A REMARK ON REGULARLY CONVEX SETS

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Let E be a normed (not necessarily complete) linear space and E^* its conjugate space. Then the regularly convex sets in E^* are defined by M. Krein and V. Šmulian as follows: a set $K^* \subset E^*$ will be called regularly convex if for every $f_0 \in E^*$ not belonging to K^* there exists an element $x_0 \in E$ such that

$$\sup_{f \in K^*} f(x_0) < f_0(x_0).^{1)}$$

This conception was studied in detail in their paper, but as they did not consider it on the standpoint of weak topology, their beautiful results were mainly restricted in separable case. Therefore we shall intend to simplify some of their results in general case. After that we shall give the proof of M. Krein and D. Milman's theorem on the existence of extreme points by means of Zorn's lemma.[2)]

Theorem 1. Every regularly convex set K^* in E^* is convex and weakly closed.

Let g, $h \in K^*$ and α, β satisfy the condition

$$\alpha > 0, \quad \beta > 0, \quad \alpha + \beta = 1.$$

If $\alpha g + \beta h$ were not contained in K^*, there should exist an element $x_0 \in E$ such that

$$\sup_{f \in K^*} f(x_0) < \alpha g(x_0) + \beta h(x_0).$$

On the other hand $g(x_0)$ and $h(x_0) \leqq \sup_{f \in K^*} f(x_0)$, and hence

$$\alpha g(x_0) + \beta h(x_0) \leqq (\alpha+\beta) \sup_{f \in K^*} f(x_0) = \sup_{f \in K^*} f(x_0).$$

This is a contradiction, and it means K^* is convex.

Next we shall show that K^* is weakly closed. Let $f_0 \in E^*$ be a limiting point of K^* in the weak topology and not contained in K^*. Then we can select an element $x_0 \in E$ such that

$$\sup_{f \in K^*} f(x_0) < f_0(x_0),$$

and hence if we choose a positive number $\varepsilon > 0$ satisfying the condition

$$f_0(x_0) - \sup_{f \in K^*} f(x_0) > \varepsilon,$$

Then the weak neighbourhood $U(f_0 ; x_0 ; \varepsilon)$ contains no elements of K^*. Therefore K^* must be weakly closed.

The next theorem which we shall prove in this paper is the inverse of the theorem 1, and as we mentioned above it has already been proved by M. Krein and V. Smulian when E is a separable space.[3)]

For this purpose we shall state the next two lemmas.

Lemma 1. Let $F_0(f)$ be a weakly continuous linear functional on E^*. Then there exist an element $x_0 \in E$ such that

$$F_0(f) = f(x_0)$$

for all $f \in E^*$.

This lemma has been proved by Banach[4)] making use of regularly closed set when E is separable, and that proof is also applied to general case if we notice to the fact that a linear subspace in E^* is regularly closed if and only if it is weakly closed. But we shall give here a direct proof without use of regularly closed set.

From the assumption $\{ f ; f \in E^*, |F_0(f)| < 1 \}$ is an open set in the weak topology containing the zero functional 0, hence there exist a finite number of elements x_1, x_2, ..., x_n of E such that

$$U(0; x_1, x_2, \dots, x_n ; 1) \subset \{ f ; f \in E^*, |F_0(f)| < 1 \},$$

and furthermore without loss of generality we may assume x_1, x_2, ..., x_n are linearly independent.

If we denote by Γ the set of all $f \in E^*$ which satisfy the conditions

$$f(x_i) = 0 \quad (i = 1, 2, \ldots, n),$$

then obviously Γ is a weakly closed linear subspace, and further if we consider the factor space E^*/Γ and denote by $\bar{f}$ the class which contains $f \in E^*$ then to every element $x_i \in E$ corresponds a linear functional $\bar{F}_{x_i}$ continuous on E^*/Γ by the formulae

$$\bar{F}_{x_i}(\bar{f}) = f(x_i) \quad (i = 1, 2, \ldots, n).$$

Since x_1, x_2, ..., x_n are linearly independent, there exists a positive number $\gamma > 0$ such that

$$\mathrm{Max}\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_n|\} \leqq \gamma \| \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \|$$

for any real numbers α_1, α_2, ..., α_n, and hence we can find f_1, f_2, ..., $f_n \in E^*$ satisfying the conditions

$$f_j(x_i) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

for i, j = 1, 2, ..., n. Then $\bar{f}_1$, $\bar{f}_2$, ..., $\bar{f}_n$ constitute a base of E^*/Γ, namely for any $\bar{f} \in E^*/\Gamma$ we have

$$\bar{f} = \sum_{i=1}^{n} \xi_i \bar{f}_i$$

where

$$\xi_i = f(x_i) \quad (i = 1, 2, \ldots, n).$$

and therefore for any linear functional $\bar{F}$ on E^*/Γ

$$\bar{F}(\bar{f}) = \sum_{i=1}^{n} \lambda_i \xi_i = \sum_{i=1}^{n} \lambda_i \bar{F}_{x_i}(\bar{f})$$

where

$$\lambda_i = \bar{F}(\bar{f}_i) \quad (i = 1, 2, \ldots, n).$$

Thus we have proved that every linear functional on E^*/Γ is a linear combination of $\bar{F}_{x_1}$, $\bar{F}_{x_2}$, ..., $\bar{F}_{x_n}$.

