TYPICAL FUNCTIONS OF SUMS OF NON-NEGATIVE INDEPENDENT RANDOM VARIABLES

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1. Let the distribution function of a random variable X be $F(x)$ and

$$\Phi(h) = \int_{-\infty}^{\infty} \frac{h^2}{x^2 + h^2} dF(x), \qquad h > 0 \tag{1.1}$$

which we shall call the typical function of X and is defined by K. Kunisawa [1]. It is evident that

$$\Phi(h) \geqq 0 \quad (h > 0), \qquad \Phi(+0) = 0, \qquad \Phi(+\infty) = 1.$$

The function plays certain important roles in the theory of sums of independent random variables.

We consider a sequence of independent random variables

$$X_1, X_2, \cdots \tag{1.2}$$

and let $F_n(x)$ be the distribution function of X_n. We form the typical function $\Phi_{F_1 * \cdots * F_n}(h)$ of

$$S_n = \sum_{k=1}^{n} X_k,$$

i. e.,

$$\Phi_{F_1 * \cdots * F_n}(h) = \int_{-\infty}^{\infty} \frac{h^2}{x^2 + h^2} d(F_1 * \cdots * F_n(x)). \tag{1.3}$$

This is not necessarily non-decreasing with increasing n, but converges to 0 as $n \to \infty$ for every $h > 0$.

The aim of the present paper is to discuss the behavior of (1.3) for large n. We assume throughout that

$$X_i \geqq 0, \qquad i = 1, 2, \cdots. \tag{1.4}$$

Let

$$f_i(s) = \int_0^{\infty} e^{-sx} dF_i(x), \qquad i = 1, 2, \cdots. \tag{1.5}$$

be the Laplace transform of $F_i(x)$ and set

$$\varphi_n(s) = \prod_{i=1}^{n} f_i(s). \tag{1.6}$$

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I have shown the following fact in proving a renewal theorem [2] which is stated as

LEMMA 1. *Let $\{X_i\}$ be a sequence of non-negative independent random variables, and suppose that*

$$0 < m_i = E(X_i) < \infty, \qquad i = 1, 2, \cdots, \tag{1.7}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} m_i = m, \tag{1.8}$$

and

$$\lim_{A\to\infty} \int_A^\infty x\, dF_n(x) = 0 \tag{1.9}$$

holds uniformly with respect to n. Then we have

$$\lim_{s\to+0} s \sum_{n=1}^{\infty} \varphi_n(s) = \frac{1}{m}. \tag{1.10}$$

This lemma is an essential part in proving the renewal theorem I have got and we shall consider the consequence of it concerning the typical function (1.3), and discuss about the behavior of (1.3) with $h = h(n)$ ($h(n) \to \infty$) as $n \to \infty$.

2. It seems convenient to state certain facts of elementary nature as lemmas.

LEMMA 2. *Besides the hypotheses of Lemma* 1, *we further suppose that*

$$F_i(x) - F_i(0) \leqq Ax^p, \qquad \textit{for} \quad 0 < x < \delta, \tag{2.1}$$

where δ, A are constants independent of $1 \leqq i < \infty$ and $p > 1$. Then there exist s_0 and B independent of i such that

$$f_i(s) \leqq Bs^{-q}, \qquad \textit{for} \quad 0 < s_0 < s, \tag{2.2}$$

q being any positive number less than p.

Proof. Put $\alpha = q/p$. Without loss of generality we can suppose $q > 1$. Then $\alpha < 1$, $\alpha p > 1$. We have

$$f_i(s) = \int_0^\infty e^{-sx} dF_i(x)$$

$$= \int_0^{s^{-\alpha}} + \int_{s^{-\alpha}}^\infty$$

$$\leqq \int_0^{s^{-\alpha}} dF_i(x) + e^{-s^{1-\alpha}} \int_{s^{-\alpha}}^\infty dF_i(x). \tag{2.3}$$

Thus by (2.1) there exists an $s_0' > 0$ such that the last expression does not exceed

$$As^{-\alpha p} + e^{-s^{1-\alpha}}$$

$$= As^{-q} + e^{-s^{1-\alpha}}.$$

Since $e^{-s^{1-\alpha}} = o(s^\beta)$, β being any positive number, there exist s_0 and B independently of i such that

$$f_i(s) \leqq Bs^{-q}.$$

LEMMA 3. *Under the conditions and notations of Lemma* 2

(i) *there exist* s_0 *and* A *such that*

$$\sum_{n=1}^{\infty} \varphi_n(s) \leqq As^{-q}, \qquad (s > s_0), \tag{2.4}$$

q being any positive number less than p,

(ii) *we have, for every* $s > 0$,

$$-\sum_{n=1}^{\infty} \varphi_n'(s) < \infty. \tag{2.5}$$

Proof. (i) By Lemma 2, there exists an s_0' (> 0) such that

$$\varphi_n(s) \leqq B^n s^{-nq}.$$

Therefore we have

$$\sum_{n=1}^{\infty} \varphi_n(s) \leqq \sum_{n=1}^{\infty} B^n s^{-nq} = \frac{Bs^{-q}}{1 - Bs^{-q}} \leqq As^{-q}, \qquad s > s_0,$$

for some positive constants A and s_0.

