DIFFERENTIAL EQUATIONS WITH INVARIANT PFAFFIAN FORMS

By Toshiya SAITO

1. Let Ω be a compact n-dimensional analytic manifold without torsion. We consider a following system of differential equations,

$$(1)\quad \begin{cases} \dfrac{dx_1}{dt} = X_1(x_1, \cdots, x_n), \\ \cdots\cdots \\ \dfrac{dx_n}{dt} = X_n(x_1, \cdots, x_n), \end{cases}$$

where $x_1, \ldots, x_n$ are analytic local coordinates and $X_1, \ldots, X_n$ are one-valued real holomorphic functions in Ω. General solutions of this system can be written down in the following form,

$$x_i = f_i(x_{10}, \cdots, x_{n0}, t), \quad i = 1, \cdots, n,$$

where

$$x_{i0} = f_i(x_{10}, \cdots, x_{n0}, 0), \quad i = 1, \cdots, n,$$

and f_i 's are analytic functions with respect to their arguments.

If we define a transformation S_t by

$$P = S_t P_0,$$
$$P = (x_1, \cdots, x_n), \quad P_0 = (x_{10}, \cdots, x_{n0}),$$
$$x_i = f_i(x_{10}, \cdots, x_{n0}, t), \quad i = 1, \cdots, n,$$

the totality of such transformations forms a one-parameter group. Hence differential equations (1) can be regarded as defining a one-parameter stationary flow S_t in Ω.

We suppose that (1) admits $n-1$ linearly independent (with respect to numerical coefficients) invariant Pfaffian forms (in the sense of E.Cartan)[1],

$$(2)\quad \varpi_i = \sum_{k=1}^{n} A_{ik}(x_1, \cdots, x_n)\, dx_k, \quad i = 1, \cdots, n-1,$$

where A_{ik} 's are one-valued real holomorphic functions in Ω. Then we have

$$(3)\quad \sum_{k=1}^{n} A_{ik} X_k \equiv 0, \quad i = 1, \cdots, n-1 \text{ [2]}$$

Moreover we assume that ϖ_i 's are exact, i.e.

$$d\varpi_i = 0, \quad i = 1, \cdots, n-1,$$

or, in other words,

$$(4)\quad \frac{\partial A_{ik}}{\partial x_j} = \frac{\partial A_{ij}}{\partial x_k},$$
$$i = 1, \cdots, n-1, \quad j, k = 1, \cdots, n.$$

Under these assumptions, we want to study the behavior of the trajectories of (1). Our main result is the Theorem 3 of § 5 which states the necessary and sufficient condition for every trajectory of (1) to be everywhere dense in Ω. Then we apply this result to the flow in n-dimensional toroid and establish a sufficient condition for the ergodicity of S_t.

2. Let p be a one-dimensional Betti number of Ω, and $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ be its independent cycles. We put

$$\int_{\Gamma_k} \varpi_i = \omega_{ik},$$
$$i = 1, \cdots, n-1, \quad k = 1, \cdots, p.$$

Since ϖ_i 's are exact, we can find $n-1$ holomorphic functions $u_1, \ldots, u_{n-1}$ such that

$$du_i = \varpi_i, \quad i = 1, \cdots, n-1.$$

According to the relation (3),

$$\frac{du_i}{dt} = 0, \quad i = 1, \cdots, n-1$$

Hence u_i 's are integrals of (1) and the trajectory of (1) is generally given as an intersection of $n-1$ hypersurfaces

$$du_1 = 0, \quad \cdots, \quad du_{n-1} = 0.$$

u_i 's are, in general, not one-valued since they are additive functions with $\omega_{i1}, \ldots, \omega_{ip}$ as fundamental periods.

We first prove the following

THEOREM 1. If there exist $n-1$ real numbers $\lambda_1, \ldots, \lambda_{n-1}$, not simultaneously zero, such that the ratios between

$$\sum_{i=1}^{n-1}\lambda_i\omega_{i1}, \quad \cdots, \quad \sum_{i=1}^{n-1}\lambda_i\omega_{ip}$$

are all rational numbers (the case $\sum\lambda_i\omega_{i1} = \cdots = \sum\lambda_i\omega_{ip} = 0$ is included), S_t has an invariant closed analytic submanifold whose dimension is not greater than $n-1$.

Proof. If $\lambda_1, \ldots, \lambda_{n-1}$ can be so chosen that

$$\sum\lambda_i\omega_{i1} = \cdots = \sum\lambda_i\omega_{ip} = 0,$$

the function $\sum_{i=1}^{n-1}\lambda_i u_i$ is evidently one-valued since its fundamental periods

$$\int_{\Gamma_k}\sum_{i=1}^{n-1}\lambda_i\varpi_i, \quad k=1, \cdots, p$$

all vanish. As u_i's are integrals of (1), so is $\sum\lambda_i u_i$. Hence the hypersurface defined by

$$\sum\lambda_i u_i = const.$$

is a closed analytic invariant submanifold whose dimension is not greater than $n-1$.

Except this case, we may suppose, without loss of generality, that

$$\sum\lambda_i\omega_{i1} \neq 0.$$

As

$$\frac{\sum\lambda_i\omega_{ik}}{\sum\lambda_i\omega_{i1}}, \quad k=1, \cdots, p$$

are all rational, we can find p integers $m_1, \ldots, m_p$ such that

$$\frac{\sum\lambda_i\omega_{ik}}{\sum\lambda_i\omega_{i1}} = \frac{m_k}{m_1}, \quad k=1, \cdots, p.$$

Then the function

$$u = \frac{m_1}{\sum\lambda_i\omega_{i1}}\sum\lambda_i u_i$$

is an additive function with fundamental periods

$$\frac{m_1}{\sum\lambda_i\omega_{i1}}\int_{\Gamma_k}\sum\lambda_i\varpi_i = m_1\frac{\sum\lambda_i\omega_{ik}}{\sum\lambda_i\omega_{i1}} = m_k.$$

$m_1, \ldots, m_p$ being integers, a function

$$e^{2\pi i u}$$

is one-valued on Ω. Since u is an integral of (1), so is $\exp(2\pi i u)$. Hence the hypersurface

$$e^{2\pi i u} = const.$$

defines a closed analytic invariant submanifold of S_t whose dimension is not greater than $n-1$.

If $p \leq n-1$, it is evident that the assumption of the Theorem is always satisfied. So we have

COROLLARY. If $p \leq n-1$, S_t has an invariant closed analytic submanifold whose dimension is not greater than $n-1$.

3. THEOREM 2. If there exist $n-1$ real numbers $\lambda_1, \ldots, \lambda_{n-1}$, not simultaneously zero, such that the set

$$I(\lambda_1, \cdots, \lambda_{n-1}) \equiv \left\{P\,;\ \sum_{i=1}^{n-1}\lambda_i A_{i1}(P) = \cdots = \sum_{i=1}^{n-1}\lambda_i A_{in}(P) = 0\right\}$$

is not empty, S_t leaves invariant a closed analytic submanifold whose dimension is not greater than $n-1$.

