ON EXTENSION OF SCHWARZ'S THEOREM

By Sanzō AIKAWA

§1. Schwarz's theorem in the theory of functions of several complex variables is as useful as in one complex variable.

In this paper we are going to extend the former, and to illustrate simple examples.

(I) Schwarz's theorem in one complex variable.

Let $F(t)$ be analytic in the closed unit circle $|t| \leqq 1$ and if $F(0) = 0$, then we have

$$|F(t)| \leqq |t| \max_{|w|=1} |F(w)|, \quad |t| \leqq 1 \quad (1)$$

the equality sign holding if and only if $F(t) = ct$, where c is a constant.

(II) Schwarz's theorem in several complex variables.[1]

Let $f_i(z_1, \cdots, z_k) \equiv f_i(z) \quad (i = 1, \cdots, n)$ be n functions which are analytic in the closed unit hypersphere $\|z\| \equiv \left[\sum_{i=1}^{k} |z_i|^2\right]^{\frac{1}{2}} \leqq 1$, and vanish at the origin. Then for any value of $p > 1$, we have

$$\|f(z)\|_p \leqq \|z\| \max_{\|w\|=1} \|f(w)\|_p \quad (2)$$

for every value of z in $\|z\| \leqq 1$, where

$$\|f(z)\|_p \equiv \left[\sum_{i=1}^{n} |f_i(z)|^p\right]^{1/p}. \quad (3)$$

§2. Our main theorem is as follows:

Let $f_i(z) \quad (i = 1, \cdots, n)$ be n functions which are analytic in the closed unit hypersphere $\|z\| \leqq 1$. Assume that

$$f_i(tz) \equiv f_i(tz_1, \cdots, tz_k) = t^m F_i(t, z) \quad (4)$$

$$(i = 1, \cdots, n;\ m: \text{a non-negative integer}).$$

where $F_i(t; z)$ are analytic and vanish on $t = 0$ for every z in $\|z\| \leqq 1$. Then, for any value of $p > 1$, we have

$$\|f(z)\|_p \leqq \|z\|^{m+1} \cdot \max_{\|w\|=1} \|f(w)\|_p \quad (5)$$

for every value of z in $\|z\| \leqq 1$.

Proof. (i) For $n = 1$. We omit the suffix of $f(z)$. Putting

$$\frac{z}{t} = \left(\frac{z_1}{t}, \cdots, \frac{z_k}{t}\right) = (w_1, \cdots, w_k) = (w), \quad (6)$$
$$(t \neq 0),$$

$\|z\| = \rho$ and $|t| = \rho$ imply $\|w\| = 1$, and conversely $\|w\| = 1$, $|t| = \rho$ imply $\|z\| = \|tw\| = \rho$.

We introduce the functions

$$M(\rho, z) \equiv \max_{|t|=\rho} |f(tz)| = \max_{|t|=\rho} |t^m F(t, z)|, \quad (7)$$

$$M(\rho) \equiv \max_{\|z\|=\rho} |f(z)| = \max_{\|w\|=1} M(\rho; w), \quad 0 < \rho \leqq 1, \quad (8)$$

$$N(\rho; z) \equiv \max_{|t|=\rho} |F(t, z)|, \quad (9)$$

$$N(\rho) \equiv \max_{\|z\|=1} N(\rho; z). \quad (10)$$

Since $F(0, z) = 0$ for every z in $\|z\| \leqq 1$, Schwarz's theorem yields, for each fixed $\|z^0\| \leqq 1$,

$$N(\rho; z^0) \leqq \rho N(1, z^0) \leqq \rho N(1) \quad (11)$$

so that

$$N(\rho) \leqq \rho N(1), \quad 0 < \rho \leqq 1. \quad (12)$$

From (7) we have

$$M(\rho; z) = \rho^m N(\rho; z), \quad 0 < \rho \leqq 1, \quad \|z\| \leqq 1 \quad (13)$$

and so

$$M(\rho) = \rho^m N(\rho) \leqq \rho^{m+1} N(1) = \rho^{m+1} M(1), \quad 0 < \rho \leqq 1, \quad (14)$$

since $M(1; z) = N(1; z)$. (14) means

$$\max_{\|z\|=\rho} f(z) \leqq \rho^{m+1} \max_{\|w\|=1} |f(w)|, \qquad (15)$$

which proves our theorem for $f(z)$.