(ii) We have

$$0 \leqq -f_i'(s) = \int_0^\infty xe^{-sx} dF_i(x) \leqq \frac{1}{s} \int_0^\infty dF_i(x) = \frac{1}{s}, \tag{2.6}$$

and

$$f_i(s) = \int_0^\delta + \int_\delta^\infty \leqq F_i(\delta) - F_i(0) + e^{-\delta s},$$

which does not exceed, by (2.1),

$$A\delta^p + e^{-\delta s}, \tag{2.7}$$

where δ is the one in (2.1). If we take $\delta_1 = \delta_1(s)$ such that $Ap\delta_1^{p-1} = se^{-\delta_1 s}$, then (2.7) has a minimum value at $\delta = \delta_1$ and $A\delta_1^p + e^{-\delta_1 s} = \theta < 1$, $\theta = \theta(s)$. Thus we have

$$f_i(s) < \theta. \tag{2.8}$$

Hence by (2.6) and (2.8) we finally have

$$- \varphi_n'(s) = - \sum_{\iota=1}^{n} f_1(s) \cdots f_{\iota-1}(s) f_\iota'(s) f_{\iota+1}(s) \cdots f_n(s)$$

$$\leqq \frac{1}{s} n\theta^{n-1},$$

which proves (2.5).

3. We shall now prove the following

THEOREM 1. *Let $\Phi_{\sigma_n}(h)$ be the typical function of $\sigma_n(x) = F_1 * F_2 * \cdots * F_n(x)$, $F_\iota(x)$ being a distribution function. If conditions in Lemma 2 are satisfied, then we have*

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=1}^{\infty} \Phi_{\sigma_n}(h) = \frac{\pi}{2m}. \tag{3.1}$$

Proof. Since

$$\int_0^\infty \sin hy e^{-sy} dy = \frac{h}{s^2 + h^2}, \qquad s > 0,$$

we have

$$\Phi_{\sigma_n}(h) = h \int_0^\infty \left(\int_0^\infty e^{-xs} \sin hs ds \right) d\sigma_n(x)$$

$$= h \int_0^\infty \sin hs ds \int_0^\infty e^{-xs} d\sigma_n(x)$$

$$= h \int_0^\infty \sin hs \varphi_n(s) ds. \tag{3.2}$$

and hence we have

$$\frac{1}{h} \sum_{n=1}^{\infty} \Phi_{\sigma_n}(h) = \sum_{n=1}^{\infty} \int_0^\infty \sin hs \varphi_n(s) ds$$

$$= \int_0^\infty \sin hs \sum_{n=1}^{\infty} \varphi_n(s) ds. \tag{3.3}$$

The interchange of $\int$ and $\sum$ is legitimate here, because for small s, by Lemma 1,

$$s \sum_{n=1}^{\infty} \varphi_n(s)$$

is bounded and $\varphi_n(s) \geqq 0$, and for large s, by Lemma 3, $\sum \varphi_n(s)$ is integrable, taking $p > q > 1$. We divide the right hand side integral into two parts

$$\left(\int_0^\delta + \int_\delta^\infty \right) \sin hs \sum_{n=1}^{\infty} \varphi_n(s) ds = I_1 + I_2,$$

say, δ being some positive constant. Then since $\sum\varphi_n(s)$ is integrable over (δ, ∞), Riemann–Lebesgue lemma shows

$$\lim_{h\to\infty} I_2 = 0. \tag{3.4}$$

Since

$$\lim_{h\to\infty}\int_0^\delta \frac{\sin hs}{s}\, ds = \frac{\pi}{2},$$

we get

$$I_1 - \frac{\pi}{2m} = \int_0^\delta \frac{\sin hs}{s}\left(s\sum_{n=1}^\infty \varphi_n(s) - \frac{1}{m}\right)ds + o(1), \tag{3.5}$$

as $h\to\infty$.