Proof. If the assumption of the Theorem is satisfied, $I(\lambda_1, \cdots, \lambda_{n-1})$ is a non-empty closed analytic submanifold whose dimension is not greater than $n-1$. We will show that $I(\lambda_1, \cdots, \lambda_{n-1})$ is an invariant manifold of S_t. For that purpose, it suffices to prove that

$$\frac{d}{dt}\sum_{i=1}^{n-1}\lambda_i A_{ik} = 0, \quad k=1, \cdots, n$$

on $I(\lambda_1, \cdots, \lambda_{n-1})$.

$$\frac{d}{dt}\sum_i\lambda_i A_{ik} = \sum_j X_j\frac{\partial}{\partial x_j}\sum_i\lambda_i A_{ik} = \sum_{i,j}\lambda_i X_j\frac{\partial A_{ik}}{\partial x_j},$$

then, by the formula (4) of § 1,

$$= \sum_{i,j}\lambda_i X_j\frac{\partial A_{ij}}{\partial x_k}$$

$$= \sum_i\lambda_i\frac{\partial}{\partial x_k}\left(\sum_j A_{ij}X_j\right) - \sum_{i,j}\lambda_i A_{ij}\frac{\partial X_j}{\partial x_k}$$

The second term of the above formula evidently vanishes on $I(\lambda_1, \cdots, \lambda_{n-1})$, because we have

$$\sum_i\lambda_i A_{ij} = 0$$

on $I(\lambda_1, \cdots, \lambda_{n-1})$. The first term also vanishes according to (3) of § 1. Therefore

$$\frac{d}{dt}\sum_i\lambda_i A_{ik} = 0, \quad k=1, \cdots, n$$

on $I(\lambda_1, \cdots, \lambda_{n-1})$.

4. To simplify the statement, we say that S_t is non-singular if the assumption of the Theorem 2 is not satisfied. Otherwise it is said to be singular.

If S_t is non-singular, for any point P of Ω, there exists a uniquely determined hypersurface containing P on which

$$du_1 = 0$$

is satisfied. We denote by $F_1[P]$ the connected component of this surface determined by P. In the same way, we can also uniquely define

$$(F_i \wedge F_j \wedge \cdots \wedge F_l)[P]$$

as a connected component of $F_i[P] \wedge F_j[P] \wedge \cdots \wedge F_l[P]$ determined by P.

Obviously, $F_i[P]$, $(F_i \wedge F_j)[P]$, $(F_i \wedge F_j \wedge F_k)[P]$, ..., are all invariant sets of S_t. Especially, $(F_1 \wedge F_2 \wedge \cdots \wedge F_{n-1})[P]$ is a trajectory passing through P.

For the non-singular flow S_t, we now prove the following $n-1$ Lemmas.

LEMMA 1^1. Let P, Q be any two points of Ω which can be joined by an arc C of finite length such that

$$\int_C \varpi_1 = 0 .$$

Then we can join P and Q by an arc $\tilde{C}$ lying on $F_1[P]$ and homologous to C.

LEMMA 1^2. Let P be an arbitrary point of Ω and Q be a point on $F_1[P]$. If P and Q can be joined by an arc C of finite length lying on $F_1[P]$ such that

$$\int_C \varpi_2 = 0 ,$$

then P and Q can be joined by an arc $\tilde{C}$ lying on $(F_1 \wedge F_2)[P]$ and homologous to C.

- - - - - - - - -

- - - - - - - - -

LEMMA 1^{n-1}. Let P be an arbitrary point of Ω and Q be a point on $(F_1 \wedge F_2 \wedge \cdots \wedge F_{n-2})[P]$. If P and Q can be joined by an arc C of finite length lying on $(F_1 \wedge F_2 \wedge \cdots \wedge F_{n-2})[P]$ such that

$$\int_C \varpi_{n-1} = 0$$

then P and Q be joined by an arc $\tilde{C} \subset (F_1 \wedge F_2 \wedge \cdots \wedge F_{n-1})[P]$ homologous to C. In other words, the trajectory passing through P contains Q.

Proof of the LEMMA 1^1. As we consider only the arc of finite length, we omit the words "of finite length" for simplicity's sake. So, hereafter, the word "arc" always means "arc of finite length".

Since Ω is a compact analytic manifold, we can introduce in Ω a metric $d(P,Q) \geq 0$, $P, Q \in \Omega$, in such a way that the topology determined by this metric is equivalent to the original one. We first notice that for any P we can find a positive number $\varepsilon_1(P)$ such that our Lemma is valid in a sphere

$$K[P, \varepsilon_1(P)] = \{Q ;\ d(P,Q) < \varepsilon_1(P)\}.$$

Strictly speaking, if $A, B \in K[P; \varepsilon_1(P)]$ can be connected by an arc $\gamma = \widehat{AB}$ contained in $K[P; \varepsilon_1(P)]$ in such a way that

$$\int_\gamma \varpi_1 = 0,$$

then A and B can be joined by an arc $\tilde{\gamma}$ entirely contained in $F_1[A] \wedge K[P; \varepsilon_1(P)]$ and homologous to γ.

In fact, if this is not the case, the derivative of u_1 in any direction must be zero at P. But this contradicts with our assumption of the non-singularity.

Ω being compact,

$$\inf_{P \in \Omega} \varepsilon_1(P) = \varepsilon_1 > 0$$

Therefore our Lemma is valid in any sphere of radius ε_1.

Now let A be an arbitrary point on C, and we consider a function

$$f(A) = \int_{C(PA)} \varpi_1$$

where $C(MN)$ is an arc of C between two points M, N on C. Evidently

$$f(P) = f(Q) = 0$$

Suppose that the Lemma has been proved for the case when $f(A)$ increases from P to M, remains constant from M to N, and then decreases from N to Q where M and N are two (not necessarily different) points on C. Then we can prove the Lemma for the case when

$$f(A) > f(P) = f(Q)$$

for every A between P and Q. In fact, in this case $f(A)$ has a finite number of maxima and minima on C. Let

$$m_1 > m_2 > \cdots > m_\nu > 0$$

be these extremal values. The set

$$I = [A ; f(A) \geqq m_2]$$

is made up of a finite number of arcs, $C(A_1 A_2)$, $C(A_3 A_4)$, ..., $C(A_{k-1} A_k)$, $A_1, \ldots, A_k \in C$. On each of these arcs $f(A)$ possesses the property stated above. So we can connect A_1 and A_2, A_3 and A_4, ..., A_{k-1} and A_k by the curves $C_1, \ldots, C_\ell$ each contained in $F_1[A_1] = F_1[A_2]$, $F_1[A_3] = F_1[A_4]$, ..., $F_1[A_{k-1}] = F_1[A_k]$. If we replace $C(A_1 A_2)$, ..., $C(A_{k-1} A_k)$ by $C_1, \ldots, C_\ell$ respectively, we obtain a new arc C' joining P and Q for which we have

$$\int_{C'} \varpi_1 = 0 .$$

If we consider on C' a function

$$f'(A) = \int_{C'(PA)} \varpi_1 ,$$

extremal values of $f'(A)$ are

$$m_2 > m_3 > \cdots > m_\nu > 0 .$$

Then we consider the set

$$I' = [A ; f'(A) \geqq m_3]$$

and repeat the same procedure. Repeating such a process $\nu - 1$ times, we finally arrive at the curve $C^{(\nu-1)}$ on which a function

$$f^{(\nu-1)}(A) = \int_{C^{(\nu-1)}(PA)} \varpi_1$$

possesses the property stated above. Hence P and Q can be joined by an arc $\tilde{C}$ contained in $F_1[P]$. As is evident from the procedure we have adopted, $\tilde{C}$ is homologous to C.

In the same way we can easily show that our Lemma holds when

$$f(A) < f(P) = f(Q)$$

for every A between P and Q. Combining these results we can prove the Lemma for general cases. So we have only to show that our Lemma is true when $f(A)$ increases from P to M, remains constant from M to N, and decreases from N to Q.

