ON COMMUTATIVE T-CLOSURE OPERATORS*)

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1. Let S be a complete lattice and Φ be the set of all T-closure operators φ defined on S, that is, each element a of S is mapped by φ on an element a^{φ} of S such that $a \subset a^{\varphi}$, $(a \cup b)^{\varphi} = a^{\varphi} \cup b^{\varphi}$, $(a^{\varphi})^{\varphi} = a^{\varphi}$, $o^{\varphi} = o$, where o is the zero element of S. Let φ, ψ be two elements of Φ. When $a^{\varphi} \subset a^{\psi}$ for any element a of S, we define $\psi < \varphi$. Then it is known that Φ is a complete lattice by this partial order $<$ [1]. When $(a^{\varphi})^{\psi} = (a^{\psi})^{\varphi}$ for any element a of S, we say that φ and ψ are commutative. Denoting $(a^{\varphi})^{\psi}$ by $a^{\varphi\psi}$, we say $\varphi\psi$ the product of φ and ψ.

In this note, we will obtain the condition that φ and ψ should be commutative, and as an application, we will next investigate what properties a maximal subset Ψ of Φ has, of which any two elements are commutative.

2. In the first place, we will consider the condition for commutability, by the partial order of Φ.

<u>Theorem 1.</u> If $\varphi\psi = \psi\varphi$, then $\varphi\psi = \varphi \cap \psi$.

<u>Proof.</u> If $a^{\varphi} = a$, a is said to be φ-closed. Let C_{φ}, C_{ψ} be the set of all φ-closed elements and ψ-closed elements respectively. Then C_{φ} and C_{ψ} are both complete sublattices over S with respect to meet [2], and $\varphi \cap \psi$ is a T-closure operator determined by $C_{\varphi} \cap C_{\psi}$ [3].

Let a be any element of S, then

$$a^{\varphi\psi} = \bigcap_x (x \in C_{\psi} ; x \supset a^{\varphi}) = \bigcap_x (x \in C_{\psi} ; x \supset a^{\varphi} \cup a^{\psi}).$$

Similarly

$$a^{\psi\varphi} = \bigcap_x (x \in C_{\varphi} ; x \supset a^{\varphi} \cup a^{\psi}).$$

Since, by hypothesis $a^{\varphi\psi} = a^{\psi\varphi}$ and C_{φ}, C_{ψ} are both complete sublattices over S with respect to meet,

$$a^{\varphi\psi} = a^{\psi\varphi} = \bigcap_x (x \in C_{\varphi} \cap C_{\psi} ; x \supset a^{\varphi} \cup a^{\psi}).$$

On the other hand,

$$a^{\varphi \cap \psi} = \bigcap_x (x \in C_{\varphi} \cap C_{\psi} ; x \supset a) = \bigcap_x (x \in C_{\varphi} \cap C_{\psi} ; x \supset a^{\varphi} \cup a^{\psi}).$$

Hence $\varphi\psi = \psi\varphi$. Q.E.D.

<u>Remark.</u> It is easy to see by giving an example that the converse of this theorem does not always hold. But the following corollary is easily verified.

<u>Corollary.</u> When $\varphi\psi < \psi\varphi$, then $\varphi\psi = \psi\varphi$ if and only if $\varphi\psi = \varphi \cap \psi$.

In the next place, we will consider the condition for commutability, by the product of elements of Φ.

<u>Theorem 2.</u> $\varphi\psi = \psi\varphi$ if and only if $\varphi\psi\varphi = \varphi\psi$ and $\psi\varphi\psi = \psi\varphi$.

<u>Proof.</u> As the necessity is obvious, we will prove the sufficiency. Since $a^{\varphi\psi\varphi} = a^{\varphi\psi}$ for any element a of S, $a^{\varphi\psi} \in C_{\varphi}$.

On the other hand, $a^{\psi\varphi} = \bigcap_x (x \in C_{\varphi} ; x \supset a^{\psi})$, hence $a^{\psi\varphi} \subset a^{\varphi\psi}$.

Similarly, from $a^{\psi\varphi\psi} = a^{\psi\varphi}$, we get $a^{\varphi\psi} \subset a^{\psi\varphi}$. Q.E.D.