(ii) For $n>1$. Let $a_i\ (i=1,\cdots,n)$ be n arbitrary constants. Define

$$f_{(a)}(z) \equiv \sum_{i=1}^{n} a_i f_i(z), \qquad (16)$$

then $f_{(a)}(z)$ has the same properties as $f(z)$, and then by (i), we get

$$\max_{\|z\|=\rho} \left|\sum_{i=1}^{n} a_i f_i(z)\right|$$
$$\leqq \rho^{m+1} \max_{\|w\|=1} \left|\sum_{i=1}^{n} a_i f_i(w)\right|. \qquad (17)$$

For $p>1$, Hölder's inequality states

$$\left|\sum_{i=1}^{n} a_i f_i(w)\right| \leqq \|a\|_q \cdot \|f(w)\|_p, \qquad (18)$$

where $\frac{1}{p}+\frac{1}{q}=1$. Therefore we have

$$\max_{\|z\|=\rho} \left|\sum_{i=1}^{n} a_i f_i(z)\right| \leqq \rho^{m+1} \|a\|_q \max_{\|w\|=1} \|f(w)\|_p. \quad (19)$$

Now denote by z' a value of z on $\|z\|=\rho$ for which $f(z)$ attains its maximum on $\|z\|=\rho$, i.e.

$$\|f(z)\|_p \leqq \|f(z')\|_p, \quad \|z\|=\rho. \qquad (20)$$

and then determine $a_i\ (i=1,\cdots,n)$ as follows:

$$a_i = \begin{cases} |f_i(z')|^p / f_i(z'), & \text{if } f_i(z') \neq 0, \\ 0, & \text{if } f_i(z')=0. \end{cases}$$

Then we have

$$\|a\|_q = (\|f_i(z)\|_p)^{p/q},$$

$$\left|\sum_{i=1}^{n} a_i f_i(z')\right| = (\|f(z')\|_p)^p$$

and so

$$\|f(z')\|_p = (\|f(z')\|_p)^{p(1-\frac{1}{q})}$$
$$= \left|\sum_{i=1}^{n} a_i f_i(z')\right| / \|a\|_q.$$

Inserting this into (19) and (20), we obtain

$$\|f(z)\|_p \leqq \rho^{m+1} \cdot \max_{\|w\|=1} \|f(w)\|_p, \quad \|z\|=\rho,$$

or

$$\|f(z)\|_p \leqq \|z\|^{m+1} \cdot \max_{\|w\|=1} \|f(w)\|_p, \quad \|z\| \leqq 1,$$

which proves the theorem.

§3. <u>Example 1.</u> Consider an analytic transformation

$$z_i' = z_i + f_i(z) \qquad (i=1,\ldots,n)$$

where $f_i(z)$ are analytic in $\|z\| \leqq 1$ and have only the terms of order $\geqq m$ in the power-series expansions at the origin. Then for any value of $p>1$, we have

$$\|z\|_p - \|z\|^m \max_{\|z\|=1} \|f(z)\|_p \leqq \|z'\|_p$$
$$\leqq \|z\|_p + \|z\|^m \max_{\|z\|=1} \|f(z)\|_p \qquad (21)$$

for every value of z in $\|z\| \leqq 1$.

First we give a lemma whose proof is omitted here.

<u>Lemma.</u> For any complex numbers A, B, and for any value of $p>1$, the following inequality holds good

$$|A+B|^p \leqq (1+c)^{p-1}|A|^p + \left(1+\frac{1}{c}\right)^{p-1}|B|^p, \qquad (22)$$

where c is any positive number.[2]

<u>Proof of the Example 1.</u> Due to the above lemma, we have

$$|z_i'|^p = |z_i + f_i(z)|^p$$
$$\leqq (1+c)^{p-1}|z_i|^p + \left(1+\frac{1}{c}\right)^{p-1}|f_i(z)|^p$$

so that

$$\|z'\|_p^p \leqq (1+c)^{p-1}\|z\|_p^p + \left(1+\frac{1}{c}\right)^{p-1}\|f(z)\|_p^p.$$

Putting $c=\|f(z)\|_p / \|z\|_p$, and take the p-th root, we obtain

$$\|z'\|_p \leqq \|z\|_p + \|f(z)\|_p. \qquad (23)$$

By the same consideration for $z_i = z_i' - f_i(z)$, we have

$$\|z'\|_p \geqq \|z\|_p - \|f(z)\|_p. \qquad (24)$$

From the hypothesis on $f_i(z)$, we have, by our main theorem

$$\|f(z)\|_p \leqq \|z\|^m \cdot \max_{\|z\|=1} \|f(z)\|_p, \quad (25)$$

and inserting (25) into (23) and (24), we finally obtain (21), which completes the proof of our example 1.

§4. Example 2. An inequality for $\dfrac{A\|f(z)\|_p}{B+C\|f(z)\|_p}$.

