ON A MEROMORPHIC FUNCTION IN THE UNIT CIRCLE WHOSE NEVANLINNA'S CHARACTERISTIC FUNCTION IS BOUNDED

By Kihachiro ARIMA

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Let E be a Borel set on the unit circle $|z|=1$ and let its capacity be positive. Then there exists a mass distribution $\mu(a)$ on E so that

$$u(z)=\int_E \log\frac{1}{|z-a|}\,d\mu(a)$$

is harmonic and bounded in the unit circle;

$$|u(z)|<K$$

We suppose that $f(z)$ is a meromorphic function in the unit circle and its characteristic function $T(r,f)$ is bounded. Let a_i be a pole of order $m(a_i)$. Applying Green's formula to $\log(1+|f(z)|^2)$ and $u(z)$, we have

$$4\int_0^r\int_0^{2\pi} u(z)\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,r\,dr\,d\theta$$
$$= r\int_0^{2\pi}\frac{\partial}{\partial r}\left(\log\left(1+|f(z)|^2\right)\right)\cdot u(z)\,d\theta$$
$$- r\int_0^{2\pi}\frac{\partial u}{\partial r}\log(1+|f(z)|^2)\,d\theta+4\pi\sum_{|a_i|<r}u(a_i)m(a_i),$$

where $z=re^{i\theta}$. Dividing by $4\pi r$ and then integrating, we have

$$\frac{1}{\pi}\int_0^r\frac{1}{r}dr\left[\int_0^r\int_0^{2\pi}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\,r\,dr\,d\theta\right]$$
$$=\frac{1}{4\pi}\int_0^{2\pi}\log(1+|f(z)|^2)\,u(z)\,d\theta-\frac{2}{4\pi}\int_0^r\frac{dr}{r}\int_0^{2\pi}\log(1+|f(z)|^2)\frac{\partial u(z)}{\partial r}d\theta$$
$$+\int_0^r\frac{1}{r}\left[\sum_{|a_i|<r}u(a_i)m(a_i)\right]dr+O(1),$$

$$\left|\frac{1}{\pi}\int_0^r\frac{dr}{r}\left[\int_0^r\int_0^{2\pi}\frac{|f'(z)|^2}{(1+|f(z)|^2)^2}u(z)\,r\,dr\,d\theta\right]\right|$$
$$\leqq KT(r,f)+O(1),$$

$$\left|\frac{1}{4\pi}\int_0^{2\pi}\log(1+|f(z)|^2)\,u(z)\,d\theta\right|$$
$$\leqq\frac{K}{4\pi}\int_0^{2\pi}\log(1+|f(z)|^2)\,d\theta$$
$$\leqq Km(r,f)+O(1)\leqq KT(r,f)+O(1),$$

$$\left|\int_0^r\frac{dr}{r}\left(\sum_{|a_i|<r}u(a_i)m(a_i)\right)\right|\leqq\int_0^r\frac{dr}{r}K\sum_{|a_i|<r}m(a_i)$$
$$\leqq K\int_0^r\frac{n(r,\infty)}{r}dr=KN(r,\infty)$$
$$\leqq KT(r,f).$$

Hence

$$\left|\int_0^r\frac{dr}{r}\int_0^{2\pi}\log(1+|f(z)|^2)\frac{\partial u(z)}{\partial r}d\theta\right.$$

is bounded. Thus we get the following theorem:

Theorem 1. Let $f(z)$ be a meromorphic function whose characteristic function is bounded, then

$$\int_0^r\int_0^{2\pi}\log^+|f(z)|\frac{\partial u(z)}{\partial r}\,r\,dr\,d\theta$$

is bounded.

We put

$$L(\varphi)=\int_{|z-\frac{3}{4}e^{i\varphi}|<\frac{1}{4}}\cos\psi\,d\psi\int_0^{\cos\frac{\psi}{2}}\log^+\frac{1}{|f-a|}dt,$$

where $z=re^{i\theta}=e^{i\varphi}-te^{-i\psi}$, $0<r<1$, and $t=|e^{i\varphi}-z|$. Then applying Tuji's method[1], from Theorem 1, we have the following theorem:

Theorem 2. The set of $e^{i\varphi}$ where $L(\varphi)=\infty$, is of capacity 0.

In this note we denote by $g(t)$ a positive function of t such that

$$\lim_{t\to 0}\int_t^1 g(t)\,dt=\infty.$$

Then we tet the following theorem.

Theorem 3. Let $f(z)$ be a meromorphic function whose characteristic function is bounded. We suppose that E is a Borel set on the unit circle $|z|=1$ and E is of capacity positive. If at each point P $(z=e^{i\varphi})$ belonging to E, there exists an angular domain with vertex at P in which

$$\log^+\frac{1}{|f(z)-a|}\geqq g(t),\ \text{where}\ t=|e^{i\varphi}-z|$$

then $f(z)\equiv a$.

Corollary. We suppose that $f(z)$ is regular in the unit circle and $|f(z)|<1$. We put

$$\sigma(\varphi)=\lim_{r\to 1}\int_0^{\varphi}\log\left|\frac{1-\bar{a}f(z)}{f(z)-a}\right|d\theta,$$

where $z=re^{i\theta}$. We suppose that, for $e^{i\varphi}$ belonging to a Borel set E on the unit circle, as $z\to e^{i\varphi}$ in an angular domain with vertex at $e^{i\varphi}$,

$$\int_0^{2\pi} R\left(\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\right) d\sigma(\varphi) > g(t),$$

where $t=|e^{i\varphi}-z|$, and we denote by $R(w)$ the real part of w. If E is of capacity positive, $f(z)\equiv a$.

Corollary. Let $u(z)$ be a positive harmonic function in the unit circle. Let E be a set of $P\,(z=e^{i\varphi})$ where $u(z)>g(z)$ as $z\to e^{i\varphi}$ in an angular domain with vertex at P, where $t=|e^{i\varphi}-z|$. If E is of capacity positive, then $u(z)\equiv\infty$.

(*) Received October 10, 1950.

(1) M. Tuji. Beurling's theorem on exceptional sets. (To appear shortly.)

Kagoshima University.