<u>Corollary.</u> $\varphi\psi = \psi\varphi$ if and only if $\varphi\psi$ and $\psi\varphi$ are both T-closure operators.

<u>Proof.</u> As the necessity follows from Theorem 1, we will prove the sufficiency. By the assumption that $\varphi\psi$ is a T-closure operator, we have

$$(a^{\varphi\psi})^{\varphi\psi} = \bigcap_x (x \in C_{\psi} ; x \supset a^{\varphi\psi\varphi}) = a^{\varphi\psi}.$$

Then $a^{\varphi\psi} \supset a^{\varphi\psi\varphi}$.

On the other hand, $a^{\varphi\psi} \subset a^{\varphi\psi\varphi}$, hence $\varphi\psi = \varphi\psi\varphi$.

Similarly $\psi\varphi = \psi\varphi\psi$. Therefore, by Theorem 2, $\varphi\psi = \psi\varphi$. Q.E.D.

In the above reasonings, we considered from the point of view of the partial order and the product of elements of Φ. We will next consider according to the relation between C_φ and C_ψ. Let C'_φ, C'_ψ be the sets of elements of C_φ, C_ψ which do not belong to $C_\varphi \frown C_\psi$ respectively, then we have the following theorem.

Theorem 3. $\varphi\psi = \psi\varphi$ if and only if any element a of C'_φ and any element b of C'_ψ are not comparable, except the case when there exists at least one element of $C_\varphi \frown C_\psi$ between a and b [4].

Proof. Necessity. Suppose, if possible, that there exist $a \in C'_\varphi$, $b \in C'_\psi$ such that $a \subset b$ and no element of $C_\varphi \frown C_\psi$ is between a and b. Then

$$(a^\varphi)^\psi = a^\psi = \bigcap_x (x \in C_\psi;\ x \supset a) \subset b.$$

Denoting $(a^\varphi)^\psi$ by c, then $c \neq a$.

On the other hand, since $b \supset c$, then $c \,\bar{\in}\, C_\varphi \frown C_\psi$. Hence $(a^\psi)^\varphi = c^\varphi \supset c$ and $c^\varphi \neq c$. Therefore $a^{\psi\varphi} \supset a^{\varphi\psi}$ and $a^{\psi\varphi} \neq a^{\varphi\psi}$.

Sufficiency. Let A be the set of all elements of $C_\varphi \frown C_\psi$ which contains all elements of C'_φ and C'_ψ. Then $A \neq \theta$ follows from $A \ni I$, where I is the unit element of S. Let B denote the set of all elements of $C_\varphi \frown C_\psi$ which are contained in all elements of C'_φ and C'_ψ. Then $B \neq \theta$ follows from the fact that $B \ni o$. And let C be the set of all elements of $C_\varphi \frown C_\psi$ which do not belong to A and B.

Let a_o be any element of S. When we denote by $+$ or $-$ the cases according to whether a_o is contained in some element of B (or C, C'_φ, C'_ψ) or not, it is obvious that there arise the following seven cases to be considered. In the following, we will prove $a_o^{\varphi\psi} = a_o^{\psi\varphi}$. In the cases (i), (iii), (iv), (vii), we can easily see that $a_o^{\varphi\psi} = a_o^{\psi\varphi}$, so we will prove the other cases.

	B	C	C'_φ	C'_ψ
(i)	−	−	−	−
(ii)	− −	− −	+ −	− +
(iii)	−	−	+	+
(iv)	−	+	−	−
(v)	− −	+ +	+ −	− +
(vi)	−	+	+	+
(vii)	[illegible] +	− +	+ +	+ +

Case (ii). As we can consider the latter case similarly, we will prove only the above case. Let $a_\alpha \in C'_\varphi$ such that $a_\alpha \supset a_o$, then $\bigcap_\alpha a_\alpha \in C'_\varphi$. Hence $a_o^\varphi = \bigcap_\alpha a_\alpha \in C'_\varphi$. Therefore $a_o^{\varphi\psi} = \bigcap_x (x \in A;\ x \supset \bigcap_\alpha a_\alpha)$. On the other hand, $a_o^\psi = \bigcap_x (x \in A;\ x \supset a_o)$. Since any element of A contains any element of C'_φ and C'_ψ, $a_o^\psi \supset \bigcap_\alpha a_\alpha$. Hence $a_o^\psi \supset a_o^{\varphi\psi}$. Therefore $a_o^{\varphi\psi} = a_o^\psi$. From $a_o^\psi \in A$, it follows that $a_o^{\psi\varphi} = a_o^\psi$. Therefore $a_o^{\varphi\psi} = a_o^{\psi\varphi}$.

