EXAMPLES OF ERGODIC DYNAMICAL SYSTEMS

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INTRODUCTION. Recently the author has established the following criterion for ergodicity of the measure-preserving flow on the torus[1].

Let Ω be a torus whose points can be represented by the coordinates (x, y), $0 \leq x, y < 2\pi$. Let S_t be a one-parameter stationary flow on Ω defined by

$$(1) \quad \begin{cases} \dfrac{dx}{dt} = X(x, y), \\ \dfrac{dy}{dt} = Y(x, y), \end{cases}$$

where X and Y are one-valued real functions on Ω having continuous first derivatives. If

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0,$$

S_t is measure-preserving, or, in other words, differential equations (1) admit an integral invariant

$$\iint dx\, dy.$$

In this case, S_t is ergodic if and only if

1) X and Y have no common zero points, and

2) $\int_0^{2\pi}\int_0^{2\pi} X\, dx dy \neq 0$, $\int_0^{2\pi}\int_0^{2\pi} Y\, dx dy \neq 0$, and $\int_0^{2\pi}\int_0^{2\pi} X\, dx dy \Big/ \int_0^{2\pi}\int_0^{2\pi} Y\, dx dy$ is an irrational number.

This criterion can be immediately generalized to the case when S_t admits an integral invariant

$$\iint \rho(x, y)\, dx\, dy$$

where $\rho(x, y)$ is a positive definite function on Ω having continuous first derivatives. In such a case, we have

$$\frac{\partial(\rho X)}{\partial x} + \frac{\partial(\rho Y)}{\partial y} = 0.$$

Hence by the similar discussion, we can easily show that S_t is ergodic if and only if

1) X and Y have no common zero points, and

2) $\int_0^{2\pi}\int_0^{2\pi} \rho X\, dx dy \neq 0$, $\int_0^{2\pi}\int_0^{2\pi} \rho Y\, dx dy \neq 0$, and $\int_0^{2\pi}\int_0^{2\pi} \rho X\, dx dy \Big/ \int_0^{2\pi}\int_0^{2\pi} \rho Y\, dx dy$ is an irrational number.

In this paper, we apply this criterion to several dynamical systems of two degrees of freedom and prove the existence of ergodic orbits.

EXAMPLE 1. Combination of two simple harmonic motions.

Consider the particle of unit mass on (x, y) -plane moving under the force $(-\alpha^2 x, -\beta^2 y)$. In this case, the particle generally describes a complicated path known as Lissajous' figure.

If we write

$$p_x = \frac{dx}{dt}, \quad p_y = \frac{dy}{dt},$$

the Hamiltonian of the system is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2 + \alpha^2 x^2 + \beta^2 y^2).$$

As is well known, this system admits two independent integrals

$$p_x^2 + \alpha^2 x^2 = c_1^2,$$
$$p_y^2 + \beta^2 y^2 = c_2^2,$$

where $c_1 > 0$, $c_2 > 0$ are integration constants. The integral surface defined by the above formulae is evidently homeomorphic to the torus. If we put

$$p_x = c_1 \cos\theta, \quad \alpha x = c_1 \sin\theta,$$
$$p_y = c_2 \cos\varphi, \quad \beta y = c_2 \sin\varphi,$$

the position of a point on this surface is represented by (θ, φ) , $0 \leq \theta, \varphi < 2\pi$. Equations of motion are

$$\frac{d\theta}{dt} = \alpha, \quad \frac{d\varphi}{dt} = \beta.$$

Hence the system is ergodic if and

only if α/β is an irrational number.

EXAMPLE 2. Planetary motion in special relativity theory.

In Newtonian dynamics, compact planetary orbit is always an ellipse. Therefore every motion is non-ergodic. But if we treat the problem relativistically, this is not the case. So let us consider the planetary motion in special relativity theory.

Let a particle of rest mass M be fixed at the origin. We consider the motion of another particle of rest mass m moving around the first particle under the Newton's law of universal attraction.

Using polar coordinates on the plane, its Hamiltonian is given by

$$H = c\sqrt{m^2c^2 + p_r^2 + \frac{p_\theta^2}{r^2}} - \frac{kmM}{r},$$

$$p_r = \frac{m\dot{r}}{\sqrt{1 - v^2/c^2}}, \quad p_\theta = \frac{mr^2\dot{\theta}}{\sqrt{1 - v^2/c^2}},$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2,$$

where $k > 0$ is a gravitational constant and c is the velocity of light in vacuo.