In this case,

$$\int_I \varpi_1 = 0 ,$$

for any arc I on $C(MN)$. Hence $C(MN)$ is itself contained in $F_1[M] = F_1[N]$.

$A_{11}, \ldots, A_{1n}$ have no common zero points because of the non-singularity of S_t. So the function u_1 never takes an extremal value. Hence, for any point P in Ω, a positive number $\delta_1(P)$ can be so determined that for any real number α, $|\alpha| < \delta_1(P)$, there always exists a point Q contained in $K[P, \varepsilon_1/2]$ for which

$$\int_P^Q \varpi_1 = \alpha ,$$

where the integration is carried out along the arc contained in $K[P, \varepsilon_1/2]$. Since Ω is compact,

$$\inf_{P \in \Omega} \delta_1(P) = \delta_1 > 0 .$$

We can divide the arcs $C(MP)$, $C(NQ)$ by a finite number of points

$$M = P_0, P_1, \cdots, P_k, P_{k+1} = P;$$

$$N = Q_0, Q_1, \cdots, Q_k, Q_{k+1} = Q,$$

in such a way that

$$f(P_j) = f(Q_j),$$

$$f(P_j) - f(P_{j+1}) = f(Q_j) - f(Q_{j+1}) = \delta ,$$

$$|\delta| < \delta_1 ,$$

$$d(P_j P_{j+1}) < \frac{\varepsilon_1}{2} ,$$

$$d(Q_j Q_{j+1}) < \frac{\varepsilon_1}{2}, \quad j = 0, 1, \cdots, k+1.$$

Next we divide the arc $C(MN)$ by the points

$$M = K_0, \ K_1, \ \cdots, \ K_\nu, \ K_{\nu+1} = N,$$

$$d(K_j K_{j+1}) < \frac{\varepsilon_1}{2}, \quad j = 0, 1, \cdots, \nu,$$

and consider the spheres

$$S_j = K[K_j ; \frac{\varepsilon_1}{2}], \quad j = 0, 1, \cdots, \nu+1.$$

Then from what we have stated above, we can find in each S_j a point B_j and an arc γ_j joining K_j and B_j in such a way that

$$\int_{\gamma_j} \varpi_1 = \delta.$$

In particular, we may suppose that

$$B_0 = P_1, \qquad B_{\nu+1} = Q_1,$$

$$\gamma_0 = C(MP_1), \quad \gamma_{\nu+1} = C(NQ_1).$$

Since the radius of S_j is $\varepsilon_1/2$ and

$$d(K_j K_{j+1}) < \frac{\varepsilon_1}{2}$$

we can construct a sphere V_j whose radius is ε_1 and containing B_j, B_{j+1}, $C(K_j K_{j+1})$, γ_j, γ_{j+1} in its interior. Then the arc

$$-\gamma_j + C(K_j K_{j+1}) + \gamma_{j+1}$$

connects B_j and B_{j+1} in V_j and

$$\int_{-\gamma_j} \varpi_1 + \int_{C(K_j K_{j+1})} \varpi_1 + \int_{\gamma_{j+1}} \varpi_1 = -\delta + 0 + \delta = 0$$

Therefore, from the fact stated at the beginning of this proof, we can join B_j and B_{j+1} by an arc σ_j contained in $F_1[B_j] \frown V_j$. Since

$$B_0 = P_1, \quad B_{\nu+1} = Q_1,$$

we can join P_1 and Q_1 by an arc

$$\sigma_0 + \sigma_1 + \cdots + \sigma_\nu$$

which is contained in $F_1[P_1]$ and obviously homologous to $C(P_1 Q_1)$. Since the length of this arc is finite, we can repeat the same discussion replacing $C(MN)$ by $\sigma_0 + \sigma_1 + \cdots + \sigma_\nu$, and connect P_2 and Q_2 by an arc contained in $F_1[P_2]$. Proceeding in this way, we can finally join P and Q by an arc $\widetilde{C}$ contained in $F_1[P]$ and homologous to C.

Proof of the LEMMA 1^2. Since S_t is non-singular, $F_1[P]$ and $F_2[P]$ are never tangent to each other for any point P of Ω. Therefore the derivative of u_2 at P in the direction tangential to $F_1[P]$ and normal to $(F_1 \frown F_2)[P]$ is different from zero for any point P. So we can find a positive number $\varepsilon_2(P)$ such that if for an arc

$$\widehat{AB} \subset F_1[P] \frown K[P, \varepsilon_2(P)]$$

we have

$$\int_{\widehat{AB}} \varpi_2 = 0,$$

A and B can be joined by an arc contained in $(F_1 \frown F_2)[P] \frown K[P; \varepsilon_2(P)]$. Since Ω is compact,

$$\inf_{P \in \Omega} \varepsilon_2(P) = \varepsilon_2 > 0.$$

Once such a number ε_2 can be found, the remaining part of the proof is quite similar to that of the LEMMA 1^1. So we do not repeat the rather lengthy proof here.

Proceeding in this way, we can successively prove the Lemmas 1^3, 1^4, ..., 1^{n-1}.

As is evident from the proof above, our Lemmas are also valid if $P = Q$. So we have

Corollary of the LEMMA 1^k. If C is a simple closed curve on $(F_1 \frown F_2 \frown \cdots \frown F_{k-1})[P]$ such that

$$\int_C \varpi_k = 0,$$

where P is a point on C, then we can find a simple closed curve $\widetilde{C}$ lying on $(F_1 \frown F_2 \frown \cdots \frown F_k)[P]$ and homologous to C.

5. THEOREM 3. For every trajectory of (1) to be everywhere dense in Ω, it is necessary and sufficient that S_t is non-singular and the assumption of the Theorem 1 does not hold.

Proof. By Theorems 1 and 2,

the necessity of the condition is obvious. So we have only to prove the sufficiency.

Let P and Q be two arbitrary points in Ω and V be an arbitrary neighborhood of Q. We connect P and Q by an arbitrary arc C_1 and put

$$\int_{C_1} \varpi_1 = \alpha_1$$

As S_t is non-singular, A_{11}, ..., A_{1n} can never be simultaneously zero. So u_1 never takes an extremal value. Therefore there exists a positive number δ_1 such that for any number ε, $|\varepsilon| < \delta_1$, we can find an arc $\widehat{QQ'}$, contained in V for which

$$\int_{\widehat{QQ'}} \varpi_1 = \varepsilon.$$

Since the assumption of the Theorem 1 does not hold, at least one of the ratios between ω_{11}, ..., ω_{1p} must be irrational. So we can find p integers m_1, ..., m_p such that

$$\sum_{i=1}^{p} m_i\, \omega_{1i} = -\varepsilon - \alpha_1, \quad |\varepsilon| < \delta_1.$$