Next if we suppose $f(x_i) = 0$ ($i = 1, 2, \ldots, n$), then we have $F_0(f) = 0$. Indeed if we have $F_0(f) \neq 0$, we can choose a positive number $\delta > 0$ such that $|F_0(\delta f)| \geq 1$, then δf does not contained in $U(0; x_1, x_2, \ldots, x_n; 1)$, namely

$$|\delta f(x_i)| \geq 1, \quad |f(x_i)| \geq \frac{1}{\delta} > 0$$

for some x_i ($i = 1, 2, \ldots, n$). This contradicts to the assumption. From this fact if we define

$$\bar{F}_0(\bar{f}) = F_0(f)$$

for any class $\bar{f}$ in E^*/Γ, then we have a linear functional $\bar{F}_0$ on E^*/Γ, and therefore $\bar{F}_0$ is a linear combination of $\bar{F}_{x_1}$, $\bar{F}_{x_2}$, ..., $\bar{F}_{x_n}$, namely we can find λ_1, λ_2, ..., λ_n such that

$$\bar{F}_0 = \sum_{i=1}^{n} \lambda_i \bar{F}_{x_i}.$$

From this relation we obtain

$$F_0(f) = \sum_{i=1}^{n} \lambda_i f(x_i)$$

and hence if we put

$$x_0 = \sum_{i=1}^{n} \lambda_i x_i$$

we have

$$F_0(f) = f(x_0)$$

for all $f \in E^*$.

<u>Lemma</u> 2. If a convex set K^* in E^* is weakly closed, then for every $f_0 \in E^*$ not belonging to K^* there exists a weakly continuous linear functional F_0 on E^* such that

$$\sup_{f \in K^*} F_0(f) < F_0(f_0).$$

We can prove this lemma almost similarly as the analogous theorem[5)] with respect to the strongly closed convex set in E, but for completeness we shall state the proof.

Without loss of generality we can suppose that K^* contains 0. From our assumptions there exists a weak neighbourhood $U(f_0; x_1, x_2, \ldots, x_n; \varepsilon)$ of f_0 which is disjoint with K^*. Then if we denote the union of all weak neighbourhoods $U(f, x_1, x_2, \dots, x_n; \varepsilon/2)$ for $f \in K^*$ by K_1^* we can easily see that K_1^* is also convex and has a boundary point $f_1 \in K_1^*$ such that $f_1 = \beta f_0$, $0 < \beta < 1$. Now for any $f \in E^*$ there exist a positive number $\rho > 0$ such that $1/\rho\, f \in K_1^*$ and hence putting

$$p(f) = \inf \{\rho \,;\, \rho > 0, \tfrac{1}{\rho} f \in K_1^*\},$$

we have a convex functional $p(f)$ on E^*. From this definition it is easy to see that $p(f) \leqq 1$ for every $f \in K_1^*$ and

$$p(f_0) = \frac{1}{\beta} p(f_1) = \frac{1}{\beta} > 1 .$$

Denote by E_1^* the linear subspace generated by f_1, then for this linear subspace E_1^* if we define a linear functional F_1 on E_1^* by

$$F_1(f) = \gamma$$

where $f = \gamma f_1$, we have for every $f \in E_1^*$

$$F_1(f) \leqq p(f).$$

Therefore by Hahn-Banach's extension theorem there exists a linear functional F_0 on E^* such that

$$F_0(f) \leqq p(f)$$

for every $f \in E^*$ and

$$F_0(f) = F_1(f)$$

in E_1^*. For this linear functional F_0 clearly

$$\begin{aligned} \sup_{f \in K^*} F_0(f) &\leqq \sup_{f \in K_1^*} F_0(f) \\ &\leqq \sup_{f \in K_1^*} p(f) = 1 \\ &< p(f_0) = F_0(f_0). \end{aligned}$$

On the other hand since K_1^* contains the weak neighbourhood $U(0\,;\, x_1, x_2, \dots, x_n\,;\, \varepsilon/2)$, we have $p(f) \leqq 1$ for any f in $U(0\,;\, x_1, x_2, \dots, x_n\,;\, \varepsilon/2)$ and therefore $|F_0(f)| \leqq 1$. From this we can easily conclude that $F_0(f)$ is weakly continuous at 0, and hence weakly continuous everywhere since it is linear.

Combining these two lemmas we have immediately

<u>Theorem</u> 2. If a convex set K^* in E^* is weakly closed, then K^* is regularly convex.

Theorem 1 and 2 give an important and simple criterion for regularly convex set, and from the well known result of L. Alaoglu [6] we obtain the next corollary.