Since

$$\frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} = 0,$$

the integral surface is given by

$$(2) \quad \begin{cases} p_\theta = h, \\ H = c\sqrt{m^2c^2 + p_r^2 + \frac{h^2}{r^2}} - \frac{kmM}{r} = E. \end{cases}$$

If we put

$$\frac{1}{r} = u,$$

the above formulae are written in the form

$$(3) \quad \begin{cases} p_\theta = h, \\ c^2p_r^2 + (h^2c^2 - k^2m^2M^2)\left(u - \frac{kmME}{h^2c^2 - k^2m^2M^2}\right)^2 \\ = \frac{h^2c^2E^2}{h^2c^2 - k^2m^2M^2} - m^2c^4. \end{cases}$$

To consider the ergodic orbit, we must restrict ourselves to the case

$$0 < r < +\infty,$$

i.e.

$$0 < u < +\infty.$$

Therefore following inequalities must hold.

$$h^2c^2 - k^2m^2M^2 > 0,$$

$$\frac{h^2c^2E^2}{h^2c^2 - k^2m^2M^2} - m^2c^4 = R^2 > 0,$$

$$\frac{kmME}{h^2c^2 - k^2m^2M^2} - \frac{|R|}{\sqrt{h^2c^2 - k^2m^2M^2}} > 0.$$

From these inequalities; we have

$$(4) \quad \begin{cases} E^2 < m^2c^4, \\ 1 < \frac{h^2c^2}{k^2m^2M^2} < \frac{m^2c^4}{m^2c^4 - E^2}. \end{cases}$$

Under these conditions, the integral surface determined by (3) is evidently a torus in $(p_r, p_\theta, u, \theta)$-space.

Putting

$$cp_r = |R| \cos\varphi,$$

$$\sqrt{h^2c^2 - k^2m^2M^2}\left(u - \frac{kmME}{h^2c^2 - k^2m^2M^2}\right) = |R| \sin\varphi,$$

we have following equations of motion.

$$\frac{d\varphi}{dt} = -\frac{\sqrt{h^2c^2 - k^2m^2M^2}}{\sqrt{m^2c^2 + p_r^2 + h^2u^2}}u^2 = X(\varphi, \theta),$$

$$\frac{d\theta}{dt} = \frac{hc}{\sqrt{m^2c^2 + p_r^2 + h^2u^2}}u^2 = Y(\varphi, \theta).$$

As X and Y have no common zero points and

$$\rho(\theta, \varphi) = \frac{\sqrt{m^2c^2 + p_r^2 + h^2u^2}}{u^2}$$

is a positive definite function satisfying

$$\frac{\partial(\rho X)}{\partial \varphi} + \frac{\partial(\rho Y)}{\partial \theta} = 0,$$

our system is ergodic if and only if $\sqrt{1 - \frac{k^2m^2M^2}{h^2c^2}}$ is an irrational number. Non-ergodic initial conditions are represented by the set of first category in (E, h)-space.

EXAMPLE 3. A particle constrained on a paraboloid in gravitational field.

We consider the particle of unit mass constrained on the paraboloid

$$z = a(x^2+y^2), \quad a>0,$$

moving under the gravitational force given by the potential function gz where g is the gravity constant. If we use the cylindrical coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z,$$

the Lagrangian of the system is

$$L = \frac{1}{2}\left[\dot{r}^2(1+4a^2r^2) + r^2\dot{\theta}^2\right] - agr^2$$

Hence,

$$p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}(1+4a^2r^2), \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta},$$

and the Hamiltonian is

$$H = \frac{1}{2}\left(\frac{p_r^2}{1+4a^2r^2} + \frac{p_\theta^2}{r^2}\right) + agr^2.$$

Since

$$\frac{dp_\theta}{dt} = -\frac{\partial H}{\partial \theta} = 0,$$

the integral surface is given by

$$(5) \qquad p_\theta = h$$

$$(6) \qquad \frac{p_r^2}{1+4a^2r^2} + \frac{h^2}{r^2} + 2agr^2 = E > 0.$$

To avoid the singular case $r = 0$, we assume

$$h^2 \neq 0.$$

We then change the variables from (p_r, r) to (v, w) by the following one-to-one transformation

$$\frac{r p_r}{\sqrt{1+4a^2r^2}} = v, \quad \sqrt{2ag}\left(r^2 - \frac{E}{2ag}\right) = w.$$