Thus for an arc $C_1' \sim C_1 + \sum m_i \Gamma_i$ ($A \sim B$ means A is homologous to B) joining P and Q, we have

$$\int_{C_1'} \varpi_1 = -\varepsilon - \alpha_1 + \alpha_1 = -\varepsilon$$

Then from the fact just proved above, we can find an arc $\widehat{QQ'}$ in V such that

$$\int_{\widehat{QQ'}} \varpi_1 = \varepsilon.$$

If we put

$$C_1' + \widehat{QQ'} = C_1''$$

we have

$$\int_{C_1''} \varpi_1 = 0.$$

S_t being non-singular, Lemma 1[1] is valid. So Q' is contained in $F_1[P]$

We then join P and Q' by an arbitrary arc C_2 lying on $F_1[P]$, and put

$$\int_{C_2} \varpi_2 = \alpha_2.$$

Since $F_1[Q']$ and $F_2[Q']$ are not tangent to each other at Q' because of the non-singularity of S_t, we can find a positive number δ_2 such that for any ε, $|\varepsilon| < \delta_2$ there exists an arc $\widehat{Q'Q''}$ contained in $V \cap F_1[Q']$ for which

$$\int_{\widehat{Q'Q''}} \varpi_2 = \varepsilon.$$

If we replace C_2 by an arc C_2' lying on $F_1[P]$,

$$\int_{C_2'} \varpi_2 = \alpha_2 + \sum_{i=1}^{p} m_i\, \omega_{2i}$$

where m_1, ..., m_p are integers satisfying a relation

$$(5) \qquad \sum_{i=1}^{p} m_i\, \omega_{1i} = 0$$

Conversely, for any p integers m_1, ... m_p satisfying (5), we can find an arc C_2' lying on $F_1[P]$ and

$$C_2' \sim C_2 + \sum m_i \Gamma_i.$$

In fact, let us first construct a closed curve C (not necessarily on $F_1[P]$) containing P and homologous to $\sum m_i \Gamma_i$. Then, we have

$$\int_C \varpi_1 = \sum m_i\, \omega_{1i} = 0.$$

So, by the corollary of the Lemma 1[1], we can find a closed curve $\tilde{C}$ lying on $F_1[P]$ and homologous to C. So, if we put

$$C_2' = \tilde{C} + C_2,$$

we have

$$C_2' \sim C + C_2 \sim \sum m_i \Gamma_i + C_2.$$

Let m_1, ..., m_p be the integers satisfying (5). Without loss of generality, we may suppose that

$$\omega_{11} \neq 0.$$

So, we have

$$m_1 = -\frac{1}{\omega_{11}} \sum_{i=2}^{p} m_i\, \omega_{1i},$$

and

$$\sum_{i=1}^{p} m_i\, \omega_{2i} = -\frac{\omega_{21}}{\omega_{11}} \sum_{i=2}^{p} m_i\, \omega_{1i} + \sum_{i=2}^{p} m_i\, \omega_{2i}$$
$$= \sum_{i=2}^{p} m_i \left(\omega_{2i} - \frac{\omega_{21}}{\omega_{11}}\, \omega_{1i} \right).$$

Since the assumption of the Theorem 1 does not hold, at least one of the ratios between

$$\omega_{21}-\lambda\omega_{11}, \cdots, \omega_{2p}-\lambda\omega_{1p}, \lambda = \frac{\omega_{21}}{\omega_{11}}$$

must be an irrational number. However, as

$$\omega_{21} - \lambda\omega_{11} = 0,$$

at least one of the ratios between

$$\omega_{22}-\lambda\omega_{12}, \cdots, \omega_{2p}-\lambda\omega_{1p}$$

must be an irrational number. So we can choose p integers m_1, ..., m_p in such a way that

$$\sum_{i=1}^{p} m_i\omega_{2i} = \sum_{i=2}^{p} m_i(\omega_{2i}-\lambda\omega_{1i})$$
$$= -\alpha_2 - \varepsilon, \quad |\varepsilon| < \delta_2,$$
$$\sum_{i=1}^{p} m_i\omega_{1i} = 0.$$

Therefore there exists an arc $C_2' \sim C_2 + \sum m_i\Gamma_i$ joining P and Q' on $F_1[P]$ for which

$$\int_{C_2'} \varpi_2 = \alpha_2 - \alpha_2 - \varepsilon = -\varepsilon.$$

Then from what we have shown above, we can find an arc $\widehat{Q'Q''}$ contained in $F_1[Q'] \cap V$ for which

$$\int_{\widehat{Q'Q''}} \varpi_2 = \varepsilon.$$

If we put

$$C_2'' = C_2' + \widehat{Q'Q''},$$

C_2'' is contained in $F_1[P] = F_1[Q']$, and

$$\int_{C_2''} \varpi_2 = 0.$$

Thus, according to the Lemma 1[2], P and Q'' can be joined by an arc contained in $(F_1 \cap F_2)[P]$.

Repeating such a process $n-1$ times, we can finally show that there exists a point $Q^{(n-1)}$ in V which can be joined with P by the curve $(F_1 \cap F_2 \cap \cdots \cap F_{n-1})[P]$. This means that the trajectory passing through P interesects with V. Since P, Q, and V have been chosen arbitrarily, every trajectory of (1) must be everywhere dense in Ω, and the proof of the Theorem 3 is complete.

If it is known that S_t has no invariant closed analytic submanifold whose dimension is not greater than $n-1$, S_t must be non-singular and the assumption of the Theorem 1 must not hold as a consequence of Theorem 1 and Theorem 2. So the Theorem 3 can also be stated in the following form.

THEOREM 3'. For every trajectory of (1) to be everywhere dense in Ω, it is necessary and sufficient that S_t has no invariant closed analytic submanifold whose dimension is not greater than $n-1$.

6. We have thus established the criterion for every trajectory to be everywhere dense under the assumption that (1) has $n-1$ invariant exact Pfaffian forms. For such a flow, it is very desirable to establish the ergodicity. Unfortunately we could not solve this problem in general cases. However, if the manifold in question is of a comparatively simple topological character, we can expect to go a little further in this direction.

As a simplest example, we will treat the flow of the above stated type in n-dimensional toroid. In the following sections, we will show that we can establish the ergodicity of the flow if it admits an analytic surface of section.

Hereafter Ω is supposed to be an n-dimensional toroid. Therefore its one-dimensional Betti number p must be equal to n.

7. An analytic $(n-1)$-dimensional closed submanifold S is said to be a surface of section of S_t if following conditions are satisfied,

(1) no trajectory of S_t is tangent to S,

(2) for any point P of Ω, the trajectory starting from P cuts S after a finite t-interval.

LEMMA 2. If S_t has a surface of section, we can construct in Ω a family of surfaces of section

$$\{ S(\alpha) ; \ 0 \leq \alpha < 1 \}$$

with following properties,

(1) $S(\alpha)$ and $S(\beta)$ have no points in common if $\alpha \neq \beta$,

(2) for any point P in Ω we

can find $S(\alpha)$ of this family which contains P

Proof. By assumption S_t has a surface of section. Let this be called $S(0)$. Consider the trajectory starting from an arbitrary point P on $S(0)$. From the condition (2) of the surface of section, this trajectory returns to $S(0)$ after the finite t - interval $T(P)$. Let $S(\alpha)$ be an analytic hypersurface defined by

$$S(\alpha) = \{ Q \,;\; Q = S_{\alpha T(P)} P, \\ P \in S(0), \quad \alpha = \text{const.} \}$$

We can easily show that $S(\alpha)$ is a surface of section. Varying α from 0 to 1, we obtain a family of surfaces of section with desired properties.

In what follows, we restrict our attention to the flow with a surface of section, and for the flow of this type we prove the following,

THEOREM 4. If S_t is non-singular and the assumption of the Theorem 1 does not hold, S_t is ergodic.

