ON FUNCTIONS OF BOUNDED DIRICHLET INTEGRAL

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In 1939, H.Grunsky [1] gave a perfect condition for a meromorphic function to be univalent in terms of the local coefficients. Later S.Bergman-M.Schiffer [1] also has given a condition under a somewhat different formulation.

It is the aim of the present paper to establish a perfect criterion in terms of the local coefficients in order that a single-valued regular function has an image domain whose area does not exceed π.

1. Basic notations. Let B be a planar schlicht n-ply connected domain with a boundary Γ consisting of analytic curves Γ_ν ($\nu = 1, \cdots, n$). For simplicity's sake, we shall assume that B contains the origin.

Let $P(z; z_0)$ be a polynomial with respect to $t = 1/(z - z_0)$:

$$P(z, z_0) = x_N t^N + x_{N-1} t^{N-1} + \cdots + x_0, \quad z_0 \in B.$$

Let $f_P(z, z_0; \alpha)$ be a single-valued meromorphic function satisfying the following conditions:

i) $f_P(z, z_0; \alpha) - P(z, z_0)$ is regular in B;

ii) all the images of Γ_ν ($\nu = 1, \cdots, n$) by $f_P(z, z_0; \alpha)$ are the segments with inclination α to the real axis.

Existence and uniqueness of $f_P(z, z_0; \alpha)$ for any given P is well-known. Cf. H.Grunsky [1].

Let $F_P(z, z_0)$ be

$$\frac{1}{2}\left(f_P(z, z_0; \alpha) - f_P\left(z, z_0; \alpha + \frac{\pi}{2}\right)\right),$$

then we have

$$F_P(z, z_0) = \exp(2i\alpha) \sum_{m=0}^{N} \bar{x}_m \mathfrak{f}_m(z, z_0),$$

$\mathfrak{f}_m(z, z_0)$ being defined by

$$\mathfrak{f}_m(z, z_0) = \frac{1}{2}\left(f_{t^m}(z, z_0; 0) - f_{t^m}\left(z, z_0; \frac{\pi}{2}\right)\right),$$

$$\mathfrak{f}_0 \equiv 0.$$

Let further

$$\phi_m(z, z_0) = \frac{1}{2}\left(f_{t^m}(z, z_0; 0) + f_{t^m}\left(z, z_0; \frac{\pi}{2}\right)\right),$$

$$\phi_0 \equiv 1.$$

Obviously the local expansions of ϕ_m and $\mathfrak{f}_m$ ($m \geq 1$) about z_0 are of the form

$$\phi_m(z, z_0) = \frac{1}{(z - z_0)^m} + \sum_{\nu=1}^{\infty} B_{m\nu}(z - z_0)^\nu$$

and

$$\mathfrak{f}_m(z, z_0) = \sum_{\nu=1}^{\infty} S_{m\nu}(z - z_0)^\nu,$$

respectively. The coefficients S_{mn} may be called the generalized (m,n)-span. Obviously we have along Γ: $d\mathfrak{f}_m = d\bar{\phi}_m$.

We shall define the Dirichlet norm $\sqrt{D_m(f, f)} = \| f' \|_B$ by $\left(\iint_B |f'(z)|^2 d\sigma_z\right)^{1/2}$. Let $L^2(B)$ be a family of functions $\psi(z)$ satisfying the following conditions:

i) $\int^z \psi(z)\, dz$ is a single-valued regular function in B,

ii) $\| \psi \|_B < \infty$.

Let now $f'(z)$ belong to $L^2_{,}(B)$, then we have

$$\begin{aligned} D_B(f, \mathfrak{f}_m) &= \frac{i}{2}\int_\Gamma f\, d\bar{\mathfrak{f}}_m \\ &= \frac{i}{2}\int_\Gamma f\, d\phi_m \\ &= \frac{i}{2}\int_\Gamma f(z)\left(\frac{-m}{(z - z_0)^{m+1}} + \sum_{\nu=1}^{\infty} \nu B_{m\nu}(z - z_0)^{\nu-1}\right) dz \\ &= \frac{i}{2}\int_\Gamma f(z) \frac{-m}{(z - z_0)^{m+1}}\, dz. \end{aligned}$$

Here if $f(z)$ has an expansion $\sum_{\nu=1}^{\infty} c_\nu (z - z_0)^\nu$, then we have

$$D_B(f, \varphi_m) = \pi m c_m .$$

Especially we have

$$D_B(\varphi_n, \varphi_m) = \pi m S_{nm} = \pi n \overline{S}_{mn} ,$$

and

$$D_B(\varphi_m, \varphi_m) = \pi m S_{mm} \gneqq 0 .$$

2. Preparatory considerations. Let z_0 be the origin.

(a) $\{\varphi_n'\}$, $n = 1, 2, \cdots$, is linearly independent.