Then (6) is transformed into

$$(6') \qquad v^2 + w^2 = \frac{E^2}{2ag} - h^2.$$

Thus, if

$$(7) \qquad \frac{E^2}{2ag} - h^2 > 0,$$

the integral surface determined by (5) and (6') is a torus in (v, w, p_θ, θ) -space. If we put

$$v = \sqrt{\frac{E^2}{2ag} - h^2}\cos\varphi, \quad w = \sqrt{\frac{E^2}{2ag} - h^2}\sin\varphi,$$

equations of motion are

$$\frac{d\varphi}{dt} = \frac{2\sqrt{2ag}}{\sqrt{1+4a^2r^2}} = X(\varphi, \theta),$$

$$\frac{d\theta}{dt} = \frac{h}{r^2} = Y(\varphi, \theta).$$

Since X and Y have no common zero points, and $\rho(\varphi, \theta) = \sqrt{1+4a^2r^2}$ is a positive definite function satisfying

$$\frac{\partial(\rho X)}{\partial \varphi} + \frac{\partial(\rho Y)}{\partial \theta} = 0,$$

the system is ergodic if and only if

$$h\int_0^{2\pi}\int_0^{2\pi} \frac{\sqrt{1+4a^2r^2}}{r^2}\,d\theta\,d\varphi \Big/ 2\sqrt{2ag}\int_0^{2\pi}\int_0^{2\pi} d\theta\,d\varphi$$
$$= \frac{h}{2\pi}\sqrt{\frac{ag}{2}}\int_0^{2\pi} \frac{\sqrt{1+\frac{2aE}{g}+\frac{2a}{g}\sqrt{E^2-2agh^2}\sin\varphi}}{E+\sqrt{E^2-2agh^2}\sin\varphi}\,d\varphi$$

is an irrational number. Non-ergodic initial conditions are represented by the set of first category in the domain determined by (7) in (E, h) -space.

EXAMPLE 4. A particle attracted by two centres of gravitation.

A particle of unit mass is moving on the (x, y) -plane attracted by two fixed particles whose masses are μ and μ' respectively under the Newton's law of universal attraction. Let us call these two centres of gravitation A and B respectively. Let A be situated at $(1, 0)$ and B at $(-1, 0)$. Moreover we make an assumption

$$\mu + \mu' > 1, \quad \mu > \mu'.$$

Then the Lagrangian is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\mu}{\sqrt{(x-1)^2+y^2}} + \frac{\mu'}{\sqrt{(x+1)^2+y^2}}.$$

If we put,

$$z = \cosh w, \quad z = x+iy, \quad w = \xi + i\eta,$$

(x, y) -plane is in one-to-one correspondence with the half-cylinder

$$\xi \geq 0, \quad 2\pi > \eta \geq 0,$$

except the segment $\overline{AB}$. In these new variables,

$$L = \frac{1}{2}(\dot{\xi}^2 + \dot{\eta}^2)(\cosh^2\xi - \cos^2\eta)$$

$$+\frac{(\mu+\mu')\cosh\xi+(\mu-\mu')\cos\eta}{\cosh^2\xi-\cos^2\eta}$$

Hence

$$p_\xi=\frac{\partial L}{\partial\dot\xi}=\dot\xi(\cosh^2\xi-\cos^2\eta),$$

$$p_\eta=\frac{\partial L}{\partial\dot\eta}=\dot\eta(\cosh^2\xi-\cos^2\eta),$$

and the Hamiltonian is

$$H=\frac{1}{\cosh^2\xi-\cos^2\eta}\Big[\frac{1}{2}(p_\xi^2+p_\eta^2)-(\mu+\mu')\cosh\xi -(\mu-\mu')\cos\eta\Big].$$

By a simple calculation, we have[2]

$$(8)\quad \frac{1}{2}p_\xi^2=E\cosh^2\xi+(\mu+\mu')\cosh\xi+\gamma,$$

$$(9)\quad \frac{1}{2}p_\eta^2=-E\cos^2\eta-(\mu'-\mu)\cos\eta-\gamma,$$

where E is the energy constant and γ is another integration constant.

(8) and (9) determine the integral surface of the system. The topological character of this surface changes according to the values of E and γ. We must search for the values of E and γ which make our integral surface homeomorphic to torus.

