ON A HOMOTOPY CLASSIFICATION PROBLEM

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As is shown by S.Eilenberg in his paper "On the problems of topology", Annals of Math., 1949, it may well be said that the homotopy classification problem is a central problem in modern topology. It is to enumerate effectively all the homotopy classes of mappings f of one space X into the other space Y by some computable invariants of f, X and Y. This general problem has not yet been solved, but in several special cases various kinds of brilliant results are known.

Some of them have much to do with the problem discussed in this paper and therefore are shown as follows. Witold Hurewicz reported in 1936 that all the homotopy classes of mappings of an n-dimensional connected finite polyhedron X into an arcwise connected topological space Y with $\pi_i(Y)=0$ for $i>1$ are in one-to-one correspondence with all the classes of equivalent-homomorphisms of $\pi_1(X)$ to $\pi_1(Y)$. The homotopy classification problem in case where X is an n-dimensional finite connected complex X and Y an n-sphere, was originally solved by H.Hopf and reproduced by H.Whitney with use of cohomology group. S.Eilenberg generalized the Hopf-Whitney's Theorem to get far reaching results that when X is also an n-dimensional geometrical cell complex and Y an arcwise connected topological space with $\pi_i(Y)=0$ for $n>i\geqq 1$, all the homotopy classes are in (1 - 1) correspondence with an n-dimensional cohomology group $H_n(X,\pi_n(Y))$ with the coefficient group $\pi_n(Y)$.

It is reported that the homotopy classification problem of an n-dimensional finite connected complex with a fixed decomposition into an arcwise connected topological space with $\pi_i(Y)=0$ for $n>i>1$, has recently been solved completely by several topologists, P.Olum (Bull. Amer. Math. Soc., 53 (1947)), M.M.Postnikov (Doklady Akad. Nauk SSSR, 66 (1949)), and S.T.Hu (Akad. Sinica Science Record 2, (1949)). But details are not yet to hand in Japan. I also intend to impart a solution to this problem by the aid of Eilenberg Mac-Lane's cohomology group of abstract group and of Steenrod's cohomology theory with local coefficients. I am deeply grateful to Mr. Nobuo Shimada for his helpful criticisms and suggestions.

1. Reproduction of Hurewicz's result and its application to this problem.

Let Ω be the totality of all the mappings of X into Y and of an point x_0 of X into a fixed point y_0 of Y. Ω is usually designated by the symbol $Y^X(x_0,y_0)$. For two maps $f,g\in\Omega$, f is said to be homotopic to g (in notation; $f\sim g$) if there exists a homotopy $h_t\in Y^X$ (for $1\geqq t\geqq 0$) such that $h_0=f$ and $h_1=g$. Ω is divided by this equivalent relation into mutually disjoint homotopy classes of mappings.

A mapping $f\in\Omega$ induces a homomorphism h_f of $\pi_1(X,x_0)$ into $\pi_1(Y,y_0)$. If $f\sim g$ for two mappings $f,g\in\Omega$, there exists a homotopy h_t. Let $\eta\in\pi_1(Y,y_0)$ be represented by the closed path $h_t(x_0)$ (for $1\geqq t\geqq 0$), then we have $h_f(\xi)=\eta\, h_g(\xi)\,\eta^{-1}$ for any $\xi\in\pi_1(X,x_0)$. Thus it is proved that if $f\sim g$, h_f and h_g are equivalent (in notation: $h_f\sim h_g$).

Two mappings $f,g\in\Omega$ are said to be r-homotopic (in notation: $f\underset{r}{\sim}g$) if $f|X^r\sim g|X^r$, where X^r is the r-skeleton of X. Then n-homotopy is the usual homotopy defined above. Since r-homotopic relation is an equivalent relation, Ω is also divided into r-homotopy classes. Then we have

<u>Theorem 1.</u> The set of all the $(n-1)$-homotopy classes is in one to one correspondence with the set

of all the classes of equivalent homomorphisms of $\pi_1(X)$ into $\pi_1(Y)$.

In order to prove this it is necessary and sufficient to show that for a given homomorphism φ there exists a mapping $f \in \Omega$ such that $h_f = \varphi$ and that if $f \overset{n-1}{\simeq} g$ for two mappings $f, g \in \Omega$, we have $h_f \sim h_g$, and vice versa.

The former statements are obvious. Before we prove the inverse relation, some preliminary remarks are given. Let B be the topological tree in the complex X^1, which starts x_0 and involves all the vertices of X. Since B is contractible to a point x_0 in itself, for any mapping $f \in \Omega$ there exists a mapping f' such that $f \simeq f'$ and $f'(B) = y_0$. It is easily verified that $h_f = h_{f'}$. Therefore, in order to prove that if $h_f \sim h_g$, we have $f \overset{n-1}{\simeq} g$, it is sufficient to show that if $h_{f'} \sim h_{g'}$, we have $f' \overset{n-1}{\simeq} g'$, where f', g' are such mappings as referred to above.