Case (v). We will only consider the above case. Let $a_\alpha \in C'_\varphi$ such that $a_\alpha \supset a_o$, then $\bigcap_\alpha a_\alpha \in C_\varphi$. Then there arise the following two cases: (1) $\bigcap_\alpha a_\alpha \in C$, (2) $\bigcap_\alpha a_\alpha \in C'_\varphi$.

Case (1). Let $c_\gamma \in C$ such that $c_\gamma \supset a_o$, then $\bigcap_\gamma c_\gamma \in C$. Hence $a_o^\varphi = (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma) \in C$. Therefore $a_o^{\varphi\psi} = a_o^\varphi$. On the other hand, since $a_o^\psi = (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma)$, $a_o^{\psi\varphi} = (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma)$. Thus $a_o^{\varphi\psi} = a_o^{\psi\varphi}$.

Case (2). Since $a_o^\varphi = (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma)$, $a_o^{\varphi\psi} = \bigcap_x (x \in C_\psi;\ x \supset a_o^\varphi) = \bigcap_x (x \in C_\psi;\ x \supset (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma))$. Then, in any case between $(\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma) \in C_\varphi \frown C_\psi$ and $(\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma) \,\bar{\in}\, C_\varphi \frown C_\psi$, $\bigcap_\gamma c_\gamma$ is the least element of C_ψ containing $(\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma)$. Hence $a_o^{\varphi\psi} = \bigcap_\gamma c_\gamma$. On the other hand, $a_o^{\psi\varphi} = \bigcap_\gamma c_\gamma$ follows from $a_o^\psi = \bigcap_\gamma c_\gamma$.

Case (vi). In this case, it is sufficient to consider the following two cases: (α) There exists $c_\gamma \in C$ such that $\bigcap_\alpha a_\alpha \supset c_\gamma \supset a_o$, ($\beta$) There exists no element of C between $\bigcap_\alpha a_\alpha$ and a_o and between $\bigcap_\beta b_\beta$ and a_o, where $b_\beta \in C'_\psi$ and $b_\beta \supset a_o$.

Case (α). In this case, $a_o^\varphi = \bigcap_\gamma c_\gamma$, $a_o^\psi = (\bigcap_\gamma c_\gamma) \frown (\bigcap_\beta b_\beta)$, hence $a_o^{\varphi\psi} = \bigcap_x (x \in C_\psi;\ x \supset \bigcap_\gamma c_\gamma) = \bigcap_\gamma c_\gamma$, $a_o^{\psi\varphi} = \bigcap_x (x \in C_\varphi;\ x \supset (\bigcap_\gamma c_\gamma) \frown (\bigcap_\beta b_\beta))$. Suppose that $a_o^{\psi\varphi} \subseteq \bigcap_\gamma c_\gamma$ and $a_o^{\psi\varphi} \neq \bigcap_\gamma c_\gamma$, then $a_o^\varphi \subset \bigcap_\gamma c_\gamma$, $a_o^\varphi \neq \bigcap_\gamma c_\gamma$. This contradicts $a_o^\varphi = \bigcap_\gamma c_\gamma$. Hence $a_o^{\psi\varphi} = \bigcap_\gamma c_\gamma$.

Case (β). Since $a_o^\varphi = (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma)$, $a_o^{\varphi\psi} = \bigcap_x (x \in C_\psi;\ x \supset (\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma))$, where $c_\gamma \supset a_o$, $c_\gamma \in C$. Then $(\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma) \in C'_\varphi$ by the hypothesis. On the other hand, since any element of C'_φ is not comparable with any element of C'_ψ, no element of C'_ψ is between $\bigcap_\gamma c_\gamma$ and $(\bigcap_\alpha a_\alpha) \frown (\bigcap_\gamma c_\gamma)$. Hence $a_o^{\varphi\psi} = \bigcap_\gamma c_\gamma$. Similarly $a_o^{\psi\varphi} = \bigcap_\gamma c_\gamma$. Q.E.D.