Now if we put

$$\chi(s) = s\sum_{n=1}^\infty \varphi_n(s) - \frac{1}{m},$$

then it holds by Lemma 1 that

$$\lim_{s\to+0}\chi(s) = 0. \tag{3.6}$$

And $s\chi(s)$ is a function of bounded variation, and the total variation over $(0, u)$ is

$$\begin{aligned}\int_0^u |d(s\chi(s))| &\leqq \int_0^u \left|d\left(s^2\sum_{n=1}^\infty \varphi_n(s)\right)\right| + \frac{u}{m}\\ &= \int_0^u \left|2s\sum_{n=1}^\infty \varphi_n(s) + s^2\frac{d}{ds}\sum_{n=1}^\infty \varphi_n(s)\right| ds + \frac{u}{m}.\end{aligned} \tag{3.7}$$

The differentiability of $\sum\varphi_n(s)$ is easily verified, for the series (2.5) of Lemma 3 (ii) is uniformly convergent in every finite interval not containing the origin.

The series $s\sum\varphi_n(s)$ is bounded for small s, and it follows, putting $|s\sum\varphi_n(s)|\leqq M$ for small s, that (3.7) does not exceed

$$\begin{aligned}&2Mu + \int_0^u s^2\left|\frac{d}{ds}\sum_{n=1}^\infty \varphi_n(s)\right| ds + \frac{u}{m}\\ &= 2Mu - \int_0^u s^2\frac{d}{ds}\sum_{n=1}^\infty \varphi_n(s)\, ds + \frac{u}{m}\\ &= 2Mu - \left[s^2\sum_{n=1}^\infty \varphi_n(s)\right]_0^u + 2\int_0^u s^2\sum_{n=1}^\infty \varphi_n(s)\, ds + \frac{u}{m}\\ &\leqq 2Mu + 2M\int_0^u ds + \frac{u}{m}\\ &\leqq \left(4M + \frac{1}{m}\right)u.\end{aligned}$$

Hence we get

$$\int_0^u |d(s\chi(s))| = O(u) \qquad (u \to 0). \tag{3.8}$$

(3.8) with (3.6) is nothing but the Young's condition for the convergence of Fourier series. Thus we have shown that

$$\lim_{h\to\infty} I_1 = \frac{\pi}{2m},$$

which is, with (3.4), the required conclusion.

4. In this section we shall prove the theorem.

THEOREM 2. *Let $N(h)$ be any integral valued function such that*

$$\frac{N(h)}{h} \to \infty \qquad (h \to \infty). \tag{4.1}$$

Then under the conditions of Theorem 1, we have

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=1}^{N(h)} \Phi_{\sigma_n}(h) = \frac{\pi}{2m}. \tag{4.2}$$

For the proof, we show some lemmas.

LEMMA 4. *Under the conditions of Theorem 1, there exists a $\theta = \theta(\delta, A)$ less than 1, such that*

$$\sum_{i=n+1}^{\infty} \varphi_i(s) \leqq C\theta^n, \qquad \text{for} \quad \delta \leqq s \leqq A, \tag{4.3}$$

where δ, A are any positive constants and C is a constant independent of n.

By (2.8), there exists a $\theta_1 = \theta_1(s)$ such that $f_i(s) < \theta_1$. $\theta_1(s)$ is a continuous function of s and $\theta_1(s) < 1$ *for* $\delta \leqq s \leqq A$. Let $\max_{\delta \leqq s \leqq A} \theta_1(s) = \theta(\delta, A) = \theta$. Then $\theta(\delta, A) < 1$. Hence

$$\sum_{i=n+1}^{\infty} \varphi_i(s) \leqq \sum_{i=n+1}^{\infty} \theta^i = \frac{\theta^n}{1-\theta} = C\theta^n.$$

LEMMA 5. *Assume the conditions of Lemma 1 and let ε be an arbitrary positive number. Then there exist $\delta = \delta(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ such that*

$$\varphi_n(s) = e^{-sn(m+\delta_n+\eta_n)}, \qquad \text{for} \quad s < \delta, \quad n > n_0, \tag{4.4}$$

where $\eta_n = \eta_n(s)$, δ_n is independent of s and $|\delta_n| < \varepsilon$, $|\eta_n| < \varepsilon$.

This was proved in my former paper [2].

LEMMA 6. *Under the conditions of Lemma 1, there exist positive constants m_1 and D such that for $s < 1$,*

$$\sum_{i=n+1}^{\infty} \varphi_i(s) \leqq Ds^{-1}e^{-snm_1}, \qquad n > n_0. \tag{4.5}$$

This is immediate from Lemma 5, because, for $n > n_0$

$$\sum_{i=n+1}^{\infty} \varphi_i(s) \leqq \sum_{i=n+1}^{\infty} e^{-sn(m-2\varepsilon)} = \frac{e^{-nm_1}}{1-e^{-sm_1}} \leqq D\frac{e^{-snm_1}}{s},$$

where we have put $m - 2\varepsilon = m_1$.