<u>Proof of Theorem 1.</u> Put $X \times I = Z$, $Z' = (X \times 0) \cup (X \times 1)$, and $Z^{r+1} = (X^r \times I) \cup Z'$. Let σ_{ij}^1 be a 1-simplex $\overrightarrow{x_i x_j}$ and let u_i be a path joining in B a vertex x_0 to a vertex x_i of X. Putting $v_{ij} = u_i \sigma_{ij}^1 u_j^{-1}$, the element $[v_{ij}]$ of $\pi_1(X, x_0)$ which is represented by the closed path v_{ij} is a generator of $\pi_1(X, x_0)$. As $h_{f'} \sim h_{g'}$ we have $h_{f'}([v_{ij}]) = \mathfrak{z}\, h_{g'}([v_{ij}])\, \mathfrak{z}^{-1}$ for any generator $[v_{ij}]$ of $\pi_1(X, x_0)$, where $\mathfrak{z} \in \pi_1(Y, y_0)$. In order to prove Theorem 1, it is sufficient to show that there exists a mapping $F: Z^n \to Y$ such that $F(x,0) = f'(x)$ and $F(x,1) = g'(x)$ for any $x \in X$. Now, let $\tau(t)$ (for $1 \geq t \geq 0$) be a representative of $\mathfrak{z}$ and we define a mapping $F: Z' \to Y$ as follows:

$$\begin{cases} F(x,0) = f'(x), & \text{for } x \in X, \\ F(x,1) = g'(x), & \text{for } x \in X, \\ F(x_i, t) = \tau(t), & \text{for } x_i \in X^0 \text{ and } 1 \geq t \geq 0. \end{cases}$$

Then, for any σ_{ij}^1 of X^1, $F|\partial(\sigma_{ij}^1 \times I)$ is homotopic to zero, because

$$\begin{aligned} [f'(\sigma_{ij}^1)] &= [f'(v_{ij})] = h_{f'}([v_{ij}]) \\ &= \mathfrak{z}\, h_{g'}([v_{ij}])\, \mathfrak{z}^{-1} = \mathfrak{z}\, [g'(v_{ij})]\, \mathfrak{z}^{-1} \\ &= \mathfrak{z}\, [g'(\sigma_{ij}^1)]\, \mathfrak{z}^{-1}. \end{aligned}$$

Therefore F can be extended to a mapping $F: Z^2 \to Y$. For any 2-simplex σ_i^2 of X, $F|\partial(\sigma_i^2 \times I)$ is also inessential in virtue of $\pi_2(Y) = 0$, so that F is again extended to a mapping $F: Z^3 \to Y$. Through the same arguments we have an extended mapping $F: Z^n \to Y$, using the assumptions that $\pi_3(Y) = \cdots = \pi_{n-1}(Y) = 0$. It follows that we have $f' \overset{n-1}{\simeq} g'$. Thus Theorem 1 has been established.

2. A generalized obstruction theory with use of Steenrod's cohomology group with local coefficients.

It is the main aim of the rest part in this paper to find a necessary and sufficient condition that two maps in an $(n-1)$-homotopy class are n-homotopic each other. To do this Eilenberg's obstruction theory should be slightly retouched to apply to our case, because the space Y is not assumed to be n-simple. In this section § 2 this point is clarified with use of Steenrod's cohomology group with local coefficients and, moreover, an n-cohomology class $C_{\mathfrak{z}}(f)$ (refer to 2.1) which plays an eminent rôle in this problem, is discussed in § 3 in connection with Eilenberg-Mac Lane's cohomology group of abstract group. In the last section § 4 a general theory established will be reduced to the results obtained by Hopf and Eilenberg as special cases.

2.1. Definition of an n-cohomology class $C_{\mathfrak{z}}(f)$ and a formula concerning .

All the mappings considered in the rest part of this paper are assumed to belong to an $(n-1)$-homotopy class U^λ, so that without loss of generality they may be assumed to coincide on X^{n-1}. It may be also assumed that they map the topological tree B into y_0 because B is contractible in B to a point x_0. Now, let f be such a mapping and h_f an induced homomorphism of $\pi_1(X)$ into $\pi_1(Y)$. We denote by $\mathfrak{Z}_f^\lambda$ the centralizer of the subgroup $h_f(\pi_1(X))$ of $\pi_1(Y)$; in notation $\mathfrak{Z}_f^\lambda = \{\mathfrak{z}\,;\ \mathfrak{z}a = a\mathfrak{z}, \text{ for every } a \in h_f(\pi_1(X))\}$. Then for any two mappings f, g in an $(n-1)$-homotopy class U^λ we have $\mathfrak{Z}_f^\lambda = \mathfrak{Z}_g^\lambda$ from the assumptions referred to above, and therefore $\mathfrak{Z}_f^\lambda$ can be merely designated by $\mathfrak{Z}^\lambda$. Let $\tau(t)$ (for $1 \geq t \geq 0$) be a representative of an element $\mathfrak{z} \in \mathfrak{Z}^\lambda$. Now a mapping $F: Z' \to Y$ is defined as follows:

