ON THE COMPACTIFICATION OF TOPOLOGICAL SPACES

By Yosinao MISONOU and Ziro TAKEDA

There are two kinds of compactification of a complete regular space due to H.Wallman [5] and E. Čech [2] .

The Wallman compactification $\mathfrak{W}$ is the set of all maximal collections $\mathfrak{p}$ of the lattice $\mathcal{L}$ consisting of all closed subsets F of a completely regular space S and its topology is defined by taking the elements of $\mathcal{L}$ as the basis of the closed sets of $\mathfrak{W}$ associating with all maximal collections which contain the elements as coordinates.

On the other hand, the Čech compactification $\mathfrak{M}$ is the set of all maximal ideals M of the Banach algebra R consisting of all realvalued and bounded continuous functions x(s) defined on S. (c. f.I.Gelgand and G.Silov [3]). The topology of $\mathfrak{M}$ is introduced by the following method, known as Stone-topology: M belongs to the closure of subset $\mathfrak{A}$ of $\mathfrak{M}$ if and only if M divides the intersection of all M'_α s belonging to $\mathfrak{A}$.

These two compactifications are not equivalent in general since the former gives a compact T_1 -space and the later a compact Hausdorff space. But the relation of them is given by the following theorems due to P.Alexandroff [1] and A.Komatu [4] .

Theorem 1. <u>A continuous mapping from $\mathfrak{W}$ to $\mathfrak{M}$ exists preserving all points of S.</u>

Theorem 2. <u>$\mathfrak{W}$ coincides with $\mathfrak{M}$ if and only if S is normal.</u>

In this note we give an alternative proof of them using the method due to Gelfand and Šilov.

Proof of Theorem 1. If we put for a closed set F in S

$$(1) \qquad I(F) = \{ x \mid s \in F \rightarrow x(s) = 0 \},$$

then I(F) is an ideal of R. The join ideal $I(\mathfrak{p})$ of I(F) for all $F \in \mathfrak{p}$ is a proper ideal of R since $\mathfrak{p}$ has the finite intersection property. Hence by Zorn's Lemma a maximal ideal M divides $I(\mathfrak{p})$. By this definition, M belongs to the closure (in $\mathfrak{M}$) of F for each $F \in \mathfrak{p}$, whence

$$(2) \qquad H = \{ s \mid \varepsilon \geq |x(s)| \ \text{for } x \in M, \varepsilon > 0 \}$$

meets for any $F \in \mathfrak{p}$, and so H belongs to $\mathfrak{p}$, for $\mathfrak{p}$ is maximal.

If M and M' divide $I(\mathfrak{p})$, then an element x of M exists such that $x \equiv 1(M')$ and $0 \leqq x \leqq 1$. Hence

$$(3) \qquad H' = \{ s \mid x(s) \leqq \tfrac{1}{3} \}, \ H'' = \{ s \mid x(s) \geqq \tfrac{2}{3} \}$$

belong to $\mathfrak{p}$ with $H' \frown H'' = \phi$. This is a contradiction. Hence only one maximal ideal M divide $I(\mathfrak{p})$, namely, it defines a mapping $M(\mathfrak{p})$ from $\mathfrak{W}$ into $\mathfrak{M}$. Evidently, this mapping preserves all points of S.

Next, suppose that $\mathfrak{f}$ is the set of all F such as $F \in \mathfrak{p}_\alpha$ for all $\mathfrak{p}_\alpha \in A \subset \mathfrak{W}$, and suppose further a maximal collection $\mathfrak{p}$ contains $\mathfrak{f}$, namely, $\mathfrak{p}$ belongs to the closure (in $\mathfrak{W}$) of A. If we put $M(\mathfrak{p}) = M$, $M(\mathfrak{p}_\alpha) = M_\alpha$ and M does not belong to the closure (in $\mathfrak{M}$) of $\{M_\alpha\}$, then by the normality of $\mathfrak{M}$ there exists an element x in M such that $x \equiv 1(M_\alpha)$ for all α and $0 \leqq x \leqq 1$. Hence H'' of (3) belongs to $\mathfrak{f}$, and so to $\mathfrak{p}$, on the other hand $H' \frown H'' = \phi$. This is clearly a contradiction since $H' \in \mathfrak{p}$ and $\mathfrak{p}$ has the finite intersection property. Therefore this mapping is continuous. Furthermore, as S is dense in $\mathfrak{W}$ and also in $\mathfrak{M}$, $M(\mathfrak{p})$ maps $\mathfrak{W}$ onto $\mathfrak{M}$. This completes the proof of Theorem 1.

Proof of Theorem 2. Since a lemma of H. Wallmann [5] states that $\mathfrak{M}$ is normal if and only if S is normal, it suffices to show the sufficiency. To prove this, it is sufficient to show the one-to-one correspondence of the mapping under the normalily of S. Assume the contrary and suppose that $M(\mathfrak{p}) = M(\mathfrak{p}') = M$. Then there exists a pair of disjoint closed sets $F \in \mathfrak{p}$ and $F' \in \mathfrak{p}'$. Hence a element x of R exists by the nomality such that $x(s) = 0$ for $s \in F$ and $x(s) = 1$ for $s \in F'$. This is a contradiction since by the later excludes x.

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Math. Inst. Tohoku Univ.,
Sendai.