ON POSITIVELY INFINITE SINGULARITIES OF A SOLUTION OF THE EQUATION $\Delta u + k^2 u = 0$

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In the present paper we shall show that a theorem analogous to that of G. C. Evans [1] on harmonic functions remains valid also for solutions of the equation $\Delta u + k^2 u = 0$ with $k > 0$.

THEOREM. *Let D be a bounded domain in the plane, S the exterior frontier of a closed set F lying in D, and D' the portion of D exterior to S. Then a necessary and sufficient condition that there exists a function $u(\mathrm{M})$ finite in D' which satisfies the differential equation*

$$\Delta u + k^2 u = 0 \quad in\ D',$$

where k is a positive constant, and possesses the boundary behavior

$$\lim_{\substack{\mathrm{M} \to \mathrm{Q} \\ \mathrm{M} \in D'}} u(\mathrm{M}) = +\infty$$

for all $\mathrm{Q} \in S$, is that S be of zero capacity. Moreover, it will be seen that F and S are identical provided the condition is fulfilled.

Proof. Let S_n be the boundaries, approximating to S, of a sequence of regular nested domains D_n approximating to D' and containing the part of D exterior to S_n.

In the first step, we shall show that F and S are identical provided the condition is fulfilled. When such a function $u(\mathrm{M})$ is given, we define a continuous function $u_n(\mathrm{M})$ as follows:

$u_n = u$ in D_n, therefore u_n satisfies the equation $\Delta u + k^2 u = 0$ in D_n. $\Delta u_n = 0$ in $D - D_n$, $u_n = u$ on the boundary S_n.

For sufficiently large m exceeding n, we have an inequality

$$(\mathrm{I}) \qquad u_n \leqq u$$

in the region between S_n and S_m. In fact, we have $u_n = u$ on S_n and $u_n \leqq u$ on S_m so that (I) holds on the boundary of the region between S_n and S_m. Next we can show, that this property remains to hold also in the interior of the domain. From the beginning, we may take, as the domain D', a domain in which $u > 0$ since our problem is concerned with local property. Hence $\Delta u = -k^2 u < 0$ in the above-mentioned region, that

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is, the function u is superharmonic, while u_n is harmonic in the same region. So the inequality on the boundary remains to hold in the region.

Now the functions u_n become superharmonic in the domain D, since, by virtue of our assumption $u_n > 0$, we have $\Delta u_n = -k^2 u_n < 0$ in D_n, $\Delta u_n = 0$ in $D - D_n$, and for any point Q on the boundary S_n

$$u_n(\mathrm{Q}) \geqq \mathfrak{M}_{\mathrm{Q}}(u_n)$$

where $\mathfrak{M}_{\mathrm{Q}}(u_n)$ denotes the mean value of u_n on a small circle with Q as center. Therefore $\{u_n\}$ is really a monotone increasing sequence of functions superharmonic in the domain D. The limit function $v \equiv \lim_{n\to\infty} u_n$ remains to be superharmonic in D. On the other hand, we have the identities

$$v \equiv u \quad \text{in } D',$$
$$v \equiv \infty \quad \text{in } D - D'.$$

Our argument after this can be reduced to that of Evans [1]. Following Evans we thus conclude that $F \equiv S$.

In the second step, we can readily recognize that our condition is *necessary* by exactly the same reasoning as that of Evans.

In the third step, we shall show that our condition is *sufficient*, that is, if a closed bounded set S of capacity zero is given, there exists a function which is infinite at every point of S but at no other point and satisfies the equation $\Delta u + k^2 u = 0$ in D'.

Now Evans [1] has introduced a potential in case of space which, in case of plane, may be stated as follows. Given n points $\mathrm{P}_1, \cdots, \mathrm{P}_n$ on S, we choose an $(n+1)$st point P on S so that the potential

$$v(\mathrm{P}, \mathrm{P}_1, \cdots, \mathrm{P}_n) = \frac{1}{n}\sum_{\nu=1}^{n} \log \frac{1}{\mathrm{PP}_\nu}$$

is a minimum which is denoted by $w(\mathrm{P}_1, \cdots, \mathrm{P}_n)$. Let further the upper bound of $w(\mathrm{P}_1, \cdots, \mathrm{P}_n)$ as $\mathrm{P}_1, \cdots, \mathrm{P}_n$ vary on S be denoted by v_n. By definition, there exist a sequence of sets of n points $\mathrm{P}_1{}^i, \cdots, \mathrm{P}_n{}^i$ and a corresponding sequence of points P^i, all lying on S, such that

$$\lim_{i\to\infty} v(\mathrm{P}^i, \mathrm{P}_1{}^i, \cdots, \mathrm{P}_n{}^i) = v_n.$$