$$F(x,0) = f(x), \quad x \in X,$$
$$F(x,1) = f(x), \quad x \in X,$$

$$F(x_i, t) = \tau(t), \quad x_i \in X^0 \text{ and } 1 \geq t \geq 0$$

Since $[f(\sigma_i^1)] = [f(v_{ij})] \in h_f(\pi_1(X))$ and so $\mathfrak{z}$ commutes $[f(\sigma_i^1)]$, F_1 can be extended to a map F: $Z^1 \to Y$. Moreover, from the assumptions that $\pi_2(Y) = \cdots = \pi_{n-1}(Y) = 0$, F can be extended to a map $F: Z^n \to Y$. Now, all the vertices of X are ordered linearly so that for any simplex of X the first vertex in this order is preassigned. Let a_i be the first vertex of an n-simplex σ_i^n and b_j the first vertex of its $(n-1)$-face σ_j^{n-1}. $F \mid \partial(\sigma_i^n \times I)$ represents an element $C(F, \sigma_i^n)$ of $\pi_n(Y, f(a_i) = y_0)$, and $C(F) = \sum C(F, \sigma_i^n) \cdot \sigma_i^n$ may be regarded as an n-cocyle of n-complex X with local group as coefficient group. An n-cocycle $C(F)$ may indeed depend on i) the choice of a representative of $\mathfrak{z}$ and also on ii) the way of extending the mapping F, but it can be shown that, independently of i), ii), $C(F)$ determines uniquely a cohomology class of $H_n(X, \pi_n(Y, y_0))$ (n-th cohomology group of X with local coefficients) for a mapping f and for $\mathfrak{z} \in \mathfrak{Z}^\lambda$, which we designated by $C_\mathfrak{z}(f)$. As to i), ii) it is sufficient to prove that when for the mapping f and for a representative $\tau'(t)$ (for $1 \geq t \geq 0$) of $\mathfrak{z} \in \mathfrak{Z}^\lambda$, another mapping $F'; Z^n \to Y$ is constructed in the same way as used in case of F, $C(F)$ is cohomologous to $C(F')$. From the homotopy extension property of a polyhedron there exists a mapping $F'': Z^n \to Y$ such that $F' \simeq F''$ and $F'' \mid Z' = F \mid Z'$. Then we have $C(F') = C(F'')$. Moreover, both the same property of a polyhedron and the assumptions $\pi_2(Y) = \cdots = \pi_{n-1}(Y) = 0$, assure the existence of a mapping $F''': Z^n \to Y$ such that $F'' \simeq_n F'''$ and $F''' \mid Z^{n-1} = F \mid Z^{n-1}$. It is clear that we have $C(F''') = C(F'') = C(F')$. Then we shall show that $C(F)$ is cohomologous to $C(F''')$. As from $F \mid Z^{n-1} = F''' \mid Z^{n-1}$ we have $F \mid \partial(\sigma_j^{n-1} \times I) = F''' \mid \partial(\sigma_j^{n-1} \times I)$ for any $\sigma_j^{n-1} \in X^{n-1}$, following Eilenberg (Annals of Math., 41, 1940), $d(F, F''', \sigma_j^{n-1}) \in \pi_n(Y, y_0 = f(b_j))$ can be defined and also we have an $(n-1)$-cochain $d^{n-1}(F, F''') = \sum d(F, F''', \sigma_j^{n-1}) \sigma_j^{n-1}$. Now, with Steenrod's cohomology theory with local coefficients (Annals of Math. 1942), we have

$$\delta d^{n-1}(F, F''')(\sigma_i^n)$$

$$= \sum_{\substack{\sigma_j^{n-1} \\ \sigma_i^n > \sigma_j^{n-1}}} [\sigma_j^{n-1} : \sigma_i^n] \, h_{\sigma_i^n \sigma_j^{n-1}} d(F, F''', \sigma_j^{n-1})$$

$$= C(F, \sigma_i^n) - C(F''', \sigma_i^n)$$

so that $\delta d^{n-1}(F, F''') = C(F) - C(F''')$. This proves that $C(F)$ is cohomologous to $C(F')$.

It should be noticed that for any element C of the cohomology class $C_\mathfrak{z}(f) = [C(F)]$ we can construct a mapping $F': Z^n \to Y$ such that $C(F') = C$; namely all the elements of $C_\mathfrak{z}(f)$ can be obtained through the procedure referred to above from f and $\mathfrak{z}$. Since $C(F)$ is cohomologous to C; there exists an $(n-1)$-cochain $d^{n-1}(\sigma_i^{n-1}) = \alpha_i$, $\alpha_i \in \pi_n(Y, y_0)$, such that $\delta d^{n-1} = C(F) - C$. As is easily seen, there exists a mapping $F': Z^n \to Y$ such that $F' \mid Z^{n-1} = F \mid Z^{n-1}$ and $d^{n-1} = d^{n-1}(F, F')$. As $\delta d^{n-1}(F, F') = C(F) - C(F') = C(F) - C$, we have $C = C(F')$. This remark is employed essentially in discussions appeared later.

Now we show a formula concerning $C_\mathfrak{z}(f)$.

$$(2.1.1) \quad C_\mathfrak{z}(f) - C_\eta(f) = C_{\eta^{-1}\mathfrak{z}}(f)^{\eta^{-1}}$$

where $\mathfrak{z}, \eta \in \mathfrak{Z}^\lambda$ and $C_{\eta^{-1}\mathfrak{z}}(f)^{\eta^{-1}}$ is represented by a cocycle $\sum_i \alpha_i^{\eta^{-1}} \sigma_i^n$, putting $C_{\eta^{-1}\mathfrak{z}}(f) = [\sum_i \alpha_i \sigma_i^n]$.