8. For the proof of the Theorem, we first prove

LEMMA 3. Let Π be an r - dimensional toroid whose points can be represented by the coordinates

$$(w_1, \cdots, w_r),$$

$$0 \leq w_i < 1, \quad i = 1, \cdots, r.$$

We consider in Π a mapping Φ

$$\Phi w_i = [w_i + \gamma_i], \quad i = 1, \cdots, r,$$

where the notation $[\alpha]$ means a value of α reduced modulo 1, and γ_i 's are real constants such that the relation

$$[\Sigma m_i \gamma_i] = 0,$$

$$m_1, \cdots, m_r \;:\; \text{integers},$$

implies

$$m_1 = \cdots = m_r = 0.$$

If A is a measurable subset of Π with positive Lebesgue measure

$$\int_A dw_1 \cdots dw_r$$

and invariant under Φ except the set of measure zero, the characteristic function of A must be equal to 1 almost everywhere.

Proof. Let $\mathfrak{H}$ be the totality of complex-valued functions Lebesgue measurable and square summable on Π. $\mathfrak{H}$ is evidently a Hilbert space with the inner product

$$(f, g) = \int_\Pi f \bar{g} \, dw_1 \cdots dw_r, \quad f, g \in \mathfrak{H}$$

and the norm

$$\| f \| = \sqrt{(f, f)}, \quad f \in \mathfrak{H}.$$

We define in $\mathfrak{H}$ a linear transformation U by

$$U f(w_1, \cdots, w_r) \\ = f([w_1 + \gamma_1], \cdots, [w_r + \gamma_r]).$$

U is evidently a unitary transformation, and the function

$$e^{2\pi i (m_1 w_1 + \cdots + m_r w_r)}, \quad m_1, \cdots, m_r : \text{integers},$$

is an eigenfunction of U corresponding to an eigenvalue

$$e^{2\pi i (m_1 \gamma_1 + \cdots + m_r \gamma_r)}.$$

Since the totality of $\exp\{2\pi i (m_1 w_1 + \cdots + m_r w_r)\}$ forms a complete orthonormal system in $\mathfrak{H}$, there can be no other eigenfunctions.

Let χ_A be a characteristic function of A, i.e.

$$\chi_A(P) = 1, \quad \text{if } P \in A, \\ = 0, \quad \text{otherwise}.$$

Then χ_A belongs to $\mathfrak{H}$ and we have

$$U \chi_A = \chi_A \neq 0,$$

(where the equality sign must be interpreted in the sense of the strong topology in $\mathfrak{H}$), which shows that χ_A is an eigenfunction of U belonging to the eigenvalue 1. Hence

$$\chi_A = e^{2\pi i (m_1 w_1 + \cdots + m_r w_r)},$$

where

$$[\Sigma m_i \gamma_i] = 0.$$

But from the assumption of the Lemma, the latter relation implies

$$m_1 = \cdots = m_r = 0.$$

Therefore

$$\chi_A = 1$$

in the sense of the strong topology in $\mathfrak{H}$. Consequently χ_A must be equal to 1 almost everywhere on Π.

9. Proof of the THEOREM 4. As S_t is supposed to have a surface of section, we can construct a family of surfaces of section $\{ S(\alpha),\ 0 \leq \alpha < 1 \}$ with the properties (1) and (2) stated in the Lemma 2. As is evident from the method of construction of this family stated in the proof of the Lemma 2, every $S(\alpha)$ is homeomorphic to $S(0)$, and Ω is homeomorphic to the topological product of $S(0)$ and the circle $0 \leq \alpha < 1$. Since Ω is an n-dimensional toroid, every $S(\alpha)$ must be homeomorphic to an $(n-1)$-dimensional toroid. Therefore we may suppose, without loss of generality, $\Gamma_1, \ldots, \Gamma_{n-1}$ form the one-dimensional homology base of $S(0)$. Then, if we regard $\varpi_1, \ldots, \varpi_{n-1}$ as Pfaffian forms on $S(0)$, their periods on $S(0)$ are

$$\omega_{i1}, \cdots, \omega_{i\,n-1}, \quad i = 1, \cdots, n-1.$$

By assumption of the Theorem, S_t does not satisfy the assumption of the Theorem 1. Hence

$$\begin{vmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1,n-1} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{n-1,1} & \omega_{n-2,2} & \cdots & \omega_{n-1,n-1} \end{vmatrix} \neq 0$$

In fact, if this is not the case, we can find $n-1$ real numbers $\lambda_1, \ldots, \lambda_{n-1}$ not simultaneously zero such that

$$\sum_{i=1}^{n-1} \lambda_i \omega_{iK} = 0, \quad K = 1, \cdots, n-1,$$

contrary to our assumption.

Consequently we can find $(n-1)^2$ real numbers λ_{iK}, $i, K = 1, \cdots, n-1$ such that

$$\begin{pmatrix} \lambda_{11} & \lambda_{21} & \cdots & \lambda_{n-1,1} \\ \lambda_{12} & \lambda_{22} & \cdots & \lambda_{n-1,2} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{1,n-1} & \lambda_{2,n-1} & \cdots & \lambda_{n-1,n-1} \end{pmatrix} \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1,n-1} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{n-1,1} & \omega_{n-1,2} & \cdots & \omega_{n-1,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

If we put

$$\pi_i = \sum_{K=1}^{n-1} \lambda_{iK} \varpi_K, \quad i = 1, \cdots, n-1,$$

we have

$$\int_{\Gamma_K} \pi_i = 0, \qquad \text{if} \quad i \neq K,$$

$$\int_{\Gamma_i} \pi_i = 1.$$

Let A, B be two points on $S(0)$ and C be any arc joining A and B. Then

$$\int_C \pi_i = w_i + m_i,$$

where m_i is an integer and $0 \leq w_i < 1$. Obviously w_i is uniquely determined by A and B, and independent of C. So let us write

$$\Big[\int_A^B \pi_i \Big] = w_i.$$

Let us correspond to every point P on $S(0)$ an ordered set of $n-1$ real numbers

$$(w_1, \cdots, w_{n-1}),$$

$$w_i = \Big[\int_{P_0}^{P} \pi_i \Big],$$

where P_0 is an arbitrarily chosen fixed point on $S(0)$. This is a bicontinuous mapping of $S(0)$ onto the subset of an $(n-1)$-dimensional toroid Π

$$\Pi = \{ (w_1, \cdots, w_{n-1});\ 0 \leq w_i < 1, \\ i = 1, \cdots, n-1 \}.$$

Moreover this correspondence is one-to-one. To show this, it is sufficient to prove that if $P \neq P_0$,

$$\Big[\int_{P_0}^{P} \pi_1 \Big], \ \cdots, \ \Big[\int_{P_0}^{P} \pi_{n-1} \Big]$$

can never be simultaneously zero.

Suppose that

$$\Big[\int_{P_0}^{P} \pi_1 \Big] = \cdots = \Big[\int_{P_0}^{P} \pi_{n-1} \Big] = 0$$

for some $P \neq P_0$. Then if we connect P_0 and P by an arbitrary curve C of finite length lying on $S(0)$,

$$\int_C \pi_i = m_i, \quad i=1,\cdots,n-1$$

where m_i 's are all integers. If we replace C by a curve

$$C' \sim C - \sum_{i=1}^{n-1} m_i \Gamma_i, \quad C' \subset S(0),$$

we have

$$\int_{C'} \pi_1 = \cdots = \int_{C'} \pi_{n-1} = 0.$$

As S_t is assumed to be non-singular, Lemmas 1^1, ..., 1^{n-1} are all valid. Therefore P_0 and P can be joined by an arc of trajectory $\tilde{C}$ which is homologous to C'. Let us then consider the function $\alpha(P)$ defined as follows,

$$\alpha(P) = \alpha, \quad \text{if} \quad P \in S(\alpha).$$

α is not a holomorphic function because of its discontinuity at $S(0)$. To avoid this, we have only to put

$$S(\alpha+m) = S(\alpha), \quad m: \text{integer},$$

and regard $\alpha(P)$ as a many-valued function

$$\alpha(P) = \alpha + m, \quad \text{if } P \in S(\alpha+m)$$

Evidently the differential $d\alpha$ is one-valued in spite of the many-valuedness of α. Since α increases when P moves along the trajectory in the increasing sense of t,

$$\int_{\tilde{C}} d\alpha \neq 0.$$

Also, C' being contained in $S(0)$, we have

$$\int_{C'} d\alpha = 0.$$

Hence

$$\int_{C'-\tilde{C}} d\alpha \neq 0.$$

On the other hand, as C' is homologous to $\tilde{C}$

$$C' - \tilde{C} \sim 0$$

So we must have

$$\int_{C'-\tilde{C}} d\alpha = 0$$

Thus we have arrived at the contradiction.

