CONFORMAL MAPPING OF MULTIPLY CONNECTED DOMAINS, II.

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4. Preparatory considerations; monodromy conditions in doubly connected case.

In previous Note we have discussed the problems of mapping any given multiply connected domain onto whole plane slit along horizontal and vertical segments as well as onto whole plane slit along radial segments and circular arcs, both under respective assigned normalizations; Komatu-Ozawa [1]. We have pointed out that general case of any connectivity can, in each of the above problems, be reduced to that of connectivity two. Consequently, in order to perform an existence proof for general case completely, it remains only to work out the last simplest case in a direct manner.

For that purpose we now establish lemmas necessary in the sequel. We first take an annulus

$$(0<)q<|z|<1$$

as a doubly connected basic domain. The exterior and interior circumferences, i.e., $|z|=1$ and $|z|=q$, will be denoted by L_1 and L_2, respectively.

We begin with lemma stating a sort of monodromy condition which will be used in case of mapping onto whole plane slit along horizontal and vertical segments.

Lemma 1. Let ζ_κ $(\kappa=1,\cdots,k)$ be any points interior to $q<|z|<1$. Then, a function, being determined uniquely except a real additive constant, which is regular analytic in $q<|z|<1$ and whose imaginary part possesses the boundary values given by

$$\mathfrak{I}\sum_{\kappa=1}^{k}\frac{a_\kappa}{z-\zeta_\kappa} \qquad (z\in L_\nu;\ \nu=1,2)$$

with constant coefficients a_κ, is one-valued in the annulus if and only if the relation holds:

$$\mathfrak{I}\sum_{\kappa=1}^{k}\frac{a_\kappa}{\zeta_\kappa}=0.$$

Proof. In view of a well-known monodromy condition, in order that the real part of a thus defined function is also single-valued in the annulus, it is necessary and sufficient that the relation

$$\int_{L_1}\mathfrak{I}\sum_{\kappa=1}^{k}\frac{a_\kappa}{z-\zeta_\kappa}\,d\theta=\int_{L_2}\mathfrak{I}\sum_{\kappa=1}^{k}\frac{a_\kappa}{z-\zeta_\kappa}\,d\theta$$

$$(\theta=\arg z)$$

does hold; cf., for instance, Komatu [5] or [6]. Now, by means of residue theorem, we get

$$\int\mathfrak{I}\sum_{\kappa=1}^{k}\frac{a_\kappa}{z-\zeta_\kappa}\,d\theta=\mathfrak{I}\int\sum_{\kappa=1}^{k}\frac{a_\kappa}{z-\zeta_\kappa}\frac{dz}{iz}$$

$$=\begin{cases}0 & (z\in L_1),\\ -2\pi\,\mathfrak{I}\displaystyle\sum_{\kappa=1}^{k}\frac{a_\kappa}{\zeta_\kappa} & (z\in L_2),\end{cases}$$

whence follows immediately the desired result.

We now establish another sort of monodromy condition which will be useful in case of mapping onto whole plane slit along a radial segment and a circular arc. Let the Green function of the basic annulus $q<|z|<1$ with singularity ζ be $g(z;\zeta)$, and the harmonic measure of L_ν $(\nu=1,2)$ be $\omega_\nu(z)$. An identity $\omega_1(z)+\omega_2(z)\equiv 1$ is obvious. Let further an analytic function whose real part coincides with $g(z;\zeta)$ or with $\omega_\nu(z)$ be denoted by $G(z;\zeta)$ or $\Omega_\nu(z)$, respectively; more precisely, we put

$$G(z;\zeta)=g(z;\zeta)+i\tilde{g}(z;\zeta),$$

$$\Omega_\nu(z)=\omega_\nu(z)+i\tilde{\omega}_\nu(z)\quad(\nu=1,2),$$

$\tilde{g}$ and $\tilde{\omega}_\nu$ being conjugate harmonics of g and ω_ν, respectively. These functions G and $\Omega_\nu(z)$ are determinate except purely imaginary additive constants.