This fact is equivalent to the strictly positive definiteness of the Hermitian form $\sum_{\mu,\nu=1}^{N} \nu S_{\mu\nu} x_\mu \overline{x}_\nu \gneqq 0$. Cf. H. Grunsky [1] .

(b) $\{\varphi_n'\}$, $n = 1, 2, \cdots$, is a complete system in $L^2(B)$.

In $L^2(B)$, introducing the inner product

$$(f', g') = \iint_B f'(z)\, \overline{g'(z)}\, d\sigma_z ,$$

$L^2(B)$ becomes a Hilbert space.

Let

$$g'(z) = \sum_{\nu=1}^{\infty} \nu g_\nu z^{\nu-1} \in L^2(B)$$

satisfy the orthogonality relations: $(g', \varphi_m') = 0$, $m = 1, 2, \cdots$, then $m g_m = 0$. Thus we have $g' \equiv 0$ on B , and hence the desired completeness in $L^2(B)$. Therefore $f(z) = \sum_{\nu=1}^{\infty} a_\nu \varphi_\nu'(z)$ for all $f'(z) \in L^2(B)$.

In $L^2(B)$ there exists a kernel function $K(z, \overline{\zeta})$. This is a well-known fact, or we can construct it from $\{\varphi_\nu'\}$ by a usual orthonormlization process due to Gram-Schmidt. But we need only the following expression of $K(z, \overline{\zeta})$:

$$K(z, \overline{\zeta}) = \sum_{\mu,\nu=1}^{\infty} l_{\mu\nu}\, \varphi_\mu'(z)\, \overline{\varphi_\nu'(\zeta)} , \quad l_{\mu\nu} = \overline{l}_{\nu\mu} .$$

Expanding $K(z, \overline{\zeta})$ in a neighborhood of the origin as an analytic function of two variables z and $\overline{\zeta}$ in the form

$$K(z, \overline{\zeta}) = \sum_{\mu,\nu=0}^{\infty} k_{\mu\nu} z^\mu \overline{\zeta}^\nu ,$$

we shall show the following Lemma.

<u>Lemma 1.</u> $k_{\mu\nu} = \frac{1}{\pi}(\mu+1) S_{\nu+1, \mu+1}$.

Proof. Evidently we have

$$\mu!\nu!\, k_{\mu\nu} = \frac{\partial^{\mu+\nu}}{\partial^\mu z\, \partial^\nu \overline{\zeta}} K(0, \overline{0})$$

$$= \sum_{m,n=1}^{\infty} l_{mn} (\mu+1)!\, S_{m,\mu+1} (\nu+1)!\, \overline{S}_{n,\nu+1} .$$

The reproducing property of the kernel function leads to the relation

$$\varphi_m'(\zeta) = \iint_B \varphi_m'(z)\, \overline{K(z, \overline{\zeta})}\, d\sigma_z$$

$$= \sum_{\kappa,\lambda=1}^{\infty} \overline{l}_{\kappa\lambda}\, \varphi_\lambda'(\zeta) \left(\varphi_m'(z), \varphi_\kappa'(z)\right)_B$$

$$= \pi \sum_{\kappa,\lambda=1}^{\infty} \overline{l}_{\kappa\lambda}\, \kappa S_{m\kappa}\, \varphi_\lambda'(\zeta).$$

Thus, in local, we have

$$\sum_{\rho=0}^{\infty} (\rho+1)\, S_{m,\rho+1} \zeta^\rho$$

$$= \pi \sum_{\kappa,\lambda=1}^{\infty} \kappa\, \overline{l}_{\kappa\lambda} S_{m\kappa} \sum_{\rho=0}^{\infty} (\rho+1)\, S_{\lambda,\rho+1} \zeta^\rho ,$$

and hence

$$S_{m,\rho+1} = \pi m \sum_{\kappa,\lambda=1}^{\infty} \overline{l}_{\kappa\lambda} S_{\lambda,\rho+1} \overline{S}_{\kappa m} ,$$

which leads to the desired result.