Now we shall prove Theorem 2. Put

$$\begin{aligned} J_n(h) &= \int_0^{\infty} \sin sh \sum_{i=n+1}^{\infty} \varphi_i(s) ds \\ &= \int_0^{\delta} + \int_{\delta}^{A} + \int_A^{\infty} \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where ε is any positive number and δ, A are those of Lemma 5. We take B such that (2.2) holds, q being a positive number less than p, and A such that $A^q > B$. Further we take n_0 such that

$$(1 - BA^{-q})^{-1}B^n < A^{nq-1} < \varepsilon, \qquad n > n_0.$$

Then since $\varphi_i(s) \leqq B^i s^{-qi}$ by (2.2) we have

$$\begin{aligned} |I_3| &\leqq \int_A^{\infty} \sum_{i=n+1}^{\infty} \varphi_i(s) ds \leqq \int_A^{\infty} \frac{B^{n+1}s^{-q(n+1)}}{1-Bs^{-q}} ds \\ &\leqq \frac{1}{1-BA^{-q}} \cdot \frac{B^{n+1}}{A^{q(n+1)-1}} < \varepsilon, \end{aligned}$$

that is, it holds that for $n > n_0$, uniformly with respect to h

$$|I_3| < \varepsilon. \tag{4.6}$$

For this ε, we take n_1, such that $C(A-\delta)\theta^n < \varepsilon$, where θ and ε are those in Lemma 4. Lemma 4 shows

$$|I_2| \leqq \int_{\delta}^{A} \sum_{i=n+1}^{\infty} \varphi_i(s) ds \leqq (A-\delta)C\theta^n < \varepsilon. \tag{4.7}$$

Finally by making use of Lemma 6

$$\begin{aligned} |I_1| &\leqq \left| \int_0^{\delta} \sin hs \sum_{i=n+1}^{\infty} \varphi_i(s) ds \right| \\ &\leqq h \int_0^{\delta} s \cdot D \cdot s^{-1} e^{-snm_1} ds \\ &\leqq D \cdot h \cdot \frac{1}{nm_1}. \end{aligned} \tag{4.8}$$

In (4.8), we put $n = N(h)$. Since $h/N(h) \to 0$, we can take h such that $Dh/(nm_1) < \varepsilon$ and we let $N(h) > \max(n_0, n_1)$. Then we get, by (4.6), (4.7) and (4.8),

$$|J_{N(h)+1}(h)| < 3\varepsilon. \tag{4.9}$$

Theorem 1 and (4.9) with

$$J_{N(h)+1} = \int_0^\infty \sin hs \sum_{n=N(h)+1}^\infty \varphi_n(s)\,ds$$
$$= \frac{1}{h} \sum_{n=N(h)+1}^\infty \Phi_{\sigma_n}(h),$$

show the validity of Theorem 2.

5. If $n(h)$ is an integral valued function such that $n(h)/h \to 0$, $(n(h) \to \infty)$, then it is evident that

$$\frac{1}{h} \sum_{n=1}^{n(h)} \Phi_{\sigma_n}(h) \to 0, \qquad h \to \infty,$$

since $\Phi_{\sigma_n}(h) \leqq 1$. Hence by Theorem 2, we have

$$\lim_{h\to\infty} \frac{1}{h} \sum_{n=n(h)+1}^{N(h)} \Phi_{\sigma_n}(h) = \frac{\pi}{2m},$$

$N(h)$ is the one in Theorem 2. This suggests the existence of $\lim \Phi_{\sigma_n}(nh)$, h being a constant. Indeed we have

$$\lim_{n\to\infty} \Phi_{\sigma_n}(nh) = \frac{h^2}{m^2 + h^2}.$$

This is an immediate consequence of the law of large numbers, under certain conditions, for

$$\Phi_{\sigma_n}(nh) = \int_{-\infty}^\infty \frac{n^2h^2}{x^2 + n^2h^2}\, d\sigma_n(x)$$
$$= \int_{-\infty}^\infty \frac{h^2}{x^2 + h^2}\, d\sigma_n(nx),$$

and $\sigma_n(nx)$ converges to $\mathcal{E}_m(x)$ (law of large numbers), where

$$\mathcal{E}_m(x) = \begin{cases} 0, & x < m, \\ 1, & x > m. \end{cases}$$

We shall, here, prove this under the conditions of Theorem 1.