An explicit expression of Green function for the annulus $q<|z|<1$ is well-known; cf., for instance, Courant-Hilbert [1], p.335-337; Komatu [7]. We have, in fact,

$$G(z;\zeta)=-\lg\left\{ i\,(q^{-1/2}z)^{1/2-\lg|\zeta|/\lg q} \times \frac{\vartheta_1\left(\frac{1}{2\pi i}\lg\frac{z}{\zeta}\right)}{\vartheta_0\left(\frac{1}{2\pi i}\lg\frac{\bar{\zeta}z}{q}\right)}\right\} + ic^*,$$

c^* being a real constant and theta-functions depending on parameter q coincident with the interior radius of our annulus. (In Courant-Hilbert [1], another normalization of singularity at the source point ζ being taken, the expression is multiplied by the factor $1/2\pi$) On the other hand, $\Omega_\nu(z)$ ($\nu=1,2$) can be elementarily expressed, namely

$$\Omega_1(z) = \lg\frac{z}{q}\Big/\lg\frac{1}{q},$$

$$\Omega_2(z) = \lg\frac{1}{z}\Big/\lg\frac{1}{q}.$$

Lemma ii. Let λ_κ $(\kappa=1,\cdots,k)$ be integers, c_2 be a real number, and further ζ_κ $(\kappa=1,\cdots,k)$ be points interior to an annulus $q<|z|<1$. Then, the function defined by

$$\Phi(z) = \exp\left(\sum_{\kappa=1}^{k}\lambda_\kappa G(z;\zeta_\kappa)-2\pi c_2\Omega_2(z)\right)$$

is single-valued in the annulus if and only if the relations hold:

$$\sum_{\kappa=1}^{k}\lambda_\kappa = 0$$

and

$$\prod_{\kappa=1}^{k}|\zeta_\kappa|^{\lambda_\kappa} = e^{c_2}.$$

Proof. It is evident that $\Re\,\Phi(z)$ is single-valued in the annulus. In fact, the possibility of many-valuedness of $\Phi(z)$ can be caused by the conjugate harmonics contained in the exponent of its expression. Since λ_κ are all supposed to be integral, the points ζ_κ are all ordinary points, eventually zeros or poles but not branch points. Hence, the monodromy conditions for $\Phi(z)$ may be expressed as in the following form:

$$\int_{L_\nu} d\arg\Phi(z) = 0 \quad (\nu=1,2).$$

But, we have

$$\int_{L_\nu} d\tilde{g}(z;\zeta) = -2\pi\,\omega_\nu(\zeta),$$

$$\int_{L_\nu} d\tilde{\omega}_2(z) = p_{\nu 2} \quad (\nu=1,2),$$

$p_{\nu 2}$ $(\nu=1,2)$ denoting the periodicity-moduli of $\tilde{\omega}_2(z) \equiv \Im\,\Omega_2(z)$ with respect to L_ν; cf., for instance, Bergman [2], p.46. Thus, the monodromy conditions become

$$\sum_{\kappa=1}^{k}\lambda_\kappa\omega_\nu(\zeta_\kappa) + c_2 p_{\nu 2} = 0 \quad (\nu=1,2).$$

In view of identities $\omega_1(z)+\omega_2(z)\equiv 1$ and $p_{12}+p_{22}=0$, we get, from these conditions, a relation

$$\sum_{\kappa=1}^{k}\lambda_\kappa = 0.$$

That the conditions stated in the lemma are necessary for single-valuedness of $\Phi(z)$ has thus been established. The sufficiency of the conditions will also be obvious.

Any function $\Phi(z)$ of the type stated in lemma ii behaves evidently analytically in the closed annulus $q\leq|z|\leq 1$, and its boundary values satisfy the relations

$$|\Phi(z)| = 1 \quad (z\in L_1),$$

$$|\Phi(z)| = e^{-2\pi c_2} \quad (z\in L_2).$$

The Laurent expansion of $\Phi(z)$ around $z=\zeta_\kappa$ $(\kappa=1,\cdots,k)$ has a beginning term

$$\varepsilon_\kappa\,(z-\zeta_\kappa)^{-\lambda_\kappa},$$

ε_κ being a coefficient.