<u>Corollary</u> 1. (V. Gantmacher and V. Smulian) A bounded convex set $K^* \subset E^*$ is regularly convex if and only if it is weakly compact.[7]

If we say a set $K^* \subset E^*$ <u>locally weakly compact</u> when the intersection of K^* with any sphere in E^* is always weakly compact, then we have

<u>Corollary</u> 2. In order that a convex set $K^* \subset E^*$ is regularly convex it is necessary and sufficient that it is locally weakly compact.[8]

The necessity of this corollary is almost obvious. Indeed for any sphere S^* in E^* the intersection $K^* \wedge S^*$ is bounded regularly convex set and hence it is weakly compact by the corollary 1.

Inversely let us suppose for any sphere S^* in E^* the intersection $K^* \wedge S^*$ is weakly compact. Then for any bounded regularly convex set K_1^* the intersection of K^* with K_1^* is clearly convex and further noting to the fact that $K^* \wedge K_1^* = K^* \wedge S^* \wedge K_1^*$ for any sphere $S^* \supset K_1^*$ it is weakly compact. Therefore it is regularly convex and this means that K^* is itself regularly convex by the Theorem 5 in M. Krein and V. Smulian's paper.[9]

On the end of this paper we shall prove M. Krein and D. Milman's theorem on the existence of extreme points by use of Zorn's lemma in place of the transfinite induction.

A point of a set K^* in E^* is called an <u>extreme point</u> if it is not an inner point of any segment joining two points in K^*. The importance of this notion has been recognized in the representation theory of group of vector lattice by many authors and hence the next theorem has become to have a great significance.

<u>Theorem</u> 3. (M. Krein and D. Milman) Let $K^* \subset E^*$ be weakly

compact, then the set Ω^* of extreme points of K^* is not empty. In particular if K^* is a convex set and hence bounded regularly convex, then the regularly convex envelope of Ω^* coincides with K^*.

We consider the totality $\mathfrak{E}$ of all pair (A , B^*) where A , B^* are those subsets of E , K^* respectively which satisfy the following two conditions:

1) B^* is not empty and weakly compact, and

2) if $f \in B^*$ are written in the form

$$f = \alpha g + \beta h$$

where g , h belong to K^* and $\alpha + \beta = 1$, $\alpha > 0$, $\beta > 0$ then g , h must belong to B^* and

$$g(x) = h(x)$$

for all $x \in A$.

To any point $x_0 \in E$ corresponds a finite continuous function $\varphi_{x_0}(f) = f(x_0)$ on the compact set K^* . If we define A as a set composed only one point x_0 and B^* as a set of those $f \in K^*$ on which the function $\varphi_{x_0}(f)$ reaches its maximum, then the pair (A , B^*) obviously fulfil the two conditions 1) 2), and hence $\mathfrak{E}$ is non-empty.

Next we introduce a semi-order in $\mathfrak{E}$. For any two pairs (A_1 , B_1^*), (A_2 , B_2^*) we define (A_1 , B_1^*) $\leqq$ (A_2 , B_2^*) if and only if $A_1 \subset A_2$ and $B_1^* \supset B_2^*$. Then it is easily seen that any linearly ordered system (A_λ , B_λ^*) ($\lambda \in \Lambda$) in $\mathfrak{E}$ has an upper bound ($\bigcup_{\lambda \in \Lambda} A_\lambda$, $\bigcap_{\lambda \in \Lambda} B_\lambda^*$), and hence there exists a maximal pair (A_0 , B_0^*) in $\mathfrak{E}$.

We now prove that A_0 coincides with E . Assuming that A_0 does not coincide with E , we take an element $x_0 \in E - A_0$. Then if we denote the set of those $f \in B_0^*$ on which the function $\varphi_{x_0}(f)$ restricted on B_0^* reaches its maximum, we have (A_0, B_0^*) < ($A_0 \cup \{x_0\}$, B_0^*), and this contradicts to the maximality of the pair (A_0 , B_0^*). Then it is easy to verify that every point in B_0^* is an extreme point of K^* .

The latter half of this theorem is proved by the same procedure as the Krein and Milman's original proof.

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(1) M.Krein and V.Šmulian: On regularly convex sets in the space conjugate to a Banach space, Annals of Math., Vol.41 (1940), pp. 556-583.

(2) M.Krein and D.Milman: On the extreme points of regular convex sets, Studia Math., 9 (1940), pp. 133-137.

(3) See 1), p. 571, Theorem 10.

(4) S.Banach: Théorie des opérations linéaires, Warszawa, 1932, p. 131, Théorème 8.

(5) If a convex set $K \subset E$ is strongly closed, then for every $x_0 \in E$ not belonging to K there exists an $f_0 \in E^*$ such that $\sup \{ f_0(x) ; x \in K \} < f_0(x_0)$.

(6) L.Alaoglu: Weak topologies of normed linear spaces, Annals of Math., Vol.41 (1940), pp. 252-267.

(7) V.Šmulian: Sur les topologies différentes d ns l'espace de Banach, Comptes Rendus de l'Acad. des Sc. de l'URSS, 23, 4(1939).

(8) See 1), p. 572, Theorem 11.

(9) See 1), p. 564, Theorem 5. In order that an unbounded set $K \subset E^*$ be regularly convex it it necessary and sufficient that the intersection of K with every bounded regularly convex set be also regularly convex.

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