<u>Proof.</u> Let $C(F) = \sum_i \beta_i \sigma_i^n$, $C(G) = \sum_i \gamma_i \sigma_i^n$ be representatives of $C_\mathfrak{z}(f)$, $C_\eta(f)$ respectively, where $\beta_i = C(F, \sigma_i^n)$ and $\gamma_i = C(G, \sigma_i^n)$. A mapping $\Phi: Z^n \to Y$ is defined such that

$$\Phi(x, t) \equiv \begin{cases} G(x, 1-2t), & \frac{1}{2} \geq t \geq 0, \\ F(x, 2t-1), & 1 \geq t \geq \frac{1}{2}. \end{cases}$$

Since we have $\Phi(x, 0) = \Phi(x, 1) = f(x)$, and $\Phi(x_i, t)$ (for $1 \geq t \geq 0$) represents an element $\eta^{-1}\mathfrak{z} \in \mathfrak{Z}^\lambda$ for any vertex x_i, $C(\Phi) = \sum \alpha_i \sigma_i^n$ represents $C_{\eta^{-1}\mathfrak{z}}(f)$. Then it is easily seen in consideration of reference points that we have $\beta_i - \gamma_i = \alpha_i^{\eta^{-1}}$. This proves that $C_\mathfrak{z}(f) - C_\eta(f) = C_{\eta^{-1}\mathfrak{z}}(f)^{\eta^{-1}}$.

2.2 Definition of $\mathfrak{F}(f, \mathfrak{z})$ and some theorems concerning $\mathfrak{F}(f, \mathfrak{z})$.

We intend to introduce a concept " $\mathfrak{z}$-homotopy". If for two mappings

f, g belonging to $\mathfrak{U}^\lambda$, f is n-homotopic to g, where $f(B) = g(B) = y_0$ is assumed, there exists a homotopy h_t (for $1 \geq t \geq 0$) such that $h_0 = f$ and $h_1 = g$. It is easily verified that for any vertex x_i of X, $h_t(x_i)$ (for $1 \geq t \geq 0$) represents an element $\mathfrak{z}$ of $\mathfrak{Z}^\lambda$. Then g is said to be "$\mathfrak{z}$-homotopic" to f (in notation: $f \underset{n}{\overset{\mathfrak{z}}{\sim}} g$, or simply $f \overset{\mathfrak{z}}{\sim} g$).

Lemma 2.2.1 For a mapping $f \in \mathfrak{U}^\lambda$ which maps B into y_0, and for an element $\mathfrak{z} \in \mathfrak{Z}^\lambda$ there exists a mapping g such that $f | X^{n-1} = g | X^{n-1}$ and $f \overset{\mathfrak{z}}{\sim} g$.

Proof. Let $\tau(t)$ (for $1 \geq t \geq 0$) be a representative of $\mathfrak{z}$. A mapping $F : (X \times 0) \cup (X^{n-1} \times 1) \cup (X^0 \times I) \to Y$ can be defined as follows:

$$\begin{cases} F(x, 0) = f(x), & x \in X, \\ F(x, 1) = f(x), & x \in X^{n-1}, \\ F(x_i, t) = \tau(t), & x_i \in X^0, t \in I. \end{cases}$$

As $[f(\sigma_{ij}^1)] \simeq [f(v_{ij})]$ commutes $\mathfrak{z}$, $F | \partial(\sigma_{ij}^1 \times I)$ is inessential. Therefore F can be extended to a mapping $F : (X \times 0) \cup (X^{n-1} \times 1) \cup (X^1 \times I) \longrightarrow Y$. By the aid of the assumptions that $\pi_i(Y) = 0$ for $1 < i < n$, we have a mapping $F : (X \times 0) \cup (X^{n-1} \times I) \longrightarrow Y$. Then from the homotopy extension property of a polyhedron a desired mapping $F : Z \longrightarrow Y$ is obtained, for $F | (X \times 1) = g$ is $\mathfrak{z}$-homotopic to f and $g | X^{n-1} = f | X^{n-1}$.

For two mappings $f, g \in \mathfrak{U}^\lambda$ which coincide on X^{n-1} and map B into y_0, we construct an n-cocycle $d^n(f, g)(\sigma_i^n) = d(f, g, \sigma_i^n)$, where $d(f, g, \sigma_i^n) \in \pi_n(Y, y_0)$, following Eilenberg. We designate by $\vartheta(f, g)$ a cohomology class of $H_n(X, \pi_n(Y))$ to which $d^n(f, g)$ belongs. Then we have

Existence Theorem 2.2.2. For any element ϑ of $H_n(X, \pi_n(Y))$ and for a mapping $f \in \mathfrak{U}^\lambda$, there exists a mapping $g \in \mathfrak{U}^\lambda$ such that $\vartheta(f, g) = \vartheta$.

Proof. Let ϑ be represented by a cocycle $\sum_i \alpha_i \sigma_i^n$, where $\alpha_i \in \pi_n(Y, y_0)$ then it is proved with Eilenberg that there exists a mapping g such that $d(f, g, \sigma_i^n) = \alpha_i$ for any σ_i^n of X.

Homotopy Theorem 2.2.3. For two mappings f, f' we have $f \underset{n}{\overset{\mathfrak{z}}{\sim}} f'$ if and only if $\vartheta(f, f') = \mathfrak{C}_{\mathfrak{z}^{-1}}(f)$.

This theorem corresponds to the Eilenberg's Homotopy Theorem. Since in his case $\pi_1(Y) = 0$ is also assumed, we have $\mathfrak{C}_{\mathfrak{z}^{-1}}(f) = 0$ so that $\vartheta(f, f') = 0$. Therefore two mappings f, f' are homotopic each other if and only if $d^n(f, f') \backsim 0$. This theorem will be again discussed in § 4 in a slightly generalized form.