Consequently the correspondence

$$P \to (w_1, \cdots, w_{n-1})$$

is a homeomorphism of $S(0)$ onto a subset of the $(n-1)$ -dimensional toroid Π. However, since $S(0)$ itself is an $(n-1)$ -dimensional toroid, this correspondence must be a homeomorphism of $S(0)$ onto whole Π. Thus, in place of $(x_1, \cdots, x_n)$, we can choose $(\alpha, w_1, \cdots, w_{n-1})$ as a coordinate system in Ω by putting

$$P = (\alpha, w_1, \cdots, w_{n-1}), \quad \text{if}$$
$$P = S_{[\alpha]T(P_0)} P_0,$$
$$P_0 = (w_1, \cdots, w_{n-1}) \in S(0),$$

where $T(P_0)$ is a function which we have used in the proof of the Lemma 2.

Let $P = (\alpha, w_1, \cdots, w_{n-1})$ be an arbitrary point on $S(\alpha)$. The trajectory starting from P returns to $S(\alpha+1) = S(\alpha)$ at some other point P'

$$P' = (\alpha+1, w_1', \cdots, w_{n-1}').$$

We define an automorphism Ψ of $S(\alpha)$ by

$$\Psi(P) = P',$$

and put

$$\gamma_i = w_i' - w_i, \quad i=1,\cdots,n-1.$$

As π_i 's are invariant forms, γ_i 's are constants independent of P. In fact, let P and Q be two different points on $S(\alpha)$, and

$$P' = \Psi(P),$$
$$Q' = \Psi(Q).$$

Let C_1, C_2, C_3, C_4 be the arcs on $S(\alpha)$ joining PP', $P'Q'$, $Q'Q$, QP, respectively. Then $C = C_1 + C_2 + C_3 + C_4$ is a closed curve on $S(\alpha)$. Therefore

$$\left[\int_C \pi_i\right] = 0, \quad i=1,\cdots,n-1.$$

As π_i 's are invariant Pfaffian forms,

$$\left[\int_P^Q \pi_i\right] = \left[\int_{P'}^{Q'} \pi_i\right], \quad i=1,\cdots,n-1,$$

or,

$$[\int_{C_2}\pi_i] + [\int_{C_4}\pi_i] = 0, \quad i=1,\cdots,n-1.$$

Consequently

$$[\int_P^{P'}\pi_i] = [\int_Q^{Q'}\pi_i], \quad i=1,\cdots,n-1.$$

Hence γ_i's are constants independent of P.

Thus Ψ induces on Π a mapping Φ

$$\Phi w_i = [w_i + \gamma_i], \quad i=1,\cdots,n-1.$$

From these facts, we can see that

$$dt\, dw_1 \cdots dw_{n-1}$$

is invariant by the transformation S_t. The function $T(P)$ can be considered as a positive definite holomorphic function of $w_1, \ldots, w_{n-1}$, and we have

$$dt = T(w_1 \cdots w_{n-1})d\alpha + \alpha \sum \frac{\partial T}{\partial w_i} dw_i,$$

So, S_t has an invariant measure

$$dm = \left(T\, d\alpha + \alpha \sum \frac{\partial T}{\partial w_i} dw_i\right) dw_1 \cdots dw_{n-1}$$

$$= T\, d\alpha\, dw_1 \cdots dw_{n-1}. \quad (3)$$

As was shown by J. von Neumann and G.D.Birkhoff, the ergodicity is equivalent to the metrical transitivity for the flow with an invariant measure.(4) So, for the proof of our Theorem, we have only to show that every m-measurable set of positive m-measure invariant by S_t must be equal to Ω itself except the set of measure zero.

By Theorem 3, every trajectory of (1) must be everywhere dense in Ω. Hence the set

$$\{\Psi^k(P);\ k=0,\pm1,\pm2,\cdots,\ P\in S(\alpha)\}$$

must be everywhere dense on $S(\alpha)$. Therefore the set

$$\{([w_1+k\gamma_1], \cdots, [w_{n-1}+k\gamma_{n-1}]),\ k=0,\pm1,\pm2,\cdots\}$$

must be everywhere dense on Π, whence we can conclude that the relation

$$[\sum_{i=1}^{n-1} m_i\gamma_i] = 0,$$
$$m_1,\cdots,m_{n-1}:\ \text{integers},$$

implies

$$m_1 = \cdots = m_{n-1} = 0.$$

Then, by the Lemma 3, if M is a Φ-invariant subset of Π with positive Lebesgue measure, the characteristic function of M must be almost everywhere equal to 1 on Π.

Let A be an S_t-invariant m-measurable subset of Ω with positive m-measure. Then $A \wedge S(\alpha)$ is a Ψ-invariant subset of $S(\alpha)$ with positive $\int_{A\wedge S(\alpha)} dw_1 \cdots dw_{n-1}$ for almost all α, which corresponds on Π to a Φ-invariant set with positive Lebesgue measure. So, from what we have shown above, the characteristic function of $A \wedge S(\alpha)$ must be almost everywhere equal to 1 on $S(\alpha)$ with respect to the measure $dw_1 \cdots dw_{n-1}$ for almost all α. Hence the characteristic function of A must be equal to 1 almost everywhere on Ω with respect to m-measure. This shows that A is equal to Ω except the set of m-measure zero. Thus we have completed the proof.

Corresponding to Theorem 3', Theorem 4 can also be stated in the following form.

THEOREM 4'. If S_t leaves invariant no closed analytic submanifold whose dimension is not greater than $n-1$, S_t is ergodic.

Also, if we use the Theorem 3, we have

THEOREM 4". If every trajectory of S_t is everywhere dense in Ω, S_t is ergodic.

10. In this section we will give several examples of the system of differential equations to which our result is applicable.

EXAMPLE 1. A measure-preserving flow on the torus.

Let Ω be a torus whose points can be represented by two angular coordinates $0 \leq x_1, x_2 < 2\pi$, and we consider an analytic flow S_t defined by

$$\begin{cases} \dfrac{dx_1}{dt} = X_1(x_1, x_2), \\[2mm] \dfrac{dx_2}{dt} = X_2(x_1, x_2). \end{cases}$$

We suppose that S_t admits an integral invariant(5)

$$\iint \rho(x_1, x_2)\, dx_1\, dx_2$$

where ρ is a positive definite holomorphic function on Ω. In this case, we have

$$(6) \qquad \frac{\partial \rho X_1}{\partial x_1} + \frac{\partial \rho X_2}{\partial x_2} = 0.^{(6)}$$

Therefore if we consider a Pfaffian form

$$\varpi = \rho X_2\, dx_1 - \rho X_1\, dx_2,$$

we have

$$d\varpi = 0,$$

and

$$\rho X_2 \cdot X_1 - \rho X_1 \cdot X_2 \equiv 0.$$

Hence ϖ is an exact invariant Pfaffian form, and the theorems hitherto established are applicable.

