NON-DISCRETE LINEARLY ORDERED GROUPS

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A group $\mathfrak{G}$ is called a linearly ordered group if $\mathfrak{G}$ satisfies the following conditions:

(1) $\mathfrak{G}$ is a linearly ordered set;
(2) $\mathfrak{G}$ is homogeneous; $a \gtreqless b$ implies $cad \gtreqless cbd$ for arbitrary elements c, d in $\mathfrak{G}$.

In a linearly ordered set, the intrinsic topology (i.e. the one defined by taking the open intervals, consisting of x such that $a < x$, $x < a$ or $a < x < b$, where a, b are elements in $\mathfrak{G}$, as a base for the open set of the space) is equivalent to the topology by the Moore-Smith convergence or interval topology and it is a T_1-normal space. Moreover, $\mathfrak{G}$ is a topological group in this topology.

But, we did not know about relations between the structures of groups and their topologies. We study about these points. Throughout this paper the letter $\mathfrak{G}$ will denote a linearly ordered group (abbreviation: l.o.g.). In §1 we show that $\mathfrak{G}$ is a topological group and classify it into two types according to be locally archimedean or not. We then show that the topological properties exert an influence upon the structures of groups, i.e. if $\mathfrak{G}$ is connected, then $\mathfrak{G}$ is locally compact, and if locally compact, then locally archimedean. In § 2 we show that if $\mathfrak{G}$ is locally archimedean, all subgroups of $\mathfrak{G}$ are open; i.e. the structures of groups exerts an influence upon its topologies. We investigate its special subgroups when $\mathfrak{G}$ is non-locally archimedean. In § 3 we show that zero-dimensionality and totally disconnectedness are equivalent, and one-dimensionality and locally compactness so is also. Consequently, if $\mathfrak{G}$ is one-dimensional then $\mathfrak{G}$ has the component homeomorphic to the additive group of real numbers. In § 4 we consider the problem of completion. If $\mathfrak{G}$ is locally archimedean, it is imbeded densely in one-dimensional l.o.g. If $\mathfrak{G}$ is non-locally archimedean, then, even when $\mathfrak{G}$ is complete, $\mathfrak{G}$ is still zero-dimensional. In § 5 we show that, if $\mathfrak{G}$ is locally archimedean, its dimension is equivalent in both Urysohn-Menger's and Lebesgue's senses. In § 6 we see uniformities of one-dimensional group under inner-automorphisms.

§1. We assume always that the topology in $\mathfrak{G}$ is intrinsic topology. We set

$$U_b(a) = \{t \,;\, ab^{-1} < t < ab \,,\ b > e\},$$

then $\{U_b(a)\,,\ b \in \mathfrak{G}\}$ is the basis of the neighbourhood (abbreviation: n.b.d.) about a. Now we assume that $\mathfrak{G}$ is non-discrete, i.e. if $p > e$ there exists q such that $p > q > e$, then we can easily prove the following theorem.

Theorem 1.1. $\mathfrak{G}$ has the equivalent following properties:

1) $e < a \longrightarrow \exists b \,;\ e < b < a$;
2) $e < a \longrightarrow \exists b, c \,;\ e < b < c < bc < a$;
3) $e < a \longrightarrow \exists b \,;\ e < b < b^2 < a$;
4) $e = \sup(x;\ x < e),\ e = \inf(x;\ x > e)$;
5) $e = \sup(x^2;\ x^2 < e),\ e = \inf(x^2;\ x^2 > e)$.

From above theorem $\mathfrak{G}$ is a topological group. Next we define some terminologies. All $\mathfrak{G}$ considered in this paper are non-discrete even if they are not explicity stated.

Definition. $\mathfrak{G}$ is called locally archimedean (abbreviation l.a.) if there exists a n.b.d. $U_b(e)$ such that if $e < c < b$ there are some integer $n = n(c, b)$ for which $c^n > b$. The n.b.d. having this property is called locally archimedean neighbourhood (abbreviation l.a. n.b.d.). A subgroup of $\mathfrak{G}$ is non-discrete closed. $\mathfrak{G}$ is called connected or totally disconnected if the component of e is $\mathfrak{G}$ or e.

Theorem 1.2. If $\mathfrak{G}$ is locally compact, $\mathfrak{G}$ is l.a.

