ON MAPPINGS DEFINED ON 2-SPHERES

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1. The following theorem has been proved by Borsuk [1]. Let f be a mapping defined on an n-dimensional sphere S^n into Euclidean n-dimensional space R^n. Then there exists a point p on S such that

$$f(p) = f(p^*)$$

where p^* denotes an antipodal point of p.

The object of the present paper is to prove the following analogies of Borsuk's theorem in the case $n=2$ (in the case $n=1$, they are trivial).

2. THEOREM 1. Let S be an 2-dimensional sphere center z in Euclidean 3-dimensional space R^3 and let f be a mapping defined on S into Euclidean 2-dimensional space R^2. Then there exist two points p, q on S such that the vectors zp, zq are perpendicular and

$$f(p) = f(q)$$

PROOF. (It is based on the method of Kakutani [2]). Let us consider S as a sphere of radius 1 in 3-space R^3, with the origin $z=(0,0,0)$ of R^3 as a center. Let us put $p_1^\circ = (1,0,0)$, $p_2^\circ = (0,1,0)$. Let further $G=\{\sigma\}$ be the group of all rotations of R^3 around its origin z.

For any $\sigma \in G$, consider the vector in R^2 defined by $\overrightarrow{f(\sigma(p_1^\circ))f(\sigma(p_2^\circ))}$. In order to prove our theorem, it suffices to show that there exists a rotation $\sigma \in G$ such that $f(\sigma(p_1^\circ)) = f(\sigma(p_2^\circ))$. We assume the contrary, and shall draw a contradiction from it. By assumption, for any $\sigma \in G$, the vector $\overrightarrow{f(\sigma(p_1^\circ))f(\sigma(p_2^\circ))}$ is not zero. Let us take an unit vector in R^2 from the origin parallel to $\overrightarrow{f(\sigma(p_1^\circ))f(\sigma(p_2^\circ))}$ and put $F(\sigma)$= the end point of this unit vector. Then $\sigma \to F(\sigma)$ is a mapping of G into S^1.

Let ℓ be the straight line $x=y$, $z=0$ and H be the subgroup of G consisting of all rotations around the line ℓ. We may denote elements of H by σ_θ ($0 \leqq \theta \leqq 2\pi$), where θ denotes the angle of rotation around the axis ℓ measured in such a sense that

$$\sigma_{\pi+\theta} = \sigma_\theta \sigma_* \text{ , where } \sigma_* \text{ denotes } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

By the above, we show easily $F(\sigma_{\pi+\theta}) = -F(\sigma_\theta)$. (for any $x \in R^2$, $-x$ be a symmetry of x about the origin). Then the fact stated above means that F maps H onto S^1. If we consider H as a S^1, by Borsuk [1], F is the antipodal mapping of S^1 in self and its degree m is not zero.

Let α be the increment of the angle of the vector $\overrightarrow{zF(\sigma_\theta)}$ in R^2 when θ runs from 0 to 2π. Then α must be of the form : $\alpha = 2m\pi$. Hence as θ runs from 0 to 2π twice time continuously, the total increment of the angle of $\overrightarrow{zF(\sigma_\theta)}$ is $4m\pi$.

On the other hand, $2H$ is homotopic to zero on G. Then $F(2H)$ is homotopic to zero on S^1. This is, however, impossible since the total increment of the angle of $\overrightarrow{zF(\sigma_\theta)}$ is $4m\pi \neq 0$. Q.E.D.

In the proof, the fact that p_1° and p_2° are perpendicular is not only essential but also we can replace it by the arbitrary different points p_1' and p_2' which subtend the angle θ at z, $0 < \theta < \pi$. Combined it with Borsuk's theorem in the case $n=2$, we have the following,

THEOREM 2. Let S be an 2-dimensional sphere center z in Euclidean 3-dimensional space R^3 and let f be a mapping defined on S into Euclidean 2-dimensional space R^2 and let θ be a given angle such that $0 < \theta < 2\pi$. Then there exist two points p and q on S such that p and q subtend the angle θ at z and

$$\underline{f(p) = f(q)}$$

Remark. By using Stiefel's manifold $V_{3,2}$ [3] and Eilenberg's theorem [4], we can prove the above theorems. But this method is not different from the above.

3. Theorem 3. <u>Let S be an 2-dimensional sphere center Z in Euclidean 3-dimensional space R^3 and let f be a mapping defined on S into an orientable 2-dimensional manifold M with the genus $\neq 0$ and θ be a given angle such that $0 < \theta < 2\pi$. Then there exist two points p and q on S such that p and q subtend the angle θ at Z and</u>

$$\underline{f(p) = f(q)}$$

PROOF. For any mapping $f: S \to M$, since $\pi_2(M) = 0$, we have a homotopy $f_t : S \to M$, $0 \leq t \leq 1$ such that

$$f_1(x) = f(x) \qquad \text{for all } x \in S$$

$$f_0(x) = \text{one fixed point } m \text{ of } M.$$

The universal covering space of M is R^2 and g denotes the projection $g: R^2 \to M$.

Using the covering homotopy theorem [5], we have a homotopy $f_t^* : S^2 \to R^2$, $0 \leqq t \leqq 1$, such that

$$g f_t^* = f_t \qquad \text{for all } t.$$

Especially we have $g f_1^* = f_1 = f$

By Theorem 2 there exist two points p, q on S such that p and q subtend the angle θ at Z and $f^*(p) = f^*(q)$. Hence we have $gf^*(p) = gf^*(q)$ i.e. $f(p) = f(q)$
Q.E.D.

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