ON PRIMAL ELEMENTS IN A MODULAR LATTICE

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The concept of primal ideals, introduced by L. Fuchs (3) for commutative rings and generalized by C. W. Curtis (2) to integral modular lattice ordered semigroups with ascending chain condition, shall be extended in this paper to modular lattices with maximum condition by our method in (4).

§ 1.

Let L be a modular lattice with ascending chain condition and Θ be a set of lattice congruences on L such that any meet of a finite number of congruences in Θ is also in Θ. We denote by $a(\theta)$ the greatest element congruent to an element a by a congruence θ on L. The set of elements x satisfying $x(\theta) = 1$ is denoted by $\chi(\theta)$. An element q is said to be primary (with respect to Θ) if $q \in \chi(\theta)$ or $q = q(\theta)$ for every θ in Θ. An element a is called to be primal (with respect to Θ) if $a(\theta_1 \wedge \theta_2) = a$ implies $a(\theta_1) = a$ or $a(\theta_2) = a$ for θ_1 and θ_2 in Θ.

Theorem 1.1. Any primary element is primal.

Proof. Let q be a primary element. $q(\theta_1) > q$ and $q(\theta_2) > q$ imply $q(\theta_1) = 1$, $q(\theta_2) = 1$ and $1 > q$, that is, $q(\theta_1 \wedge \theta_2) = 1 > q$.

Theorem 1.2. Any meet-irreducible element is primal.

Proof. If an element a is not primal then $a(\theta_1 \wedge \theta_2) = a$, $a(\theta_1) > a$ and $a(\theta_2) > a$ for some θ_1, θ_2 in Θ. Since $a(\theta_1) \wedge a(\theta_2) = a(\theta_1 \wedge \theta_2) = a$, a is meet-reducible.

In a meet of elements of L, if we can not replace any component by a element greater than it, we call the meet to be reduced. An irredundant meet of primal elements is said to be shortest if any meet of two or more components is not primal. Reduced and shortest meets are normal.

Theorem 1.3. Let $a = a_1 \wedge a_2 \wedge \cdots \wedge a_n$ be a reduced meet. Then $a(\theta) = a$ if and only if $a_i(\theta) = a_i$ for $i = 1,2,\ldots,n$.

Proof. If $a_i(\theta) = a_i$ for $i = 1,2,\ldots, n$ then $a(\theta) \leq a_i$ and hence $a(\theta) = a$. Conversely, suppose $a(\theta) = a$. Now $a_1(\theta) \wedge a_2(\theta) \wedge \cdots \wedge a_n(\theta) \geq a(\theta)$ and the left hand side is congruent to a by θ. Hence $a = a_1(\theta) \wedge a_2(\theta) \wedge \cdots \wedge a_n(\theta)$. From the reducibility, $a_i(\theta) = a_i$ for $i = 1,2,\ldots,n$.

Theorem 1.4. Let $a = a_1 \wedge a_2 \wedge \cdots \wedge a_n$ be a reduced meet of primal elements a_i. Then a is primal if and only if $a_i(\theta) = a$ implies $a(\theta) = a$ for any θ in Θ and some integer i independent to θ.

Proof. By the theorem 1.3. $a(\theta) = a$ ensures that $a_i(\theta) = a$ for $i = 1,2,\ldots,n$. If the condition is satisfied then a is primal by the definition. If it is not satisfied then there are θ_i such that $a_i(\theta_i) = a_i$ and $a(\theta_i) > a$ for $i = 1,2,\ldots,n$. $a \leq a(\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n) \leq a(\theta_i) \leq a_i(\theta_i) = a_i$. Hence $a(\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n) = a$. If a were primal then $a(\theta_i) = a$ for some i which is a contradiction.

Theorem 1.5. Any element is expressible as a normal meet of a finite number of primal elements.

Proof. First, if we represent the element as an irredundant meet of meet-irreducible elements, it is necessarily reduced. Next, by the Theorem 1.4., grouping its suitable components we obtain a shortest meet. It is easy to see the reducibility of this meet. (Only here we use the modularity of L.)

For any element a, the set of θ in Θ satisfying $a(\theta) = a$ is a M-closed subset of Θ which denoted by M(a). It is well known that the set of all M-closed subsets in Θ forms a distributive lattice, by set-inclusion, to which we refer as $M(\Theta)$. The definition shows that an element a is primal if and only if M(a) is meet-irreducible in $M(\Theta)$. If $a = a_1 \wedge a_2 \wedge \cdots \wedge a_n$ is a normal meet of primal a_i, then, by the theorem

1.3., $M(a) = M(a_1) \cap M(a_2) \cap \dots \cap M(a_n)$ which is a irredundant meet of meet-irreducible elements in $M(\Theta)$ by the theorem 1.4. From the well known theorem for distributive lattices we obtain immediately

Theorem 1.6. If both $a_1 \cap a_2 \cap \dots \cap a_n = b_1 \cap b_2 \cap \dots \cap b_m$ are normal meets of primal components, then $m = n$ and, after a suitable change of indices, $a_i(\theta) = a_i$ if and only if $b_i(\theta) = b_i$ for $i = 1, 2, \dots, n$.