If $f_i(z)$ $(i=1,2,\cdots,n)$ have the same properties as $f_i(z)$ in the main theorem, and let A, B and C be real constants such that $A>0$, $B>|C|\cdot\max\limits_{\|w\|=1}\|f(w)\|_p$. Then we have

$$\frac{A\|f(z)\|_p}{B+C\|f(z)\|_p} \leqq \|z\|^{m+1}\cdot\frac{A\cdot\max\limits_{\|w\|=1}\|f(w)\|_p}{B-|C|\max\limits_{\|w\|=1}\|f(w)\|_p} \qquad (26)$$

for $\|z\|\leqq 1$ and $p>1$.

Proof. Denote by (z') a value of (z) on $\|z\|=\rho$ for which $\|f(z)\|_p$ attains its maximum on $\|z\|=\rho$, so that

$$\|f(z)\|_p \leqq \|f(z')\|_p$$

for $\|z\|=\rho\leqq 1$.

Using this point (z'), choose

$$a_i=\begin{cases}\dfrac{|f_i(z')|^p}{f_i(z')}\|f(z')\|_p^{1-p}, & \text{if } f_i(z')\neq 0\\[2ex] 0, & \text{if } f_i(z')=0,\end{cases}$$

and put

$$f_{(a)}(z)=\sum_{i=1}^{n}a_i f_i(z).$$

Then we have

$$f_{(a)}(z')=\sum_{i=1}^{n}a_i f_i(z')=\|f(z')\|_p, \qquad (27)$$

and

$$\|a\|_q=\Big[\sum_{i=1}^{n}|f_i(z')|^{q(p-1)}\Big]^{\frac{1}{q}}\cdot\|f(z')\|_p^{1-p}$$

$$=\|f(z')\|_p^{\frac{p}{q}+1-p}=1, \qquad (28)$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Now introducing the following functions,

$$\varphi(z)\equiv\frac{A\,f_{(a)}(z)}{B+C\cdot f_{(a)}(z)},$$

$$\Phi(t,z)\equiv\frac{A\sum\limits_{i=1}^{n}a_iF_i(t,z)}{B+t^mC\sum\limits_{i=1}^{n}a_iF_i(t,z)}, \qquad (29)$$

we have

$$\varphi(tz)=\frac{A\cdot f_{(a)}(tz)}{B+C\cdot f_{(a)}(tz)}$$

$$=\frac{t^mA\cdot\sum\limits_{i=1}^{n}a_iF_i(t,z)}{B+Ct^m\sum\limits_{i=1}^{n}a_iF_i(t,z)}=:t^m\Phi(t,z)$$

Then $\varphi(z)$ and $\Phi(t,z)$ have the same properties as $f_i(z)$ and $F_i(t,z)$ respectively in the main theorem, so that

$$|\varphi(z)|\leqq\|z\|^{m+1}\cdot\max_{\|w\|=1}|\varphi(w)|,$$
$$\|z\|\leqq 1. \qquad (30)$$

For any real number $p>1$, we have by Hölder's inequality

$$|f_{(a)}(z)|=\Big|\sum_{i=1}^{n}a_if_i(z)\Big|\leqq\|a\|_q\cdot\|f(z)\|_p$$
$$=\|f(z)\|_p \qquad (31)$$

for $\frac{1}{p}+\frac{1}{q}=1$, $\|z\|\leqq 1$.

If $A>0$, $B>0$, and C is real, the function

$$\frac{Ax}{B+Cx}$$

is a monotone increasing function of a real variable x, so that for $\|z\|=\rho$, we have

$$\frac{A\|f(z)\|_p}{B+C\cdot\|f(z)\|_p}\leqq\frac{A\cdot\|f(z')\|_p}{B+C\cdot\|f(z')\|_p}$$

$$=\frac{A\ f_{(a)}(z')}{B+C\ f_{(a)}(z')} \qquad \text{(by (27))}$$

$$\leqq|\varphi(z')|\leqq\|z'\|^{m+1}\max_{\|w\|=1}|\varphi(w)| \qquad \text{(by (30))}$$

$$=\|z\|^{m+1}\max_{\|w\|=1}\left|\frac{A\ f_{(a)}(w)}{B+C\ f_{(a)}(w)}\right|$$

$$\leqq\|z\|^{m+1}\frac{A\cdot\max\limits_{\|w\|=1}|f_{(a)}(w)|}{B-|C|\cdot\max\limits_{\|w\|=1}|f_{(a)}(w)|} \qquad \text{(by (31))}$$

$$\leqq\|z\|^{m+1}\frac{A\cdot\max\limits_{\|w\|=1}\|f(w)\|_p}{B-|C|\cdot\max\limits_{\|w\|=1}\|f(w)\|_p},$$

which is the required result (26).

(*) Received July 31, 1952.

(1) S.Bochner and W.T.Martin, Several complex variables, Princeton 1948, pp.59-64.
(2) S.Takahashi, Univalent mappings in several complex variables, Annals of Math. 53 (1951), p.464.

Nagoya Institute of Technology.