If, in particular, we put

$$k=4m,\quad \lambda_\kappa=(-1)^{\kappa-1} \quad (\kappa=1,\cdots,4m),$$

then the first condition of the lemma is surely satisfied. If we further put

$$c_2 = 0, \quad \zeta_{2m+\kappa} = q/\bar{\zeta}_\kappa$$
$$(\kappa = 1, \cdots, 2m),$$

then the second condition is also satisfied. We thus obtain a function of the form

$$\Phi(z) = \exp\sum_{\kappa=1}^{2m}(-1)^{\kappa-1}\left(G(z;\zeta_\kappa)+G\left(z;\frac{q}{\bar{\zeta}_\kappa}\right)\right).$$

Now, it will easily be verified, based upon characteristic properties of Green function, that an identical relation

$$G(z;\zeta) = \overline{G(q/\bar{z}\ ;\ q/\bar{\zeta})}$$

holds good; an unessential purely imaginary additive constant being neglected. Hence, corresponding to such a choice of additive constant, we have, for the function of the last-mentioned form, an expression

$$\Phi(z) = \exp\sum_{\kappa=1}^{2m}(-1)^{\kappa-1}\left(G(z;\zeta_\kappa)+\overline{G(q/\bar{z};\zeta_\kappa)}\right),$$

the G's with the same singularity being supposed as identical, whence we immediately deduce a functional relation

$$\Phi(z) = \overline{\Phi(q/\bar{z})}.$$

By means of the explicit expression for Green function, $\Phi(z)$ can also be immediately written down; namely,

$$\Phi(z) = \prod_{\kappa=1}^{2m}\left(\frac{\vartheta_0\left(\frac{1}{2\pi i}\lg\frac{\bar{\zeta}_\kappa z}{q}\right)\vartheta_0\left(\frac{1}{2\pi i}\lg\frac{z}{\zeta_\kappa}\right)}{\vartheta_1\left(\frac{1}{2\pi i}\lg\frac{z}{\zeta_\kappa}\right)\vartheta_1\left(\frac{1}{2\pi i}\lg\frac{\bar{\zeta}_\kappa z}{q}\right)}\right)^{(-1)^{\kappa-1}}$$

In the following we shall restrict ourselves to a further special case with $m = 1$, i.e., $k = 4$, although a corresponding argument will also be valid for general case which may be left to the reader.

5. Construction of mapping functions in doubly connected case.

We now enter into our main discourse. It is well-known that any ring domain can be mapped univalently onto an annulus; cf. the remark stated at the last part of the previous Note. Consequently, we may and shall suppose that a given domain is itself an annulus, $(0<)\,Q < |z| < 1$ say. For such a canonical basic domain, the required mapping functions can explicitly be constructed, the fact which will be shown in the following lines.

Theorem 1. Let z_∞ be a point interior to the annulus $Q < |z| < 1$, and let $W(z;z_\infty)$ be a function which is regular analytic in the annulus $q = Q^2 < |z| < 1$ obtained by adjoining the inverse of the original annulus with respect to the interior circumference, and whose imaginary part possess the boundary values given by

$$-\mathfrak{J}\left(\frac{1}{z-z_\infty} + \frac{q/\bar{z}_\infty^2}{z - q/\bar{z}_\infty}\right)$$
$$(|z| = 1\ ,\ |z| = q).$$

Then, the function defined by

$$w = \Phi(z\ ;\ z_\infty)$$
$$\equiv \frac{1}{z-z_\infty} + \frac{q/\bar{z}_\infty^2}{z-q/\bar{z}_\infty} + W(z;z_\infty)$$

maps the original annulus $Q<|z|<1$ univalently onto whole plane slit along a horizontal and a vertical segment in such a manner that $|z| = 1$ and $|z| = Q$ correspond to horizontal and vertical segments of w-plane respectively and moreover that z_∞ correspond to the point at infinity, the residue of $\Phi(z\ ;\ z_\infty)$ at z_∞ being equal to unity.

Proof. $\mathfrak{J}W(z\ ;\ z_\infty)$ being the solution of Dirichlet problem, the function $W(z\ ;\ z_\infty)$ is uniquely determined except a real additive constant. It may be noticed that for an annulus the Dirichlet problem can be solved in an explicit form by means of Villat's formula; cf., for instance, Komatu [7]. In view of Lemma 1 of the preceding section, since

$$\mathfrak{J}\left(\frac{1}{z_\infty} + \frac{q/\bar{z}_\infty^2}{q/\bar{z}_\infty}\right) = \mathfrak{J}\left(\frac{1}{z_\infty} + \frac{1}{\bar{z}_\infty}\right) = 0,$$