Proof. Let $\mathfrak{G}$ be not l.a., we can take $U_a(e)$ as a l.a.n.b.d. for which $\overline{U_a(e)}$ is compact; i.e. $\exists b$; $e < b < a$, $\forall n$; $b^n < a$. We consider $\{b^n\}$, then by assumption there exists a point $p \in \overline{U_a(e)}$ such that p is accumulation point of $\{b^n\}$. Therefore $U_c(p)$, $e < c < b$, contains some b^n i.e. $pc^{-1} < b^n < pc$. However we have $p < b^n c < b^{n+1}$, hence if $m \geq n+1$, $U_c(p) \not\ni b^m$. This is a contradiction.

Theorem 1.3. If G is l.a., then G satisfy the first countable axiom.

Proof. We can take $U_a(e)$ as a l.a.n.b.d. and classify elements b, $e < b < a$, into classes $\{A_n\}$ such that

$$A_n = \{t;\ a > t > e,\ t^n > a,\ t^{n-1} < a\}.$$

We take an element t_n from each A_n, then $\{U_{t_n}(e)\}$ is the basis of e.

Corollary 1. If G is l.a. G is metrisable.

Corollary 2. If G is locally compact, G is locally bicompact.

Theorem 1.4. A necessary and sufficient condition (abbreviation: n.a.s.c.) in order that G is locally compact is that there is a n.b.d. $U(e)$ such that arbitrary subset $V \subset U(e)$ has $\sup V$ and $\inf V$.

Proof. We can assume that $\overline{U(e)}$ is compact and if $V \subset U(e)$, so is $\overline{V}$ also. Let p belong to V and put $[p] = \{x;\ x \in \overline{V}\ p \leq x\}$. Next take $q \in [p]$ and make $[q]$, and so on. Then the family of closed sets $\{[p]\}$ has the finite-intersection property, therefore $\prod [p] \neq 0$. It is easy to see that the intersection is only a point which is $\sup V$. About $\inf V$ the argument is similarly. The inverse is obvious.

Theorem 1.5. If G is connected, G is locally compact.

Proof. If G is not locally compact, we can take from above theorem some n.b.d. V in some $U(e)$ such that there does not exist $\sup V$ (or $\inf V$ (. Consider the following sets

$$G_1 = \{x;\ x < t,\ t \in V\},$$

$$G_2 = \{x; x > t, t \text{ is arbitrary in } V\}.$$

Then $G = G_1 \cup G_2$, $G_1 \cap G_2 = 0$ and G_1, G_2 are both open sets, but this is a contradiction.

§ 2. Definition. For an arbitrary element b we denote $\mathrm{Max}(b, b^{-1})$ by $|b|$. $b \circledS a$ means $|b|^n < |a|$ for all integer n, and we define the following symbols:

$A(a) = \{\ b\ ;\ b \circledS a\}$,

$A(G) = \{\ A(a)\ ;\ a \in G\}$,

$C(a) = \{$ a group generated by $A(a)\}$,

$C(G) = \{$ a group generated by $A(G)\}$,

$|N| = \{\ y\ ;\ |y| < |x|$ for some $x \in N\}$.

Theorem 2.1. If G is l.a., arbitrary subgroups are open.

Proof. Let subgroup N be not open, then there is a point a such that $N \ni a$, $\overline{G - N} \ni a$. We show that this is a contradiction. In general, we can assume $a = e$. Since G has the first countable axiom, there are $\{d_n\} \in N$, $\{b_n\} \in G - N$ such that $\{d_n\} \to e$, $\{b_n\} \to e$ and $d_1 > d_2 > \cdots\cdots$, $b_1 > b_2 > \cdots\cdots$. Now we can take $U_{d_n}(e)$ as a l.a.n.b.d. For $U_{d_n}(e)$ $\exists\ n = n(d_n)$; $m > n$, $b_m \in U_{d_n}(e)$, since $G - N$ is open $\exists\ d_l = d_l(b_m)$: $d_l < d_n$ $U_{d_l}(b_m) \cap N = 0$, and hence $U_{d_l}(b_m) d_l^{-1}$ does not contain elements of N, for if it contains an element p of N, then $U_{d_l}(b_m) \ni p d_l \in N$. This is impossible, therefore N does not contain t such that $b_m d_l^{-n} < t < b_m d_l^{-n+1}$, $n = 1, 2, \cdots\cdots$. But by assumption $\exists\ n$; $d_l^{n} > b_m$ i.e. $b_m d_l^{-n} < e < b_m d_l^{-n+1}$, $e \in N$. This is a contradiction.