Proof. Since $f \underset{n}{\overset{\mathfrak{z}}{\sim}} f'$, there exists a homotopy h_t $(1 \geq t \geq 0)$ such that $h_0 = f$ and $h_1 = f'$. Then $h_t(x_i)$ (for $1 \geq t \geq 0$) for any $x_i \in X^0$ represents an element $\mathfrak{z} \in \mathfrak{Z}^\lambda$. Let $\tau(t)$ be a representative of $\mathfrak{z}$, then we define a mapping $\mathfrak{F}$ as follows:

$$\begin{cases} \mathfrak{F}(x, 0, 0) = f(x), \ x \in X; \ \mathfrak{F}(x, 1, 0) = f'(x), \\ \qquad x \in X; \ \mathfrak{F}(x, s, 0) = f(x), \ x \in X^{n-1}, s \in I, \\ \mathfrak{F}(x, 1, t) = h_{1-t}(x), \ x \in X, \ t \in I, \\ \mathfrak{F}(x, 0, t) = f(x), \ x \in X, \ t \in I, \\ \mathfrak{F}(x_i, s, 1) = \tau(1-s), \ x_i \in X^0, \ s \in I. \end{cases}$$

Then it is easily seen that $\mathfrak{F} | \partial(x_i \times I \times I)$ is homotopic to zero, so that $\mathfrak{F}$ can be also defined on $x_i \times I \times I$ for any $x_i \in X^0$. As $\mathfrak{F} | Z^n$ has a partial homotopy on the subcomplex $(X \times 0) \cup (X \times 1) \cup (X^0 \times I) = Z'$ of Z^n, in virtue of the homotopy extension property of a polyhedron we have $\mathfrak{F} : Z^n \times I \to Y$. Since $\mathfrak{F} | \partial(\sigma_i^n \times I \times 0)$ represents $d(f, f', \sigma_i^n)$ and $\mathfrak{F} | \partial(\sigma_i^n \times I \times 1)$ represents $C_{\mathfrak{z}^{-1}}(F, \sigma_i^n)$, in consideration of the homotopy we have $d(f, f', \sigma_i^n) = C_{\mathfrak{z}^{-1}}(F, \sigma_i^n)$. This proves that $\vartheta(f, f') = \mathfrak{C}_{\mathfrak{z}^{-1}}(f)$.

Conversely, a mapping $D : Z^n \longrightarrow Y$ is defined as follows:

$$D(x, 0) = f(x), \quad x \in X,$$

$$D(x, 1) = f'(x), \quad x \in X,$$

$$D(x, t) = f(x), \quad x \in X^{n-1}, \ t \in I.$$

Then $d(f, f', \sigma_i^n)$ is represented by a mapping $D | \partial(\sigma_i^n \times I)$. If we choose suitably a representative $C(F) = \sum c(F, \sigma_i^n) \sigma_i^n$ of $\mathfrak{C}_{\mathfrak{z}^{-1}}(f)$, by the remark given in the last part of 2.1 we have $d(f,f') = c(F)$. Now we define a mapping $\Phi: Z^n \to Y$ such that

$$\Phi(x,t) = \begin{cases} F(x, 1-2t), & \frac{1}{2} \geq t \geq 0, \\ D(x, 2t-1), & 1 \geq t \geq \frac{1}{2}. \end{cases}$$

Then $\Phi(x_i, t)$ (for $1 \geq t \geq 0$), for any $x_i \in X^0$, represents ξ, and we have $\Phi(x,0) = f(x)$, $\Phi(x,1) = f'(x)$. Now, $\Phi|\partial(\sigma_i^n \times I)$ represents $(d(f,f',\sigma_i^n) - c(F,\sigma_i^n))^{\xi^{-1}}$, regarding $\Phi(a_i \times 0) = y_0$ as a base point. As $d(f,f',\sigma_i^n) = C(F,\sigma_i^n)$, it follows that $\Phi|\partial(\sigma_i^n \times I)$ is inessential for any σ_i^n. Therefore Φ is extended to a mapping $Z^{n+1} \to Y$, so that we have $f \underset{n}{\overset{\xi}{\sim}} f'$. The proof has been established.

We can mention in more generalized forms another formulaes corresponding to those shown by Eilenberg, but only several formulaes, which will be used in § 4, are given here without proof.

(2.2.4) $\quad \mathcal{D}(f,h) - \mathcal{D}(f,g) = \mathcal{D}(g,h)$

(2.2.5) $\quad \mathcal{D}(f,g) - \mathcal{D}(f,g)^{\xi} = C_\xi(f) - C_\xi(g)$

(2.2.6) $\quad$ If $f \underset{n}{\overset{\xi}{\sim}} f'$, we have

$$C_\eta(f')^{\xi} = C_{\xi\eta\xi^{-1}}(f) \quad \text{for any } \eta \in \mathfrak{Z}^\lambda.$$

3. Computation of the cocycles $C_\xi(f)$.

In this section we give some meaning to the cocycles $C_\xi(f)$. There was found an invariant cohomology class $\mathfrak{k}^{n+1}$ in the cohomology group of $H_{n+1}(\pi_1(Y), \pi_n(Y))$ by Eilenberg. Here is shown that the class is reducible from $\mathfrak{k}^{n+1}$.