the function $W(z;\ z_\infty)$ is surely single-valued, and hence so is the function $\Phi(z\ :\ z_\infty)$

also. By definition of $W(z;z_\infty)$, the function $\Phi(z;z_\infty)$ remains real along whole circumferences $|z|=1$ and $|z|=q$. The sum $\Phi(z;z_\infty)+\overline{\Phi(q/\bar{z};z_\infty)}$ represents a function analytic and single-valued in $q<|z|<1$ and moreover, as easily verified, regular there, i.e., the apparent singularities at z_∞ and $q/\bar{z}_\infty$ are removable. Since along whole boundary of the annulus $\mathcal{J}\Phi(z;z_\infty)$, and hence $\mathcal{J}\overline{\Phi(q/\bar{z};z_\infty)}$ also, vanishes everywhere, the sum must reduce to a real constant. By adjusting, if necessary, a real constant, we may suppose that the sum vanishes identically. We thus obtain a functional relation

$$\Phi(z;z_\infty)+\overline{\Phi(q/\bar{z};z_\infty)} \equiv 0.$$

It shows that $\Phi(z,z_\infty)$ remains, purely imaginary along the circumference $|z|=Q=\sqrt{q}$. The images of $|z|=1$ and $|z|=q$ lying on the real axis are symmetric each other with respect to the origin.

We shall now show that the image of the annulus $q<|z|<1$ by $w=\Phi(z)$ is two-sheeted everywhere except on the just mentioned slits on the real axis. We denote, in general, by $N(\gamma)$ the number of γ-points of $\Phi(z)$ in the annulus $q<|z|<1$. In view of boundary behavior of $\Phi(z;z_\infty)$ we see that $N(\gamma)$ remains constant unless the point γ lies on a slit originated from $|z|=1$ or $|z|=q$. And, since there exist two poles z_∞ and $q/\bar{z}_\infty$ both being of the first order, we get $N(\gamma)=2$. Thus, the required two-sheetedness has been asserted.

Hence, the annulus $q<|z|<1$ is mapped by $w=\Phi(z;z_\infty)$ onto a two-sheeted Riemann surface extended over whole plane slit along two segments lying on real axis in different sheets and being symmetric each other with respect to the origin. Both sheets cross over along a segment lying on imaginary axis. Consequently, the original annulus $Q<|z|<1$ is mapped univalently just onto a domain of a character stated in the theorem. The normalizing condition at z_∞ is also satisfied.

By adjusting a suitable additive complex constant we can normalize the mapping function in such a manner that its Laurent expansion around z_∞ is of the form

$$\frac{1}{z-z_\infty}+o(1).$$

This condition determines the function uniquely.

Theorem 2. Let z_0 and z_∞ be two different points interior to the annulus $Q<|z|<1$, and let $G(z;\zeta)$ be an analytic function whose real part coincides with the Green function of $q\equiv Q^2<|z|<1$ with singularity ζ. Then, the function defined by

$$\begin{aligned} w &= \Phi(z;z_0,z_\infty) \\ &\equiv \exp\Big(G(z;z_\infty)-G(z;z_0) \\ &\qquad +G(z;q/\bar{z}_\infty)-G(z;q/\bar{z}_0)\Big) \end{aligned}$$

maps the original annulus $Q<|z|<1$ univalently onto whole plane slit along a radial segment centred at the origin and a circular arc around the origin, which correspond to $|z|=1$ and $|z|=Q$ respectively, in such a manner that z_0 and z_∞ correspond to the origin and the point at infinity respectively.

Proof. In view of the remark subsequent to lemma ii of preceding section, the function $\Phi(z;z_0,z_\infty)$ is analytic and single-valued in $q<|z|<1$, and has the constant absolute value equal to unity along whole boundary $|z|=1$ and $|z|=q$. The function $G(z;\zeta)$ being determinate except a purely imaginary additive constant, the function $\Phi(z;z_0,z_\infty)$ does so except a constant factor with absolute value equal to unity. As already noticed, by suitable adjustment of the undetermined factor, we can suppose that the functional relation

$$\Phi(z;z_0,z_\infty)=\overline{\Phi(q/\bar{z};z_0,z_\infty)}$$

does hold good. We then have, for any point on $|z|=Q=\sqrt{q}$, i.e., for $q/\bar{z}=z$,

$$\Phi(z;z_0,z_\infty)=\overline{\Phi(z;z_0,z_\infty)}.$$

Consequently, the image of $|z|=Q$ lies on the real axis. On the other hand, for any real θ, we have

$$\Phi(e^{i\theta}; z_0, z_\infty) = \overline{\Phi(q e^{i\theta}; z_0, z_\infty)},$$

and hence the circular slits lying on doubly covered unit circumference which correspond to $|z|=1$ and $|z|=q$ are symmetric each other with respect to the real axis.