Corollary 1. If G is connected there are no subgroups.

Proof. From the above theorem, Theorem 1.5 and Theorem 1.2 in § 1.

We can easily prove the following theorem

Theorem 2.2. $C(a)$ is an open subgroup and $C(a) = |C(a)|$.

Theorem 2.3. A n.a.s.c. in order that $C(a) = C(b)$ is that if $a < b$, there exists an integer n; $a^n > b$.

Proof. Since $a < b$, $C(a) \subset C(b)$. Now, let $C(a) \not\ni p$, then $\exists\ m$; $p^m > a$ i.e. $p^{mn} > a^n > b$. This shows $C(b) \subset C(a)$. The necessarity is obvious from $a \notin C(a)$.

Corollary 2. $C(G) = \bigcup_{G \ni a} C(a) = |C(G)|$ and it is an open normal subgroup.

Corollary 3. G is archimedean if and only if $C(G) = e$

Theorem 2.4. $C(G)$ is only one subgroup such that the factor group of G by its normal subgroup is archimedean l.o.g. (where $G/C(G)$ is discrete).

Proof. From the above corollary 2 and theorem about ordered groups we see that if G/N is l.o.g., $|N| = N$.

§ 3. In this section, we consider the dimension in the sense of Uryshon-Menger. For arbitrary n.b.d. $U(e)$, $\overline{U(e)} - U(e)$ at most consists of $\sup U(e)$ and $\inf U(e)$. Hence we have the following theorem.

Theorem 3.1. G is at most one-dimensional.

In general, if a metric space X is zero-dimensional, then X is totally disconnected. But the inverse is not true and there are a topological groups which are zero-dimensional and locally compact. On the other hand there are one-dimensional totally disconnected topological groups. However, in l.o.g., we have the following special theorem.

Theorem 3.2. A n.a.s.c. in order that G is one-dimensional is that G is locally compact.

Proof. If G is not locally compact, then from Theorem 1.4 and the homogenity of groups, G is not one-dimensional. Conversely, let us suppose that G is locally compact and zero-dimensional, i.e. there exists a n.b.d. $U(e)$ such that $\overline{U(e)}-U(e)=0$. But we can assume that $\overline{U(e)}$ is bicompact, hence $U(e) \ni \sup U(e)$, $\inf U(e)$. This contradicts to $U(e)$ being open and closed. Consequently, if G is locally compact, G is one-dimensional.

Corollary 1. If G is connected then G is one-dimensional.

Theorem 3.3. If G is connected then G is homeomorphic to the additive group of real numbers.

Proof. From (1) we know that one-dimensional metric separable connected locally compact and not compact topological groups are homeomorphic to the additive group of real numbers. Since G is locally compact, G is metrisable. Hence G has the star-finite property, therefore G has the Lindelöf-property. (This follows from (2): for a connected regular space the star-finite property is equivalent to the Lindelof property.) It is easily proved that G is separable by the Lindelöf-property, i.e. G satisfies assumptions of above theorem.

Theorem 3.4. A n.a.s.c. in order that G is zero-dimensional is that G is totally disconnected.

Proof. Let G be zero-dimensional. If the component is not e, it contradicts to corollary 1. Next, we assume that G is totally disconnected and locally compact. Then we can take $U(e)$ such that $U(e)$ has the property in Theorem 1.4. Now, we take some $a \in U(e)$, $a > e$ then there exist two open sets W and V such that

$$U(e)=W \cup V,\ W \cap V=0,\ W, V \subset U(e),\ W \ni e,\ V \ni a.$$

We put $W_1(e)=\{t;\ t \in W,\ t<a\}$, then there exists $\sup W_1(e)$ which is contained in V and W. This is a contradiction.

Corollary 2. If G is one-dimensional then the component of G is homeomorphic to the additive group of real numbers.

§ 4. Theorem 4.1. If G is l.a., G is densely imbeded in one-dimensional l.o.g. G^*.