As we must exclude the case when the particle goes to infinity, E must be negative. So let us put

$$(10)\quad E=-k^2<0.$$

Then (8) is written in the form

$$p_\xi^2+2k^2\Big(\cosh\xi-\frac{\mu+\mu'}{2k^2}\Big)^2=2\gamma+\frac{(\mu+\mu')^2}{2k^2}.$$

If

$$2\gamma+\frac{(\mu+\mu')^2}{2k^2}>0,$$

the above equation defines an ellipse on $(p_\xi, \cosh\xi)$-plane. Since $\cosh\xi\geq 1$, and $\cosh\xi$ is a monotone increasing function of ξ for $\xi\geq 0$, this equation determines on (p_ξ,ξ)-plane a simple closed curve if $\cosh\xi\geq 1$ everywhere on this ellipse. To avoid the singularity of the mapping $(x,y)\to(\xi,\eta)$, we omit the case when ξ becomes zero on this curve. Thus, if

$$2\gamma+\frac{(\mu+\mu')^2}{2k^2}>0,$$

$$\frac{\mu+\mu'}{2k^2}-\frac{\sqrt{2\gamma+\frac{(\mu+\mu')^2}{2k^2}}}{\sqrt{2}\,k}>1,$$

the curve defined by the equation (8) is homeomorphic to a circle on the half-plane (p_ξ,ξ), $\xi>0$. Simplifying these inequalities we obtain

$$(11)\quad -\frac{(\mu+\mu')^2}{4k^2}<\gamma<\frac{1-2(\mu+\mu')}{4k^2}.$$

(These inequalities are mutually consistent as

$$\frac{1-2(\mu+\mu')}{4k^2}-\Big(-\frac{(\mu+\mu')^2}{4k^2}\Big)=\frac{(1-(\mu+\mu'))^2}{4k^2}>0.)$$

Now let us consider the curve (9) on the cylindrical surface (p_η,η). If the right hand member is always positive for $0\leq\eta<2\pi$, (9) gives two separate closed curves on the cylinder. Thus (8), (9), (10), (11) determine two separate tori. On the other hand, if the right hand member is not positive definite, (8), (9), (10), (11) determine the surface of more complicated topological character. So we restrict ourselves to the simple case

$$k^2\cos^2\eta-(\mu'-\mu)\cos\eta-\gamma>0.$$

As we have assumed $\mu>\mu'$, γ must satisfy the following inequality.

$$\gamma<k^2-(\mu-\mu')$$

This is consistent with (11), because the expression

$$k^2-(\mu-\mu')-\Big(-\frac{(\mu+\mu')^2}{4k^2}\Big)$$

is always positive. Thus the restriction on γ is given by

$$(12)\quad\begin{cases}-\dfrac{(\mu+\mu')^2}{4k^2}<\gamma\\ \gamma<\mathrm{Min.}\Big[\dfrac{1-2(\mu+\mu')}{4k^2},\ k^2-(\mu-\mu')\Big].\end{cases}$$

If we put

$$p_\xi=R\cos\theta,\quad \sqrt{2}\,k\Big(\cosh\xi-\frac{\mu+\mu'}{2k^2}\Big)=R\sin\theta,$$

$$R=\sqrt{2\gamma+\frac{(\mu+\mu')^2}{2k^2}},$$

equations of motion on the torus corresponding to the positive value of p_η are

$$\frac{d\theta}{dt} = \frac{\sqrt{2}k \sinh \xi}{\cosh^2\xi - \cos^2\eta} = X(\theta, \eta),$$

$$\frac{d\eta}{dt} = \frac{\sqrt{2k^2\cos^2\eta - 2(\mu' - \mu)\cos\eta - 2\gamma}}{\cosh^2\xi - \cos^2\eta}$$

$$= Y(\theta, \eta).$$

X and Y have no common zero points under the condition (12),

and $\rho(\theta,\eta) = \dfrac{\cosh^2\xi - \cos^2\eta}{\sinh\xi\sqrt{2k^2\cos^2\eta - 2(\mu'-\mu)\cos\eta - \gamma}}$

is a positive definite function satisfying

$$\frac{\partial(\rho X)}{\partial\theta} + \frac{\partial(\rho Y)}{\partial\eta} = 0,$$

Hence the system is ergodic if and only if

$$\frac{k\displaystyle\int_0^{2\pi} \frac{d\eta}{\sqrt{k^2\cos^2\eta - (\mu'-\mu)\cos\eta - \gamma}}}{\displaystyle\int_0^{2\pi} \frac{d\theta}{\sqrt{\left(\frac{K}{\sqrt{2}k}\sin\theta + \frac{\mu+\mu'}{2k^2}\right)^2 - 1}}}$$

is an irrational number. Therefore, in the domain (10) and (12) in (E, γ) -space, non-ergodic initial conditions form the set of first category.

( ) Received March 8, 1951.

(1) Forthcoming in Journ. Math. Soc. Japan. A rough statement is given in Japanese in Sugaku, Vol.1, No.4, 1949.

(2) Cf. E.T.Whittaker, A treatise on the analytical dynamics of particles and rigid bodies, 4th edition, (1937), pp.97-99.

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