3.1. Let Π be a discrete group, $K(\pi)$ an abstract closure finite complex defined as follows. An ordered $(n+1)$ -ple $[w_0, w_1, \cdots, w_n]$ of elements of Π is an n-cell of the complex $K(\Pi)$. The boundary of an n-cell is an $(n-1)$-chain defined by

$$\partial[w_0, w_1, \cdots, w_n] = \sum_0^n (-1)^i [w_0, \cdots, \hat{w}_i, \cdots, w_n].$$

By putting $w \cdot [w_0, w_1, \cdots, w_n] = [ww_0, ww_1, \cdots, ww_n]$, Π is considered as a group of automorphisms of $K(\Pi)$ without fixed cells. Let $C^n(\Pi)$ be the n-th chain group of $K(\Pi)$ with integer coefficients. Let J be an abelian group which admits Π as a group of operators. An equivariant n-cochain f^n is a homomorphism of $C^n(\Pi)$ into J such that

$$f^n(w \cdot [w_0, w_1, \cdots, w_n]) = w \cdot f([w_0, \cdots, w_n]).$$

The coboundary of f^n is defined by

$$\delta f^n([w_0, \cdots, w_{n+1}]) = f^n(\partial[w_0, \cdots, w_{n+1}])$$

By usual procedure, we can define the n-th equivariant cohomology group $H_n(\Pi, J)$.

3.2. From now on we regard $\pi_1(Y, y_0)$ as Π and $\pi_n(Y, y_0)$ as J. Let $S_1(X)$ be a closure finite complex defined by singular simplexes in Y such that all the vertices of the counter-image simplex are mapped into a fixed point y_0 in Y. Let $K^n(\Pi)$ be the n-skeleton of $K(\Pi)$. We consider mappings F of $K^n(\Pi)$ into $S_1(Y)$ as follows. All 0-cells $[w]$ are mapped into the point y_0. A 1-cell $[w_0 w_1]$ is mapped into a closed path representing the element $w_0^{-1} w_1$ of $\pi_1(Y, y_0)$. For a 2-cell $[w_0, w_1, w_2]$, $F[w_0, w_1, w_2]$ is a singular simplex defined as follows. Define a mapping T of a Euclidean 2-simplex $\sigma^2 = \langle P_0, P_1, P_2 \rangle$ into Y first on its boundary, such that

$$T(P_i) = y_0, \quad T(P_0 P_1) = F[w_0, w_1],$$

$$T(P_0 P_2) = F[w_0, w_1], \quad T(P_1 P_2) = F[w_1, w_2].$$

As easily seen, the mapping T can be extended to the interior of σ^2.

If we notice the assumption that $\pi_i(Y) = 0$ for $1 < i < n$, we can always extend the mapping given by F on the boundary of a Euclidean $(i+1)$ -simplex into its interior such that $F[ww_0, ww_1, \cdots, ww_{i+1}] \equiv F[w_0, \cdots, w_{i+1}]$ for any $v \in \Pi$, and $F(\pi[w_0, \cdots, w_{i+1}]) \equiv \varepsilon_\pi F[w_0, \cdots, w_{i+1}]$ where π denotes a permutation of $w_0, \cdots, w_{i+1}$ and ε_π equals ± 1 according as π is even or odd permutation. Thus the mapping F of $K^n(\Pi)$ into $S_1(Y)$ is defined.

3.3. We consider the set M of all such mappings F of $K^n(\Pi)$ into $S_1(Y)$. For each F and an $(n+1)$ -cell $[w_0, \cdots, w_{n+1}]$, let

$$F(\partial[w_0, \cdots, w_{n+1}]) = T(\partial\sigma^{n+1})$$

represent an element $\alpha \in \pi_n(Y, y_0)$. To every $(n+1)$ -cell $[w_0, \cdots, w_{n+1}]$ we attach the element α^{w_0}, then we obtain an equivariant $(n+1)$ - cochain k_F^{n+1}. It is easily seen that k_F^{n+1} is a cocycle and that k_F^{n+1} is cohomologous to k_G^{n+1} for any two mappings $F, G \in M$. Thus we get the invariant cohomology class $k^{n+1} \in H_{n+1}(\pi_1(Y), \pi_n(Y))$.

3.4. Suppose all the vertices of X^n are linearly ordered. A mapping f of X^n into Y, which maps X^0 into y_0, defines a singular simplex in Y on each simplex of X^n. Thus (X^n, f) is considered as a subcomplex of $S_1(Y)$.

Let R be the group ring of $\pi = \pi_1(Y)$ with integer coefficients. We construct a chain-transformation κ of the chain group $C(X^n, R)$ of X^n with coefficient group R into the chain group $C(\pi)$ of $K(\pi)$ as follows:

Let $\sigma^m = \langle p_0, \cdots, p_m \rangle$ be a simplex of X, then $f(p_i) = y_0$, and $f(p_0 p_i)$ represents an element w_i of π. Put

$$\kappa(1 \cdot \sigma^m) = [1, w_1, \cdots, w_m]$$

and $\kappa(\gamma \cdot \sigma^m) = \gamma \cdot \kappa(1 \cdot \sigma^m)$ for $\gamma \in R$, where 1 denotes the unit element of R.

If we define

$$\partial(1 \cdot \langle p_0, \cdots, p_m \rangle)$$
$$= w_1 \langle p_1 \cdots p_m \rangle + \sum_1^m (-1)^i \langle p_0, \cdots, \hat{p}_i, \cdots, p_m \rangle,$$

it follows immediately that $\kappa\partial = \partial\kappa$ and κ is a chain-transformation. From this we can define the dual homomorphism κ^* of the equivariant cohomology group $H_m(\pi, J)$ into the equivariant cohomology group $H_m(X, J)$, which may be regarded as the cohomology group with local coefficients.

In the case $m = n$, κ^* is defined on the equivariant cochain group $C_n(\pi, J)$.