Proof. Since G is locally totally bounded we can complete it to be locally compact, therefore we show that G^* is a l.o.g. with topology homeomorphic to the topology by completion. Let $G^* \ni f^*, g^*, p^*$ are pairs of foundamental sequence and p^* is a convergent sequence and f^*, g^* are not, then we give for p^* the same order in G. Let $f^*=\{f_\alpha\}$, $f_\alpha=\{x_n\}$, and give them the order in the following manner. For an arbitrary element p in G and for $\{x_n\}$, if there exists a cofinal subsequence $\{x_n'\}$ of $\{x_n\}$ such that $\{x_n'\}$ is monotone decreasing and for all n, $x_n' < p$ $(x_n' > p)$, then we put $f^* < p^*$ $(f^* > p^*)$. If there does not exist such subsequence, then we do the above process for monotone increasing subsequence. Then we can see that if f^* contains both sequence, from the property of fundamental sequences, their relations are same for p^*. Moreover, for f^*, g^* there exists some p^* between them. Thus G^* is linearly ordered set. The remainders about group operation and topology easily follow from the above process.

Corollary 1. Let G is non-l.a., then even if G can be complete, G is still non-l.a..

Proof. From above theorem.

We can give an example by the following theorem.

Theorem 4.2. If G is non-l.a. abelian, then G is complete.

Proof. From assumption $\prod_{a \in G} C(a)=e$. From (3) $G=\mathbb{P}_{\alpha \in \Lambda} R_\alpha$ where each R_α is isomorphic to the subgroup of the additive group of real numbers, $\Lambda=\{\alpha\}$ are well-ordered and G is "lexicographycally" ordered. We can take instead of $C(a)$, $C_\alpha=\{e_1, \cdots\cdots, e_\alpha, a_{\alpha+1}, \cdots\cdots, a_\beta, \cdots\}$, and this is the basis of n.b.d. Let $\{x_\alpha\}$ be a fundamental direct set, C_α be an arbitrary n.b.d. Then there exists a point $p=\{p_\alpha\}$ and an index γ such that $\beta > \gamma$ implies $x_\beta \in pC_\alpha$. This shows that $\beta > \gamma$, x_β has p_α as α-coordinate. Thus we can see that $\{x_\alpha\}$ converges to a point q which has α-coordinates determined by above method for each α.

§ 5. We consider the relation between Urysohn-Menger's and Lebesgue's dimension. If G is l.a. we can consider that G is one-dimensional (Theorem 4.1.). In this case, from corollary 2 in § 3, the component of G is a open normal subgroup and the theorem of dimension (in metric separable space, both Urysohn-Menger's and Lebesque's dimension coincides) we have the following theorem.

Theorem 5.1. If G is l.a., then the dimension of G coincides in both senses.

But, if G is non-l.a. we can not still known about their relation.

§ 6. One-dimensional l.o.g. G has the normal subgroup C homeomorphic to the additive group of real numbers as component. Is each element of C commutative to each element of G or not?. We shall study about this point using the characteristic property of linearly order and one-dimensionality.

Now we put $[a,b]=a^{-1}b^{-1}ab$, then we have B.H.Neumann s Lemma (4), i.e. let m, n be integers, then

$$[x^m,y]=e \text{ implies } xy=yx,$$

$$[x^m,y^n]=e \text{ implies } xy=yx.$$

We put $S_a = \{x ; x^m=a^i, x\in G, m, i$ are arbitrary integers$\}$.

Lemma 1. In l.o.g. G, S_a form an abelian subgroup homeomorphic to the subgroup of the additive group of rational numbers.

Proof. If $S_a \ni x, y$, there are some integers m, n, i and j such that $x^m=a^i$, $y^n=a^j$. Hence $(x^{-1})^m=a^{-i}$ implies $x^{-1}\in S_a$, by the above lemma $x^{mj}=y^{ni}$ implies $xy=yx$ and each element is commutative. $(xy)^{mn}=x^{mn}y^{mn}=a^{in+jm}$ implies $xy\in S_a$. Thus S_a form an abelian group. Next if $a^i=x^m$, we corresponds to x a symbol $(\frac{m}{i})$. From this the latter part is obviously.

Theorem 6.1. A n.a.s.c. in order that G has a solution $x\in G$ for $x^m=a^i$ where a is an arbitrary element of G and m, i are arbitrary integers, is that G is a union, in the sense of set, of disjoint (except e) abelian subgroups isomorphic to the additive group of rational numbers.

Proof. The sufficiency is obvious. Conversely, from assumption, S_a form a group isomorphic to the additive group of rational numbers. If $G-S_a\neq 0$, take $b\in G-S_a$ and make S_b, then we can easily see that $S_a\cap S_b=\{e\}$. Therefore, by Zorn s lemma, our assertion can be proved.

Lemma 2. In l.o.g. G, if $\{a^n\}$ is normal, S_a is so and a belongs to center Z and S_a is so.