We then have to show that the image of the annulus $q<|z|<1$ by $w=\Phi(z)\equiv\Phi(z;z_0,z_\infty)$ is two-sheeted everywhere except on the just mentioned slits. We denote by $N(\gamma)$ the number of γ-points of $\Phi(z)$ in the annulus $q<|z|<1$. Suppose first that $|\gamma|\neq 1$. By means of argument principle, we have

$$N(\gamma)-N(\infty) = \frac{1}{2\pi}\int_{|z|=1} d\arg(\Phi(z)-\gamma) + \frac{1}{2\pi}\int_{|z|=q} d\arg(\Phi(z)-\gamma),$$

both curvilinear integrals in the right-hand side being taken in the positive sense with respect to the domain $q<|z|<1$. In view of the fact that $|\Phi(z)|=1$ along $|z|=1$ and $|z|=q$, we see that, if $|\gamma|<1$,

$$\int_{|z|=1} d\arg\left(1-\frac{\gamma}{\Phi(z)}\right) = \int_{|z|=q} d\arg\left(1-\frac{\gamma}{\Phi(z)}\right) = 0,$$

and if $|\gamma|>1$,

$$\int_{|z|=1} d\arg\left(\frac{\Phi(z)}{\gamma}-1\right) = \int_{|z|=q} d\arg\left(\frac{\Phi(z)}{\gamma}-1\right) = 0.$$

Hence, taking the monodromy condition for $\Phi(z)$ also into account, we finally obtain

$$N(\gamma)-N(\infty) = \frac{1}{2\pi}\left(\int_{|z|=1}+\int_{|z|=q}\right) d\arg(\Phi(z)-\gamma) = 0.$$

The last relation remains to hold, in view of continuity, unless the point γ lies on a circular slit originated from $|z|=1$ or $|z|=q$. It is evident that there exist just two poles z_∞ and $q/\bar{z}_\infty$, both being of the first order, namely $N(\infty)=2$, whence it follows that

$$N(\gamma)=2$$

for any γ not lying on a circular slit.

Hence, we see, as in the proof of Theorem 1, that the original annulus $Q<|z|<1$ is mapped univalently just onto a domain of a character stated in the present theorem. That z_0 and z_∞ correspond to 0 and ∞ respectively is obvious.

By modifying $\Phi(z;z_0,z_\infty)$ by a suitable constant factor, i.e., by a suitable dilatation followed by a rotation around the origin, we can normalize the mapping function in such a manner that its residue at z_∞ becomes equal to 1. This determines the mapping function uniquely.

In Theorem 2 both points z_0 and z_∞ have been restricted in the annulus $Q\equiv\sqrt{q}<|z|<1$. But, the above argument shows more generally the following fact:

Let z_0 and z_∞ be any two distinct points contained in an annulus $q<|z|<1$. Then, the function $\Phi(z;z_0,z_\infty)$ defined in Theorem 2 maps $q<|z|<1$ also onto a two-sheeted Riemann surface extended over whole plane and slit along two arcs lying on the doubly covered unit circumference in such a manner that the points z_∞ and $q/\bar{z}_\infty$ correspond to ∞ while the points z_0 and $q/\bar{z}_0$ correspond to 0. The orders of these poles and zeros are, in general, all equal to 1, but eventually equal to 2 if z_∞ or z_0 lies exactly on the circumference $|z|=\sqrt{q}$ and hence coincides with $q/\bar{z}_\infty$ or $q/\bar{z}_0$, respectively. All the branch points of the surface lie on a unique half-line starting from the origin which bears also the image of the intermediate circumference $|z|=\sqrt{q}$.

(*) Received Dec. 16, 1951.

The detailed references have been stated in the previous Note below: Komatu,Y. and M.Ozawa [1] .

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