In this case $p = 2$, and the periods of ϖ are

$$-\int_0^{2\pi} \rho X_1\, dx_2 \quad \text{and} \quad \int_0^{2\pi} \rho X_2\, dx_1$$

respectively. If the Fourier expansions of ρX_1 and ρX_2 are given by

$$\rho X_1 = \sum a_{mn} e^{i(mx_1 + nx_2)},$$

$$\rho X_2 = \sum b_{mn} e^{i(mx_1 + nx_2)},$$

we have, by the relation (6),

$$m a_{mn} + n b_{mn} = 0, \qquad m, n = 0, \pm 1, \pm 2, \cdots.$$

Hence

$$a_{m0} = 0, \quad b_{0n} = 0, \qquad \text{for } m, n = \pm 1, \pm 2, \cdots.$$

Therefore

$$\int_0^{2\pi} \rho X_1\, dx_2 = 2\pi a_{00} + \Big[\sum_{n \neq 0} \frac{a_{mn}}{in} e^{i(mx_1 + nx_2)} \Big]_{x_2 = 0}^{x_2 = 2\pi} = 2\pi a_{00},$$

$$\int_0^{2\pi} \rho X_2\, dx_1 = 2\pi b_{00} + \Big[\sum_{m \neq 0} \frac{b_{mn}}{im} e^{i(mx_1 + nx_2)} \Big]_{x_1 = 0}^{x_1 = 2\pi} = 2\pi b_{00}.$$

Consequently the periods of ϖ are $-2\pi a_{00}$ and $2\pi b_{00}$ respectively.

For the assumption of the Theorem 1 not to hold, it is necessary and sufficient that

$$-2\pi a_{00} / 2\pi b_{00} = -a_{00} / b_{00}$$

is an irrational number. Non-singularity of S_t means that ρX_1 and ρX_2 have no common zero points. But, since ρ is positive definite on Ω, this implies that X_1 and X_2 have no common zero points.

Thus, according to our results hitherto obtained, every trajectory of S_t is everywhere dense in Ω if and only if X_1 and X_2 have no common zero points and a_{00}/b_{00} is an irrational number.

(The truth is that this statement can be replaced by a sharper one:

If X_1, X_2, and ρ all have continuous first derivatives, S_t is ergodic when and only when X_1 and X_2 have no common zero points and a_{00}/b_{00} is an irrational number.

The proof of this statement has been given elsewhere by the author.[7])

EXAMPLE 2. A separable Hamiltonian system near a formally stable equilibrium point.

Let us consider a Hamiltinian system of n degrees of freedom,

$$(7) \quad \begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H}{\partial p_i}, \\[2mm] \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial q_i}, \end{cases} \qquad i = 1, \cdots, n,$$

where H is a real-valued function of $q_1, p_1, \dots, q_n, p_n$ holomorphic in the neighborhood of the origin. We suppose that the system is separable and the origin is its equilibrium point. Then we can write

$$H = H_1(q_1, p_1) + \cdots + H_n(q_n, p_n),$$

$$H_i(q_i, p_i) = \frac{1}{2}\left(a_{i1} q_i^2 + 2 a_{i2} q_i p_i + a_{i3} p_i^2 \right) + \overline{H}_i, \qquad i = 1, \cdots, n,$$

where $\overline{H}_i$'s are power series in q_i, p_i, lacking the terms of 0th, 1st, and 2nd degree, absolutely convergent in the neighborhood of $q_i = p_i = 0$. The equations (7) can then be expressed as

$$(8,i)\begin{cases} \dfrac{dq_i}{dt} = a_{i2} q_i + a_{i3} p_i + \dfrac{\partial \bar{H}_i}{\partial p_i}, \\ \dfrac{dp_i}{dt} = -a_{i1} q_i - a_{i2} p_i - \dfrac{\partial \bar{H}_i}{\partial q_i}. \end{cases}$$

$$i=1,\cdots,n.$$

The equilibrium point

$$q_1 = p_1 = \cdots = q_n = p_n = 0$$

is said to be formally stable if the characteristic roots of the matrix

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix},$$

$$A_i = \begin{pmatrix} a_{i2} & a_{i3} \\ -a_{i1} & -a_{i2} \end{pmatrix},$$

are all purely imaginary.[8] In this case, it is obvious that the characteristic roots of each A_i are also purely imaginary. Hence $q_i = p_i = 0$ is a formally stable equilibrium point of the differential equations (8,i) for each i .

(8,i) has a one-valued integral

$$H_i = \text{const.}$$

holomorphic in the neighborhood of $q_i = p_i = 0$. Therefore the n -dimensional hypersurface $\Pi(c_1 \cdots c_n)$ defined by

$$H_1 = c_1,\ H_2 = c_2,\ \cdots,\ H_n = c_n$$

is an integral surface of the system (7). Since the characteristic roots of each A_i are purely imaginary, the formula

$$H_i = \text{const.}$$

represents a family of simple closed curves on (q_i, p_i)-plane envelopping the origin. Therefore $\Pi(c_1 \cdots c_n)$ is homeomorphic to an n -dimensional toroid in the neighborhood of the origin. Hereafter we only consider the flow S_t defined by (7) on a toroid $\Pi(c_1 \cdots c_n)$ for some fixed non-vanishing values of c_1 , ..., c_n .

Now it was proved by Ura[9] that if we put

$$\xi_i = \sqrt{|H_i|}\ \sin\theta_i,$$

$$\eta_i = \sqrt{|H_i|}\ \cos\theta_i,$$

$$\theta_i = \tan^{-1}\varphi_i,$$

where φ_i is an integral of the system of differential equations

$$\begin{cases} \dfrac{dq_i}{dt} = \dfrac{\partial H_i}{\partial q_i}, \\ \dfrac{dp_i}{dt} = \dfrac{\partial H_i}{\partial p_i}, \end{cases}$$

$(q_i, p_i) \to (\xi_i, \eta_i)$ is a one-to-one analytic transformation in the neighborhood of $q_i = p_i = 0$ and the curve

$$H_i = c_i$$

is transformed into a circle

$$\xi_i = \sqrt{|c_i|}\ \sin\theta_i,$$

$$\eta_i = \sqrt{|c_i|}\ \cos\theta_i.$$

So, as a coordinate system on $\Pi(c_1 \cdots c_n)$, we may use $(\theta_1,\cdots,\theta_n)$ thus defined.