As an application of Theorem 3, we will prove the following theorem concerning the set Ψ.

Theorem 4. Ψ is a complete sublattice over Φ with respect to meet.

Proof. As Ψ^{*} contains the unit element I and the zero element o of Φ, it is sufficient to prove that if $\mathcal{X}$ is any subset of Ψ, $\bigcap_{\varphi \in \mathcal{X}} \varphi \in \Psi$ [5]. Let γ be any element of Ψ such that $\gamma \notin \mathcal{X}$. Let $K_{\varphi\gamma} = C_{\varphi} \cap C_{\gamma}$, then it is easy to see that $(\bigcap_{\varphi \in \mathcal{X}} C_{\varphi}) \cap C_{\gamma} = \bigcap_{\varphi \in \mathcal{X}} K_{\varphi\gamma}$. Let L be the set $\bigcap_{\varphi \in \mathcal{X}} C_{\varphi} - \bigcap_{\varphi \in \mathcal{X}} K_{\varphi\gamma}$, that is, the set of all elements which belong to $\bigcap_{\varphi \in \mathcal{X}} C_{\varphi}$ but not to $\bigcap_{\varphi \in \mathcal{X}} K_{\varphi\gamma}$, then $L = \bigcap_{\varphi \in \mathcal{X}} C'_{\varphi\gamma}$, where $C'_{\varphi\gamma}$ denotes the set $C_{\varphi} - K_{\varphi\gamma}$.

On the other hand, since γ is commutative with any $\varphi \in \mathcal{X}$, by Theorem 3 any element of $C'_{\varphi\gamma}$ is not comparable with any element of $C'_{\gamma\varphi}$, with the same exception as Theorem 3, where $C'_{\gamma\varphi}$ denotes the set $C_{\gamma} - K_{\varphi\gamma}$. Then any element of L is not comparable with any element of the set $C_{\gamma} - \bigcap_{\varphi \in \mathcal{X}} K_{\varphi\gamma}$, with the same exception as Theorem 3. As $\bigcap_{\varphi \in \mathcal{X}} \varphi$ is determined by $\bigcap_{\varphi \in \mathcal{X}} C_{\varphi}$, by Theorem 3, $\bigcap_{\varphi \in \mathcal{X}} \varphi$ is a T-closure operator and is commutative with γ.

When $\gamma \in \mathcal{X}$, then $\gamma > \bigcap_{\varphi \in \mathcal{X}} \varphi$. It is easy to verify that γ and $\bigcap_{\varphi \in \mathcal{X}} \varphi$ are commutative, so we omit the proof of this case.

On the other hand, since Ψ is a maximal subset of Φ, therefore $\bigcap_{\varphi \in \mathcal{X}} \varphi \in \Psi$.

Q.E.D.

*) This note owes to the problem suggested by Prof. K. Morita of Tokyo Bunrika Daigaku.

1) G. Birkhoff; On the combination of topologies, Fund. Math. 26 (1936), pp.156-165.
2) O. Ore; Galois connexions, Trans. Amer. Math. Soc. 55 (1944), pp.493-512. According to O. Ore, a subset T of a complete lattice S is called "a complete sublattice over S with respect to meet" when T is a complete lattice such that when $a > b$ for two elements of T, then $a > b$ in S and T contains the unit element and the zero element of S, and moreover the meet operation in T coincides with the meet operation in S both for finite or infinite sets.
3) M. Nakamura; Closure in general lattices, Proc. Imp. Acad. Tokyo, 17 (1941), pp.5-6.
4) If $C'_{\varphi\psi}$ (or $C'_{\psi\varphi}$) is void, then $C_{\varphi} \subset C_{\psi}$ (or $C_{\psi} \subset C_{\varphi}$). In this case it is easy to see that $\varphi\psi = \psi\varphi$.
5) G. Birkhoff; Lattice Theory (1940), p.17, Theorem 2. 2.

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