A REMARK ON RENGEL'S THEOREM CONCERNING SZEGŐ'S CONJECTURE

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Let $w = f(z)$ $(f(0)=0,\ f'(0)=1)$ be regular and schlicht in $|z|<1$ and D be the image of $|z|<1$ on the w-plane and Γ be its boundary. We draw n equi-angular half-lines through $w=0$ and let $w_1, \ldots, w_n$ be the points of intersection of these lines with Γ, such that the segment $\overline{0w_i}$ $(i=1,2,\cdots,n)$, except w_i lies in D and put

$$d = Max(|w_1|, \cdots, |w_n|).$$

Then Rengel[1)] proved the following Szego's conjecture.

<u>Theorem.</u> $\quad d \geqq \sqrt[n]{1/4}$,

the equality holds, only when $f(z) = e^{i\theta}E_n(e^{-i\theta}z)$, where

$$E_n(z) = \frac{z}{\sqrt[n]{(1+z^n)^2}}.$$

When n is odd, Rengel assumed that any half-line through $w=0$ meets Γ at a finite distance, but by considering $\rho f(z/\rho)$ $(\rho>1)$, and making $\rho \to 1$, we see that the theorem holds in general. In the following lines, we shall simplify somewhat Rengel's original proof. We use the following Rengel's lemma.

Let $\Delta: 0 \leqq x \leqq b,\ 0 \leqq y \leqq a$ be a rectangle on the $z=x+iy$-plane and $\sigma_i:\ 0 < x_i \leqq x \leqq x_i' \leqq b$, $y = \eta_i$ $(0<\eta_i<a)$ $(i=1,2,\cdots,n)$ be segments in Δ parallel to the x-axis and Δ_0 be the complement of $\sum_{i=1}^{n}\sigma_i$ with respect to Δ.

Let $w=f(z)$ be regular in Δ_0 and J be the area of the image of Δ_0 on the w-plane by $w=f(z)$.

Lemma.[2)] If

$$\int_0^b |f'(x+iy)|\,dx \geqq L,\quad y \neq \eta_i\ (i=1,2,\cdots,n),$$

then

$$\frac{a}{b} \leqq \frac{J}{L^2},$$

the equality holds, only when $f(z) = \alpha z + \beta$.

In case the inequality holds, let I be an interval on the y-axis, such that if $y \in I$,

$$\int_0^b |f'(x+iy)|\,dx \geqq L + c,\quad c>0,$$

then

$$\frac{a}{b} \leqq \frac{J}{L^2} - \frac{2a'c}{bL}.$$

<u>Proof of the theorem.</u>

We supose that one of n equi-angular half-lines through $w=0$ coincides with the positive real axis and

$$d \leqq \sqrt[n]{1/4}. \qquad (1).$$

Then we shall prove that $f(z) = E_n(z)$, so that $d = \sqrt[n]{1/4}$. Let $|z| = q\rho$ $(\rho>0)$ be the greatest circle, which is contained in the image of $|z| \leqq \rho$ by $E_n^{-1}(f(z))$, then $\lim_{\rho\to 0} q = 1$.

By a branch of $\zeta = \xi + i\eta = \log z$, we map $\rho \leqq |z| \leqq 1$ on a rectangle $\Delta:\ \log\rho \leqq \xi \leqq 0,\ 0 \leqq \eta \leqq 2\pi$ and put $g(\zeta) = \log E_n^{-1}(f(e^\zeta))$. Then Δ may contain branch points of $g(\zeta)$. Through these branch points, we draw parallel lines to the η-axis, which we call exceptional parallels. Then Δ is divided into a finite number of rectangles $\{\Delta_i\}$. In each Δ_i, $g(\zeta)$ is one-valued and regular. As is easily seen, the sum J of areas of the images of these rectangles on the ζ_1-plane by $\zeta_1 = g(\zeta)$ is

$$J \leqq 2\pi \log\frac{1}{q\rho}.$$

In virtue of (1), as Rengel proved, the length of the image of a non-exceptional parallel on the ζ_1-plane is $\geqq 2\pi$. Hence putting $a = \log\frac{1}{\rho}$, $b = 2\pi$, $L = 2\pi$ in the Lemma, we have

$$\frac{\log 1/\rho}{2\pi} \leqq \frac{2\pi \log 1/q\rho}{(2\pi)^2}. \qquad (2)$$

If the equality in (2) does not hold, then

$$\frac{\log 1/\rho}{2\pi} \leqq \frac{2\pi \log 1/q\rho}{(2\pi)^2} - \frac{2a'c}{(2\pi)^2}$$

$$(a' > 0,\ c > 0),$$

which is impossible, since $\lim_{\rho \to 0} q = 1$.

Hence the equality in (2) holds, so that

$$\log E_n^{-1}(f(e^{\zeta})) = \alpha\zeta + \beta.$$

From this we have easily, $f(z) = E_n(z)$, so that $d = \sqrt[n]{1/4}$. If one of n equi-angular half-lines through $w = 0$ is $\arg w = \theta$, and $d \leqq \sqrt[n]{1/4}$, then $f(z) = e^{i\theta}E_n(e^{-i\theta}z)$ and $d = \sqrt[n]{1/4}$.

Hence our theorem is proved.

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1) E. Rengel : Über einige Schlitztheoreme der konformen Abbildung. Schriften des math. Seminars und des Instituts für angewandte Math. der Universität Berlin. Band 1. Heft 4 (1933); H. Grötzsch : Einige Bemerkungen zur schlichten konformen Abbildung. Jahresber. d. deutchen Math. Ver. 44 (1934).

2) E. Rengel : l. c. 1).

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