THEOREM 3. *Under the conditions of Theorem* 1, *we have, for any positive h*

$$\lim_{n\to\infty} \Phi_{\sigma_n}(nh) = \frac{h^2}{m^2 + h^2}.$$

Proof. We take as a δ the same one as in Lemma 5. Let

$$\left| nh \int_\delta^\infty \sin nhs\, \varphi_n(s)\,ds \right|$$
$$\leqq nh \left| \int_\delta^A \right| + nh \left| \int_A^\infty \right|,$$

h being a fixed positive number. Making use of Lemma 4 in the first integral and Lemma 2 in the second integral, the above does not exceed

$$nC\theta^n + nB^n\int_A^\infty s^{-nq}ds$$

$$= nC\theta^n + \frac{nB^n}{(nq-1)A^{nq-1}}.$$

Let $A^q > B$. Then this tends to zero as $n \to \infty$. Hence

$$\lim_{n\to\infty} nh\int_\delta^\infty \sin nhs\varphi_n(s)\,ds = 0. \tag{5.1}$$

Now we have obviously

$$\lim_{n\to\infty} nh\int_\delta^\infty \sin nhs e^{-nms}\,ds = 0, \tag{5.2}$$

from which it results

$$nh\int_0^\delta \sin nhs\, e^{-nms}ds = nh\int_0^\infty \sin nhs\, e^{-nms}ds + o(1)$$

$$= \frac{h^2}{m^2+h^2} + o(1). \tag{5 3}$$

We consider

$$H_n = nh\int_0^\delta \sin nhs\,(\varphi_n(s) - e^{-nsm})\,ds$$

$$= n\int_0^{\frac{k}{n}} + n\int_{\frac{k}{n}}^\delta \equiv L_1 + L_2,$$

k being any positive number. Then

$$|L_1| \leqq nh\left|\int_0^{\frac{k}{n}} nhs e^{-nms}(e^{-n\varepsilon_n s} - 1)\,ds\right|,$$

where we have put $\varphi_n(s) = e^{-n(m+\varepsilon_n)s}$, $\varepsilon_n \to 0$, by Lemma 5. Putting $\max_{0<s\leq k/n} \varepsilon_n = \varepsilon'_n$,

$$|L_1| \leqq n^3\varepsilon'_n h^2\int_0^{\frac{k}{n}} s^2e^{-nms}ds$$

$$\leqq nh^2k^2\varepsilon'_n\int_0^{\frac{k}{n}} e^{-nms}ds \leqq k^3h^2\varepsilon'_n.$$

Hence

$$\lim_{n\to\infty} L_1 = 0. \tag{5.4}$$

Next we have

$$|L_2| \leqq nh\int_{\frac{k}{n}}^{\delta} nhse^{-nms}(1-e^{-n\varepsilon_n s})ds$$

$$\leqq 2n^2h^2\int_{\frac{k}{n}}^{\delta} se^{-nms}ds$$

$$= 2n^2h^2\left\{\left[s\frac{e^{-nms}}{nm}\right]_{\delta}^{\frac{k}{n}} + \frac{1}{nm}\int_{\frac{k}{n}}^{\delta} e^{-nms}ds\right\}$$

$$\leqq 2\frac{kh^2e^{-km}}{m} + \frac{e^{-km}-e^{-nm\delta}}{n^2m^2}h^2.$$

Thus we have

$$\limsup_{n\to\infty} |L_2| \leqq \frac{2kh^2e^{-km}}{m}. \tag{5.5}$$

Using (5.1), (5.2) and (5.3), we have

$$\Phi_{\sigma_n}(nh) - \frac{h^2}{m^2+h^2} = nh\int_0^{\infty}\sin nhs\,\varphi_n(s)ds - \frac{h^2}{m^2+h^2}$$

$$= nh\int_0^{\delta}\sin nhs\,\varphi_n(s)ds + nh\int_{\delta}^{\infty}\sin nhs\,\varphi_n(s)ds - nh\int_0^{\delta}\sin nhse^{-nms}ds + o(1)$$

$$= nh\int_0^{\delta}\sin nhs\,(\varphi_n(s) - e^{-nms})ds + o(1)$$

$$= H_n + o(1).$$

(5.4) and (5.5) show

$$\limsup_{n\to\infty}\left|\Phi_{\sigma_n}(nh) - \frac{h^2}{m^2+h^2}\right| \leqq \frac{2kh^2e^{-km}}{m}.$$

since k is arbitrary, we must have

$$\lim_{n\to\infty}\Phi_{\sigma_n}(nh) = \frac{h^2}{m^2+h^2}$$

which proves our theorem.

References

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[2] T. Kawata, A renewal theorem, Journ. Math. Soc. Japan. (under the press)

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