We choose from these a subsequence P^{i_κ}; $\mathrm{P}_1{}^{i_\kappa}, \cdots, \mathrm{P}_n{}^{i_\kappa}$ converging to a limit set P^0; $\mathrm{P}_1{}^0, \cdots, \mathrm{P}_n{}^0$ on S. Let $v_n(\mathrm{M})$ be the potential defined by

$$v_n(\mathrm{M}) = \frac{1}{n}\sum_{\nu=1}^{n}\left\{\log\frac{1}{\mathrm{MP}_\nu{}^0}\right\} \quad \text{for } \mathrm{M} \in D',$$

the value $+\infty$ being admitted.

Then it is known that there holds

$$v_n(\mathrm{P}) \geqq v_n \qquad \text{for all P on } S,$$

and further that, if the capacity of S is zero, v_n tends to $+\infty$ as $n \to \infty$

and consequently, for any positive integer j there exists n_j such as

$$2^j < v_{n_j} \leqq v_{n_j}(\mathrm{P}) \qquad \mathrm{P} \text{ on } S.$$

Finally, put

$$V(\mathrm{M}) = \sum_{j=1}^{\infty} V_j(\mathrm{M}),$$

where

$$V_j(\mathrm{M}) = 2^{-j} v_{n_j}(\mathrm{M});$$

$V(\mathrm{M})$ is then a potential which is finite in D' and tends to $+\infty$ as M approaches any point of S.

We shall now define a corresponding function for the differential equation under consideration. Namely, we put

$$V'(\mathrm{M}) = \sum_{j=1}^{\infty} V_j'(\mathrm{M})$$

where

$$V_j'(\mathrm{M}) = 2^{-j} v'_{n_j}(\mathrm{M}), \qquad v'_{n_j}(\mathrm{M}) = \frac{1}{n_j} \sum_{\nu=1}^{n_j} (-Y_0(k\mathrm{MP}_\nu{}^0)).$$

Here $\mathrm{P}_\nu{}^0$ and n_j have the same meaning as in the Evans' case explained above, and Y_0 denotes the Neumann's cylindrical function.

Write a circle with a fixed radius ρ around every point of S as center where ρ is supposed to be small enough so that

$$|-Y_0(k\mathrm{MP}_\nu{}^0)| < -Y_0(k\rho) \qquad (\nu = 1, 2, \cdots, n_j)$$

for every point M lying outside all these circles. We then get for such M an estimation

$$|V_j'(\mathrm{M})| \leqq 2^{-j} \left| \frac{1}{n_j} \sum_{\nu=1}^{n_j} (-Y_0(k\mathrm{MP}_\nu{}^0) \right| < 2^{-j}(-Y_0(k\rho)).$$

Consequently, the series stated above for defining $V'(M)$ converges uniformly in the domain exterior to the above circles whenever ρ is fixed sufficiently small. Besides, it is obvious that the function $V'(M)$ satisfies the equation

$$\Delta u + k^2 u = 0 \qquad \text{in } D'.$$

Finally, we shall show that

$$\lim_{\mathrm{M} \to \mathrm{Q} \in S} V'(\mathrm{M}) = +\infty.$$

Since $Y_0(kx)$ behaves around $x = 0$ such as

$$Y_0(kx) = \frac{2}{\pi} \log x + \phi(x),$$

$\phi(x)$ being bounded, we can find a positive constant K such that

$$-Y_0(kx) > K \log(1/x)$$

for x sufficiently near to 0, $|x| < a$ say. We may assume that the given set S is contained in a square R with sides less than $a/\sqrt{2}$. Otherwise, we have only to devide a domain including S into finite number of subdomains each of which is contained in such a square.

Now, we have

$$V_j'(\mathrm{M}) \equiv \frac{1}{2^j n_j} \sum_{\nu=1}^{n_j} (-Y_0(k\mathrm{MP}_\nu{}^0)) > \frac{K}{2^j n_j} \sum_{\nu=1}^{n_j} \left(\log \frac{1}{\mathrm{MP}_\nu{}^0}\right) \equiv KV_j(\mathrm{M}),$$

and therefore, by Evans' theorem stated above,

$$V'(\mathrm{M}) > KV(\mathrm{M}) \to +\infty$$

as M tends to any point of S. Thus our theorem has been established completely.

We would notice that our theorem in the plane can be extended to the case of space, using the function $\cos kr/r$ instead of $-Y_0(kr)$ in the above argument.

REFERENCE

[1] G. C. EVANS, Potentials and positively infinite singularities of harmonic functions. Monatsh. für Math. u. Phys. 43 (1936), 419—424.

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