3.5. Let H be a subgroup $h_f(\pi_1(X^n))$ of π, and let $\mathfrak{z} \in \mathfrak{Z}^\lambda = \mathfrak{Z}_H$, then the following n -cochain of $K(H)$ becomes equivariant:

(5.1) $$k_{\mathfrak{z},F}^n[w_0, \cdots, w_n] = \sum_0^n (-1)^i k_F^{n+1}[w_0, \cdots, w_i, \mathfrak{z}w_i, \cdots, \mathfrak{z}w_n].$$

Now let $F \in M$ be an extension of f such that $F\kappa(1 \cdot \sigma^n) \equiv f(\sigma^n)$ for any $\sigma^n \in X$. Denote by $M(f)$ the set of all the mappings F which are extensions of f.

We show that

(5.2) $$\kappa^* k_{\mathfrak{z},F}^n \in C_{\mathfrak{z}}(f).$$

3.6. To prove (5.2) we make use of certain subdivision Z of the product space $X^n \times I$. The vertices of Z are those of $X^n \times 0$ and $X^n \times 1$. The order of the vertices are definite on $X^n \times 0$ and $X^n \times 1$ respectively, we set that the vertex p_i of $X^n \times 0$ is antecedent to the corresponding $\bar{p}_i$ of $X^n \times 1$. Thus the vertices of Z are partially ordered. Now define a subdivision of $\sigma^n \times I$ as follows:

(6.1) $$d(1 \cdot \sigma^n \times I) = d(1 \cdot \langle p_0 \cdots p_n \rangle \times I)$$
$$= \sum_0^n (-1)^i 1 \cdot \langle p_0 \cdots p_i \bar{p}_i \cdots \bar{p}_n \rangle,$$

where d denote subdivision operation and $(n+1)$ -cells $\langle p_0 \cdots p_i \bar{p}_i \cdots \bar{p}_n \rangle$ admit p_0 as their first vertices.

Denote by $\bar{Z}^n$ the n -skeleton of Z.

Consider a mapping $F_{\mathfrak{z}}: \bar{Z}^n \to Y$ such that $F_{\mathfrak{z}} = f$ on $X^n \times 0$ and $X^n \times 1$, and the paths $F_{\mathfrak{z}}(p_i \bar{p}_i)$ represent $\mathfrak{z} \in \mathfrak{Z}_H$.

Let the path $f(p_0 p_i)$ represent the element w_i of H, putting

(6.2) $$\kappa'(1 \cdot \langle p_0 \cdots p_i \bar{p}_i \cdots \bar{p}_n \rangle)$$
$$= [1, w_1, \cdots, w_i, \mathfrak{z}w_i, \cdots \mathfrak{z}w_n],$$

we obtain a chain-transformation κ' of $C(Z, R)$ into $C(\pi)$. Let $F \in M(f)$ be an extension of $F_{\mathfrak{z}}$ such that

(6.3) $$F\kappa' = F_{\mathfrak{z}} \quad \text{on} \quad \bar{Z}^n,$$

then by (6.1), (6.2) and (6.3) and No.3, No.5

$$C(F_{\mathfrak{z}})(1 \cdot \langle p_0, \cdots, p_n \rangle) = \{F_{\mathfrak{z}} \partial(1 \cdot \langle p_0 \cdots p_n \rangle \times I)\}$$
$$= \{F\partial\kappa' d(1 \cdot \langle p_0 \cdots p_n \rangle \times I)\}$$

$$(6.4) \quad = \sum_0^n (-1)^i k_F^{n+1}[1, w_1, \cdots, w_i, \mathfrak{z} w_i, \cdots, \mathfrak{z} w_n]$$
$$= k^n_{\mathfrak{z},F}[1, w_1, \cdots w_n]$$
$$= k^* \cdot k^n_{\mathfrak{z},F}(1 \cdot < p_0 \cdots p_n >)$$

where $\{ \ \}$ denotes the element of $\pi_n(Y)$.
This proves (5.2).

4. The classification of an $(n-1)$-homotopy class.

Select a mapping f_0 of an $(n-1)$-homotopy class U^λ which maps the topological tree B into y_0, then there exists at least one mapping g in any n-homotopy class in U^λ, such that $f_0 | X^{n-1} = g | X^{n-1}$. If we choose from each n-homotopy class involved in U^λ all the mappings, which satisfy the condition, and construct $\mathfrak{d}(f_0, g)$, it is easily seen from Existence Theorem 2.3.2 that every element of $H_n(X, \pi_n(Y))$ is obtained. Also, the analysis of the relation between $\mathfrak{d}(f_0, g)$ and $\mathfrak{d}(f_0, g')$ for two homotopic mappings g, g' belonging to U^λ gives, in some sense, a classification of an $(n-1)$-homotopy class U^λ.

__Main Theorem 4.1.__

For two maps g, g' belonging to U^λ such that $g = g' = f_0$ on X^{n-1}, g' is $\mathfrak{z}$-homotopic to g if and only if
$$\mathfrak{d}(f_0, g') - \mathfrak{d}(f_0, g)^{\mathfrak{z}^{-1}} = C_{\mathfrak{z}^{-1}}(f_0).$$

__Proof.__ The necessity of Theorem can be proved directly, but we intend to prove it here utilizing some formulaes mentioned in § 2. From (2.2.4) we have $\mathfrak{d}(f_0, g') = \mathfrak{d}(f_0, g) + \mathfrak{d}(g, g')$ and from (2.2.3) $\mathfrak{d}(g, g') = C_{\mathfrak{z}^{-1}}(g)$ holds. Thus $\mathfrak{d}(f_0, g') = \mathfrak{d}(f_0, g) + C_{\mathfrak{z}^{-1}}(g)$ and therefore we have
$$\mathfrak{d}(f_0, g') - \mathfrak{d}(f_0, g)^{\mathfrak{z}^{-1}} = \mathfrak{d}(f_0, g) - \mathfrak{d}(f_0, g)^{\mathfrak{z}^{-1}} + C_{\mathfrak{z}^{-1}}(g).$$

Lastly, from (2.2.5) it is concluded that $\mathfrak{d}(f_0, g') - \mathfrak{d}(f_0, g)^{\mathfrak{z}^{-1}} = C_{\mathfrak{z}^{-1}}(f_0)$.