3. Perfect condition for Dirichlet integral to be bounded.

<u>Theorem.</u> Let $f(z)$ be a single-valued regular function with a power series expansion $f(z) = \sum_{\mu=1}^{\infty} c_\mu z^\mu$ in a neighborhood of the origin. Then the fact that the inequalities

$$\left|\sum_{\mu=1}^{N} \mu c_\mu x_\mu\right|^2 \le \sum_{\mu,\nu=1}^{N} \nu S_{\mu\nu} x_\nu \overline{x}_\mu \quad (N = 1, 2, \cdots)$$

hold for arbitrary complex number x_μ is a perfect condition in order that $D_B(f, f) \le \pi$.

Proof. <u>Necessity.</u> Let $f(z) = \sum_{\mu=1}^{\infty} c_\mu z^\mu$ be a single-valued regular function in B satisfying $D_B(f,f) = \|f'\|_B^2 \le \pi$, then we have

$$D_B(f, F_p) = \exp(-2i\alpha) \sum_{\mu=1}^{N} x_\mu D_B(f, \varphi_\mu)$$

$$= \pi \exp(-2i\alpha) \sum_{\mu=1}^{N} \mu c_\mu x_\mu .$$

Therefore, by Schwarz's inequality, we have

$$\pi^2\left|\sum_{\mu=1}^{N} \mu c_\mu x_\mu\right|^2 \leqq D_B(f,f)\, D_B(F_p,F_p) \leqq \pi\cdot\pi\sum_{\mu,\nu=1}^{N} \nu S_{\mu\nu}\, x_\nu \bar{x}_\mu .$$

For the sufficiency proof we need a lemma with regard to the analytic continuability of an Hermitian analytic function.

Lemma 2. Let $H(z,\bar\zeta)$ be an Hermitian analytic function with local expansion $\sum_{\mu,\nu=0}^{\infty} h_{\mu\nu}\, z^\mu \bar\zeta^\nu$ around the origin. If, for every complex vector $x=(x_0,x_1,\cdots,x_N)$,

$$0\leqq\sum_{\mu,\nu=0}^{N} h_{\mu\nu}\, x_\mu \bar{x}_\nu \leqq \pi\sum_{\mu,\nu=0}^{N} k_{\mu\nu}\, x_\mu \bar{x}_\nu$$

holds, then $H(z,\bar\zeta)$ is analytic and Hermitian in the whole domain B.

Proof. This Lemma is an anlogue of a continuability theorem due to Bergman-Schiffer [1]. We can choose an orthonormal complete system $\{\chi_\nu(z)\}$ in $L^2(B)$ satisfying $\chi_\nu(z)=\sum_{\mu=\nu}^{\infty} a_{\nu\mu} z^\mu$, $a_{\nu\nu}\neq 0$ in local. Since the matrix $(a_{\nu\mu})$ is of the triangular form and $a_{\nu\nu}\neq 0$, this matrix has an inverse matrix $(b_{\nu\mu})$, in which $b_{\nu\mu}=0$ for $\nu>\mu$ and $b_{\nu\nu}\neq 0$, and we can solve $\chi_\nu=\sum_{\mu=\nu}^{\infty} a_{\nu\mu} z^\mu$, in local, in the form

$$z^\mu=\sum_{\nu=\mu}^{\infty} b_{\mu\nu}\,\chi_\nu(z)\,,\quad \mu=0,1,\cdots .$$

Then, by identifying two expressions of $K(z,\bar\zeta)$

$$\sum_{\nu=0}^{\infty}\chi_\nu(z)\overline{\chi_\nu(\zeta)}=\sum_{\mu,\nu=0}^{\infty} k_{\mu\nu}\, z^\mu\bar\zeta^\nu,$$

we get the relation

$$\sum_{\mu,\nu=0}^{\infty} k_{\mu\nu}\, b_{\mu m}\bar{b}_{\nu n}=\delta_{mn}\,,$$

in which the sums are actually of finite range and hence the left-hand side is well-defined. Similarly,

$$H(z,\bar\zeta)=\sum_{\mu,\nu=0}^{\infty} h_{\mu\nu}\, z^\mu\bar\zeta^\nu$$

can be written, in local, in the form

$$H(z,\bar\zeta)=\sum_{m,n=0}^{\infty} t_{mn}\,\chi_m(z)\,\overline{\chi_n(\zeta)},$$

where

$$t_{mn}=\sum_{\mu,\nu=0}^{\infty} h_{\mu\nu}\, b_{\mu m}\bar{b}_{\nu n} .$$

In the last expression summation is extended again actually over finite terms in number so that this is well-defined.