In these new variables, S_t is defined by

$$(9)\begin{cases} \dfrac{d\theta_i}{dt} = f_i(\theta_i), \\ f_i(\theta_i) = \dfrac{1}{1+\varphi_i^2}\left(\dfrac{\partial \varphi_i}{\partial q_i}\dfrac{\partial H_i}{\partial p_i} - \dfrac{\partial \varphi_i}{\partial p_i}\dfrac{\partial H_i}{\partial q_i}\right)^{(10)} \\ i=1,\cdots,n \end{cases}$$

It is also proved in Ura's paper that $f_i(\theta_i)$ is a holomorphic function in the neighborhood of $q_i = p_i = 0$ and never becomes zero except at this point.[11] Hence the function

$$\frac{1}{f_i(\theta_i)}$$

is also holomorphic on $\Pi(c_1 \cdots c_n)$ for the non-vanishing values of c_1 , ..., c_n . So (9) has $n-1$ invariant Pfaffian forms with holomorphic coefficients,

$$\varpi_1 = \frac{d\theta_1}{f_1(\theta_1)} - \frac{d\theta_2}{f_2(\theta_2)},$$

$$\varpi_2 = \frac{d\theta_2}{f_2(\theta_2)} - \frac{d\theta_3}{f_3(\theta_3)},$$

$$\cdots\cdots$$

$$\varpi_{n-1} = \frac{d\theta_{n-1}}{f_{n-1}(\theta_{n-1})} - \frac{d\theta_n}{f_n(\theta_n)} ,$$

$$d\varpi_1 = d\varpi_2 = \cdots = d\varpi_{n-1} = 0 .$$

Therefore the Theorem 3 is applicable. Since f_i 's never take the value zero on $\Pi(c_1 \cdots c_n)$, S_t is evidently non-singular. For the one-dimensional homology base, we may choose

$$\Gamma_1 : \quad \theta_2 = \theta_3 = \cdots = \theta_n = 0,$$

$$\Gamma_2 : \quad \theta_1 = \theta_3 = \cdots = \theta_n = 0,$$

$$- \; - \; - \; - \; -$$

$$\Gamma_n : \quad \theta_1 = \theta_2 = \cdots = \theta_{n-1} = 0.$$

If we put

$$\int_0^{2\pi} \frac{d\theta_i}{f_i(\theta_i)} = a_i, \quad i=1,\cdots,n,$$

periods of ϖ_i 's are given by the following matrix,

$$\begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ - & - & - & - \\ \omega_{n-1,1} & \omega_{n-1,2} & \cdots & \omega_{n-1,n} \end{pmatrix} = \begin{pmatrix} a_1 & -a_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -a_3 & \cdots & 0 & 0 \\ - & - & - & - & - & \\ 0 & 0 & 0 & \cdots & a_{n-1} & -a_n \end{pmatrix}$$

where

$$\omega_{ik} = \int_{\Gamma_k} \varpi_i , \quad \begin{matrix} i=1,\cdots,n-1, \\ k=1,\cdots,n. \end{matrix}$$

Hence for the assumption of the Theorem 1 not to hold, for any $n-1$ real numbers λ_1, ..., λ_{n-1}, at least one of the ratios between

$$\lambda_1 a_1, \; (\lambda_2 - \lambda_1) a_2, \; \cdots ,$$

$$(\lambda_{n-1} - \lambda_{n-2}) a_{n-1}, \; -\lambda_{n-1} a_n ,$$

must be an irrational number.

Since λ_i 's are not simultaneously zero, and a_i 's cannot be equal to zero because of the fact

$$\frac{1}{f_i(\theta_i)} \neq 0 \quad \text{on } \Pi(c_1 \cdots c_n),$$

we may suppose

$$\lambda_{n-1} a_n \neq 0 .$$

Then, if these ratios are all rational,

$$\frac{\lambda_1}{\lambda_{n-1}} \frac{a_1}{a_n}, \; \frac{\lambda_2 - \lambda_1}{\lambda_{n-1}} \frac{a_2}{a_n}, \; \cdots, \; \frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_{n-1}} \frac{a_{n-1}}{a_n}$$

must be all rational numbers. This means that there exist $n-2$ real numbers x_1, ..., x_{n-2} such that

$$r_1 = x_1 \frac{a_1}{a_n}, \; \cdots, \; r_{n-2} = x_{n-2} \frac{a_{n-2}}{a_n},$$

$$r_{n-1} = (1 - x_1 - x_2 - \cdots - x_{n-2}) \frac{a_{n-1}}{a_n}$$

are all rational. Then

$$r_{n-1} = \left\{ 1 - \left(\frac{r_1}{a_1} + \frac{r_2}{a_2} + \cdots + \frac{r_{n-2}}{a_{n-2}} \right) a_n \right\} \frac{a_{n-1}}{a_n},$$

or,

$$\frac{r_1}{a_1} + \frac{r_2}{a_2} + \cdots + \frac{r_{n-1}}{a_{n-1}} = \frac{1}{a_n} .$$

Consequently n integers m_1, ..., m_n, not simultaneously zero, can be so chosen that

$$\frac{m_1}{a_1} + \cdots + \frac{m_n}{a_n} = 0.$$

Conversely, if the above relation is satisfied for the integers m_1, ..., m_n, not simultaneously zero, we can find $n-1$ real numbers λ_1, ..., λ_{n-1} such that

$$\frac{\lambda_1}{\lambda_{n-1}} \cdot \frac{a_1}{a_n}, \; \frac{\lambda_2 - \lambda_1}{\lambda_{n-1}} \cdot \frac{a_2}{a_n}, \; \cdots$$

$$\frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_{n-1}} \cdot \frac{a_{n-1}}{a_n}$$

are all rational numbers, and the assumption of the Theorem 1 is satisfied.

Hence, by Theorem 3, every trajectory of (7) is everywhere dense in $\Pi(c_1 \cdots c_n)$ if and only if

$$\frac{1}{a_1}, \; \frac{1}{a_2}, \; \cdots, \; \frac{1}{a_n}$$

are linearly incommensurable for the integer coefficients.

As $f_1(\theta_1)$ never becomes zero on $\Pi(c_1 \cdots c_n)$, a hypersurface

$$\theta_1 = 0$$

is a surface of section of S_t. Therefore Theorem 4 is also applicable, and S_t is ergodic if

$$\frac{1}{a_1}, \; \frac{1}{a_2}, \; \cdots, \; \frac{1}{a_n}$$

are linearly incommensurable with respect to integer coefficients.

(*) Received September 27, 1951.

(1) E.Cartan: Leçons sur les invariants intégraux, Paris (1922), Chap. III.

(2) Generally speaking, an invariant Pfaffian form can be expressed as

$$\varpi = \sum_i A_i(x_1 \cdots x_n\, t)\, dx_i + A(x_1 \cdots x_n\, t)\, dt$$

where

$$\sum A_i X_i + A \equiv 0.$$

Consequently, if

$$A \equiv 0,$$

we must have

$$\sum A_i X_i \equiv 0.$$

Cf. E.Cartan, loc.cit. Chap. III.

(3) $\int dm$ is an integral invariant in the sense of Poincare, and is not an integral invariant in Cartan's sense. Cf. E.Cartan, loc.cit. Chap. III.

(4) J.von Neumann: Proof of the quasi-ergodic hypothesis, Proc.Nat.Acad.Sc., Vol. 18, No.1, (1932), pp.70-82. C.D.Birkhoff: Proof of a recurrence theorem for strongly transitive systems, Proc. Nat.Acad.Sc., Vol.17, No.12, (1931), pp.650-655, and Proof of the ergodic theorem, Proc.Nat.Acad.Sc., Vol.17, No.12, (1931), pp. 656-660.

(5) This is an integral invariant in the sense of Poincaré. Cf. E.Cartan, loc.cit. Chap.III.

(6) E.Cartan: loc.cit. Chap.XI. The function ρ is usually called the last multiplier of Jacobi.

(7) Forthcoming in Journ.Math.Soc. Japan.

(8) G.D.Birkhoff: Dynamical Systems, New York (1927), Chap. III and Chap. IV.

(9) T.Ura: On solutions near a formally stable equilibrium point, Jap.Journ.Astr., Vol.1, No.1, (1949), pp.59-67.

(10) T.Ura: loc.cit., § 3.

(11) T.Ura: loc.cit., § 4.

Department of Mathematics,
Tokyo Institute of Technology.