§ 2

Example 1. Let L be an integral modular lattice ordered semigroup with ascending chain condition. A subset k is called to be k-system if it satisfies the condition that (1) k is not empty, (2) k is J-closed and (3) k is multiplicatively closed. To every k-system k, we can make correspond a lattice congruence $\theta(k)$ of L when we define that a and b are congruent by $\theta(k)$ if $a^{-1}b \cap b^{-1}a$ is in k. Then k is a class containing 1 for $\theta(k)$. (See (4)) Now, we adopt the set of $\theta(k)$ for all k-systems k in L as our Θ. It is easy to see that any intersection of k-systems is again a k-system and corresponds to the congruence which is the intersection of congruences corresponding to each k-systems. For any element a, the totality of such elements that $ax^{-1} = a$ forms clearly a k-system to which corresponding congruence in Θ is denoted by k(a).

Theorem 2.1. $a(\theta) = a$ if and only if $\theta \leq k(a)$, for θ in Θ.

Proof. Let $\theta \leq k(a)$. $(a(\theta))^{-1}a$ is congruent to 1 by θ, hence by k(a), which shows $a((a(\theta))^{-1}a)^{-1} = a$. Since $(a(\theta))((a(\theta))^{-1}a) \leq a$, we get $a(\theta) \leq a((a(\theta))^{-1}a)^{-1} = a$ and hence $a = a(\theta)$. Conversely, if $\theta \nleq k(a)$ then there is an element x such that x is congruent to 1 by θ but not by k(a). We have $ax^{-1} > a$. Since $(ax^{-1})x \leq a$ we get $(ax^{-1})^{-1}a \geq x$ which proves that $(ax^{-1})^{-1}a$ is congruent to 1 by θ. Clearly, $a^{-1}(ax^{-1}) = 1$ is also congruent to 1 by θ. Thus ax^{-1} is congruent to a by θ. Therefore $a(\theta) \neq a$.

Theorem 2.2. An element is primal if and only if it is primal in the sense of Curtis (2).

Proof. Curtis (2) has defined the primalness of an element a by the condition that the k-system to which k(a) corresponds be a dual prime ideal of L in the sense of lattice theory. Now, let a be primal in our sense. Assume that $a(x \cup y)^{-1} = a$. The J-closure of all the powers of x is a k-system to which corresponding congruence denoted by θ_1. Similarly θ_2 for y. If t is congruent to 1 by $\theta_1 \cap \theta_2$, then $x^n \leq t$ and $y^m \leq t$ for some n and m, and so $(x \cup y)^{n+m-1} \leq t$ which shows that $at^{-1} = a$. Hence $\theta_1 \cap \theta_2 \leq k(a)$. Therefore, say, $\theta_1 \leq k(a)$ or $ax^{-1} = a$. Thus a is primal in the sense of Curtis. Conversely, if a is not primal in our sense, then $\theta_1 \nleq k(a)$, $\theta_2 \nleq k(a)$ and $\theta_1 \cap \theta_2 \leq k(a)$ for some θ_1 and θ_2 in Θ. Let x be congruent to 1 by θ_1 and not so by k(a). Similarly y for θ_2 and k(a). Then $x \cup y$ is congruent to 1 by $\theta_1 \cap \theta_2$ and hence k(a). Thus, a is not primal in the sense of Curtis.

Example 2. Let L be a modular lattice of finite length, and Θ be the totality of lattice congruences on L. It is well known that Θ is a Boolean algebra isomorphic to the lattice of all subsets of the set P of prime quotients ignoring the projectivity. The prime quotient b/a is said to be belonging to a. By k(a) we denote the element of Θ corresponding to the complement of the subset in P corresponding to all the prime quotients belonging to a. Then $a = a(\theta)$ if and only if $\theta \leq k(a)$. An element a is primal if and only if all the prime quotients belonging to a are projective. Thus, our theorems in §1 yield the well known results that, in irredundant representations by meet-irreducible elements, the prime quotients belonging to components are uniquely determined. We remark finally that an element is primary if and only if all prime quotients in a chain connecting it to 1 are projective.

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