Proof. Let $S_a\ni x$, $x^m=a^i$, $g^{-1}a^ig=a^j$, then $(g^{-1}xg)^m=g^{-1}x^mg=g^{-1}a^ig=a^j$ i.e. S_a is normal. Next, for arbitrary element $g\in G$ we put $g^{-1}ag=a^n$. If $n=1$, $a\in Z$. If $n\leq -1$, this is impossible to the homogenity of l.o.g. If $n\geq 2$, we show which is contradictory, but it is sufficient to see for $n=2$. In this case $g\{a^n\}g \subsetneqq \{a^n\}$, $g\{a^n\}g^{-1}\supsetneqq\{a^n\}$ i.e. $g^{-1}(G)g\supsetneqq G$. This is contradictory. (This method is useful for only discrete case). Therefore $a\in Z$. The latter part is obvious.

In general, if G is connected, discrete normal subgroup is contained in center. Moreover, in l.o.g. we have the following theorem.

Theorem 6.2. A n.a.s.c. in order that the component C of G is contained in center Z, is that C has a point a belongs to Z.

Proof. This follows from Theorem 6.1, Lemma 2 and $\overline{S_a}=C$.

Definition. For $g\in G$ we define $g(x)=g^{-1}xg$, then $g(x)>x$ implies $g(x^{-1})<x^{-1}$. We denote by $g(x)\gg x$ ($\ll x$) that if $x>e$, $g(x)>x$ ($<x$), therefore $g(x^{-1})<x^{-1}$ ($>x^{-1}$). Let R be a set, we denote by $g(R)\gg R$ that for every element x of R, except e, $g(x)\gg x$, and by $g(R)\equiv R$ that for every element x of R, $g(x)=x$. Then we have some uniformities for the component of one-dimensional G.

Theorem 6.3. Let g be an arbitrary element of G, $g\neq e$ and $x\neq e$ is an element of C, if $g(x) \gtreqless x$, $g(C)\gtreqless C$.

Proof. If $x\in C$ and $g(x)=x$, $g(R)\equiv R$ follows from $\overline{S_x}=C$. Hence if for an element x of C, $g(x)>x$ then there are no element such that $g(y)=y$. Therefore we can divide C into two sets in the following (except e).

$$S_1=\{x ; x\in C \quad g(x)\gg x\},$$

$$S_2=\{x ; x\in C \quad g(x)\ll x\}.$$

We assume that S_1 and S_2 are not empty. We shall show that they are open. From the definition $S_1\ni x$ implies $S_1\ni x^{-1}$. Next, we show that S_1 is open. Let $x\in S_1$; $g(x)=xc$, $x, c>e$, then since $g(z)>g(x)=xc$, $g(z)>z$ for $x<z<xc$. Now, we take some $d>e$, then $g(z)>z$; $xd^{-1}<z<x$. For if it is not true, there exists $\{p_n\}$ such that $p_n\in S_2$, $p_n\to x$, $p_1<p_2<\cdots<x$, $g(p_n)<p_n$. But this contradicts to the continuity of $g(x)$ i.e. S_1 is open. Similarly, S_2 is

also open. If we put $A = \{x : x > e, x \in C\}$ A is connected and open. Then we can decompose A into the form: $A = (S_1 \cap A) \cup (S_2 \cap A)$, $(A \cap S_1)$ and $(A \cap S_2)$ are disjoint and both open, what is contradictory.

Theorem 6.4. If $g(x) \gg x \ (\ll x)$ $x \in C$, $g(x) = a_x x, a_x > e \ (a_x < e)$, then a_x (a_x^{-1}) is monotone decreasing (decreasing), non-bounded and for arbitrary elements $b > e$, there exist elements p such that $b > a_p > e$.

Proof. Let $x > y > e$, $x, y \in S_1$, $x = yb$, $b > e$, then $g(x) = g(y)g(b)$. It is sufficient to show that $g(y)g(b)y^{-1}b^{-1} > g(y)y^{-1}$, i.e. $g(b)b^{-1} > e$, $g(b) > b$. This is obvious from above theorem: $g(C) \gg C$. Now let $\sup_{x \in C} a_x = d$ and $g(d) = pd$, $p > e$, we take some q such that $p > q > e$, then there exists a point x such that $g(x) > dq^{-1}x$, i.e. $g(xd) > dxd$. This is a contradiction. The remaining properties is obvious from the continuity of $g(x)$.

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