\frac{(x+h)^2}{2}} - \{3x-3x^3\}e^{-\frac{x^2}{2}}\right]+R \end{aligned}$$

where

$$|R| \leqq \frac{0.0217}{n^2}+\frac{0.335}{n^3}+\frac{0.111}{n^4}+\delta,$$

$$\delta = e^{-n}\left(0.064+\frac{0.383}{n}\right)+e^{-1.8n}\left(\frac{0.64}{n}\right).$$

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