Sufficiency: Let $d(f_0, g)$ and $d(f_0, g')$ be representatives of $\mathfrak{d}(f_0, g)$ and $\mathfrak{d}(f_0, g')$ respectively, then $d(f_0, g)$ and $d(f_0, g')$ are represented by mappings $D, D' : Z^n \longrightarrow Y$ respectively. Choosing suitably a representative $C(F)$ of $C_{\mathfrak{z}^{-1}}(f_0)$, we have $d(f_0, g') - d(f_0, g)^{-\mathfrak{z}} = C(F)$ in virtue of the remark given in § 2. Defining a mapping $\Phi : Z^n \longrightarrow Y$ such that
$$\Phi(x, t) = \begin{cases} D(x, 1-3t), & \frac{1}{3} \geq t \geq 0, \\ D(x, 2-3t), & \frac{2}{3} \geq t \geq \frac{1}{3}, \\ D'(x, 3t-2), & 1 \geq t \geq \frac{2}{3} \end{cases}$$
we have $\Phi(x, 0) = D(x, 1) = g(x)$, $\Phi(x, 1) = D'(x, 1) = g'(x)$ and $\Phi(x_i, t)$ (for $1 \geq t \geq 0$) represents $\mathfrak{z}$, because $F(x_i, 2-3t)$ (for $\frac{2}{3} \geq t \geq \frac{1}{3}$) represents $\mathfrak{z}$. Regarding $\Phi(a_i \times \frac{2}{3})$ as a base point, $\Phi | \partial(\sigma_i^n \times I)$ represents $d(f_0, g', \sigma_i^n) - d(f_0, g, \sigma_i^n)^{\mathfrak{z}^{-1}} - C(F, \sigma_i^n)$ so that from $d(f_0, g', \sigma_i^n) - d(f_0, g, \sigma_i^n)^{\mathfrak{z}^{-1}} - C(F, \sigma_i^n) = 0$, Φ can be extended into the interior of $\sigma_i^n \times I$ for any $\sigma_i^n \in X$. This proves that $g \underset{n}{\overset{\mathfrak{z}}{\sim}} g'$.

Now, assuming that Y is n-simple in the sense of Eilenberg, we can classify an $(n-1)$-homotopy class U^λ by a rather simple method. Since in (2.11) $C_{\eta^{-1}\mathfrak{z}}(f_0)^\eta = C_{\eta^{-1}\mathfrak{z}}(f_0)$ in virtue of n-simplicity of Y, we have $C_{\mathfrak{z}}(f_0) - C_\eta(f_0) = C_{\eta^{-1}\mathfrak{z}}(f_0)$, so that the totality $A_n(X, \pi_n(Y))$ of all the elements $C_{\mathfrak{z}}(f_0)$, for any $\mathfrak{z} \in \mathfrak{Z}^\lambda$, constitutes a subgroup of $H_n(X, \pi_n(Y))$. Because from (2.2.5) we have $C_{\mathfrak{z}}(f_0) = C_{\mathfrak{z}}(g)$ for any $g \in U^\lambda$, $A(X, \pi_n(Y))$ does not depend on f_0, but depends only on an $(n-1)$-homotopy class U^λ. This group may also be regarded as the image of the group $\mathfrak{Z}^\lambda$ by the homomorphism of $\mathfrak{Z}^\lambda$ into $H_n(X, \pi_n(Y))$. Choosing from each n-homotopy class involved in U^λ all such mappings that coincide with f_0 on X^{n-1} and constructing $\mathfrak{d}(f_0, g)$ for any g of them, it is seen from Existence Theorem 2.2.2 that every element of $H_n(X, \pi_n(Y))$ is obtained through this construction. From the two considerations that for two mappings g, g' belonging to U^λ, g' is n-homotopic to g if and only if $\mathfrak{d}(f_0, g') \equiv \mathfrak{d}(f_0, g)$ mod $A_n(X, \pi_n(Y))$ because of the main theorem 4.1 and that from Lemma 2.2.1 and from Homotopy Theorem 2.2.3 the totality of $\mathfrak{d}(f, f_0)$, for any $f \sim f_0$, coincides with $A_n(X, \pi_n(Y))$ all the n-homotopy classes involved in U^λ is in one-to-one correspondence with the factor group of $H_n(X, \pi_n(Y))$ by $A_n(X, \pi_n(Y))$.

In case where the fundamental group of Y vanishes, Y is, of course, n-simple in the sense of Eilenberg. In this case there is just one n-homotopy class and also $A_n(X, \pi_n(Y)) = 0$ by definition, so that all the n-homotopy classes are in one-to-one correspondence with $H_n(X, \pi_n(Y))$, which is so-called Eilenberg's generalized Hopf Theorem.

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