Putting $x_\nu=\sum_{\mu=0}^{N} b_{\nu\mu} y_\mu$, we have

$$0\leqq\sum_{\mu,\nu=0}^{N} t_{\mu\nu}\, y_\mu\bar{y}_\nu \leqq \pi\sum_{\mu=0}^{N}|y_\mu|^2 .$$

This leads to the relation

$$0\leqq\left|\sum_{\mu,\nu=0}^{N} t_{\mu\nu}\, y_\mu \bar{y}'_\nu\right| \leqq \pi\left(\sum_{\mu=0}^{N}|y_\mu|^2+\sum_{\nu=0}^{N}|y'_\nu|^2\right)$$

for arbitrary complex numbers y_μ and y'_μ.

Let B' be an arbitrary closed subdomain of B. Then $K(z,\bar\zeta)$ is uniformly bounded ($<M$) in B'. Putting $y_\mu=\chi_\mu(z)$ and $y'_\nu=\chi_\nu(\zeta)$, we have

$$\left|\sum_{\mu,\nu=0}^{N} t_{\mu\nu}\,\chi_\mu(z)\,\overline{\chi_\nu(\zeta)}\right|\leqq \pi\, 2M$$

for $z,\zeta\in B'$. Thus $H_N(z,\bar\zeta)=\sum_{\mu,\nu=0}^{N} t_{\mu\nu}\chi_\mu(z)\overline{\chi_\nu(\zeta)}$ is uniformly bounded in B'. Thus we can select a subsequence of H_N converging uniformly in B' and having a limit function. On the other hand, this limit function coincides with $H(z,\bar\zeta)$ in the neighborhood of the origin. Hence the whole sequence $H_N(z,\bar\zeta)$ possesses the same limit and converges uniformly in each closed subdomain of B. The limiting function is the analytic continuation of the power series $H(z,\bar\zeta)$ over the whole domain B. q.e.d.

Sufficiency proof of Theorem. Let

$$f'(z)=\sum_{\nu=1}^{\infty}\nu c_\nu z^{\nu-1}=\sum_{\mu=0}^{\infty} d_\mu z^\mu$$

satisfy the condition

$$\left|\sum_{\mu=1}^{N+1}\mu c_\mu x_\mu\right|^2\leqq\sum_{\mu,\nu=1}^{N+1}\nu S_{\mu\nu}\, x_\nu\bar{x}_\mu ,$$

then Lemma 1 leads to the inequality

$$\left|\sum_{\mu=0}^{N} a_\mu x_\mu\right|^2 \leqq \pi \sum_{\mu,\nu=0}^{N} k_{\mu\nu}\, x_\mu \bar{x}_\nu .$$

Now, $H(z,\bar{\zeta}) = f'(z)\overline{f'(\zeta)}$ satisfies the assumptions of Lemma 2. Thus $f'(z)\,\overline{f'(\zeta)}$ is analytically continuable and coincides with the expression

$$\sum_{\mu,\nu=0}^{\infty} t_{\mu\nu}\, \chi_\mu(z)\, \overline{\chi_\nu(\zeta)} ,$$

where

$$t_{\mu\nu} = \sum_{m,n=0}^{\infty} d_m \bar{d}_n\, b_{m\mu} \bar{b}_{n\nu} = \left|\sum_{m=0}^{\infty} d_m b_{m\mu}\right|^2 .$$

Thus

$$\sum_{\nu=0}^{\infty} f_\nu \chi_\nu(z), \quad f_\nu = \sum_{m=0}^{\infty} d_m b_{m\mu} ,$$

is a regular function in B coinciding with $f'(z)$ in a neighborhood of the origin.

In a similar manner as in Lemma 2, we have

$$\left|\sum_{m,n=0}^{N} f_m y_m\right|^2 \leqq \pi \sum_{m=0}^{N} |y_m|^2 .$$

Putting $y_m = \bar{f}_m$, we have

$$0 \leqq \sum_{m=0}^{N} |f_m|^2 \leqq \pi .$$

Let N tend to ∞, then we conclude

$$\sum_{m=0}^{\infty} |f_m|^2 \leqq \pi .$$

This shows that $f'(z)$ belongs to $L^2(B)$ and $D_B(f,f) = \|f'\|_B^2 \leqq \pi$ by an analogue of Riesz-Fisher theorem. Cf. S.Bergman [1] .

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