BEHAVIOR OF KERNEL FUNCTIONS ON BOUNDARIES

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For an arbitrary domain D on z-plane, Bergman and Szegö kernel functions are defined as the limiting functions which are defined for domains of exhaustions of D whose boundaries are finite number of closed analytic curves. As usual, we denote them by $K(z, \zeta)$ and $k(z, \zeta)$ respectively. We suppose that D contains at least a continuum. Then, $K(z, \zeta)$ does not vanish identically.

In this note, we will investigate the behavior of $K(z, z)$ in the neighborhood of boundary points. For the sake of simplicity, D is a simply-connected domain whose boundary is a continuum Γ.

Let z_0 be a point on Γ and suppose that there exists a triangle Az_0B, which is contained in D, i.e. there exists Stolz's paths in D.

Theorem. Let D, z_0, Γ and $K(z, \zeta)$ be defined as above. Then, $K(z, z)$ tends to positive infinity when z approaches to z_0 along any Stolz's path.

Proof. We suppose that the theorem is not true. Then, $K(z, z)$ is uniformly bounded in Az_0B.

We map D onto the unit circle E: $|w| < 1$. By a theorem due to C.Caratheodory, the image of the triangle Az_0B contains a triangle Cw_0D in E, where w_0 is one of image of z_0.

By the covariant property of the kernel function $K(z, \zeta)$, we have

$$(1) \quad K_D(z, z) = K_E(w, w)\left|\frac{dw}{dz}\right|^2 = \frac{1}{\pi}\frac{1}{(1-|w|^2)^2}\left|\frac{dw}{dz}\right|^2.$$

Integrating $K_D(z, z)$ on Az_0B, we have

$$(2) \quad \iint_{Az_0B} K_D(z, z)\,dx\,dy \geq \frac{1}{\pi}\iint_{Cw_0D}\frac{du\,dv}{(1-|w|^2)^2} \quad \begin{pmatrix} z = x+iy, \\ w = u+iv \end{pmatrix}.$$

From the hypothesis of the proof, the first integral of (2) is convergent.

On the other hand, let M be the middle point of CD. We describe a line parallel to CD, cutting through Cw_0, Dw_0 and Mw_0 at C', D' and M' respectively. The angle C'OD' is denoted by θ, and the length of $M'w_0$ by ℓ. Then, by a simple calculation, we have

$$\lim_{M \to w_0} \frac{\theta}{\ell} = \text{const.} > 0 .$$

This implies that the second integral of (2) is divergent, which contradicts the hypothesis. q. e. d.

For the case which we can describe a circle C_{r_0} touching at z_0 instead of the triangle Az_0B, where r_0 is the radius of the circle, we have the following

Corollary.[1] Under the hypotheses of the theorem,

$$\frac{1}{r_0} = \overline{\lim_{n\to\infty}}\left(\frac{e}{n}\left|\frac{\partial^{2n}}{\partial z^n\,\partial\bar{\zeta}^n}K(0,0)\right|^{\frac{1}{2n}}\right).$$

Remark. The method stated as above can be applied to multiply-connected domains. For a domain bounded by smooth curves, the theorem has already been proved by S.Bergman.

The same conclusions of the theorem and corollary remain valid also for Szegö kernel function, since the inequality

$$4\pi k^2(z, z) \geq K(z, z)$$

holds in general.

(*) Received April 14, 1952.

1) Ph.Davis and H.Pollak: A theorem for kernel functions, Proc. Amer. Math. Soc., Vol.2, 1951, pp. 686-690.
In their paper, however, the proof is proceeded for the domain which is bounded by smooth curves.

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