ON HARMONIC MEASURE FUNCTIONS IN SOME REGIONS

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(Communicated by Y. Komatu)

Chapter I.

§ 1. Let S be a region in the unit circle bounded by analytic curves C_n $(n=1,2,3,\cdots)$ and by the unit circle $|z|=1$. By $\rho(n)=\rho_n$ we denote the distance to C_n from $|z|=0$. We suppose that if z $(|z|<1)$ lies on C_n, z is not a limiting point of the point-set $\sum_{i\neq n} C_i$ and that $\rho(n) \leqq \rho(n+1)$.

We explain notations used in this note.

$n(r)$: the greatest number of n which satisfies $\rho(n) < r$.

$$\Gamma(r) = \sum_{i=1}^{n(r)} C_i, \quad \Gamma = \sum_{i=1}^{\infty} C_i$$

$E(r)$: the region bounded by $\Gamma(r)$ and by the unit circle $|z|=1$.

$\Delta(r)$: the region which is common to $E(r)$ and the circle $|z|<r$.

$A(r)$: arcs of the circle $|z|=r$ contained in S.

$B(r)$: the boundary of $E(r)$ contained in the circle $|z|<r$.

$A(1,r)$: the boundary arcs of $E(r)$ lying on the unit circle $|z|=1$.

We consider a region D whose boundary consists of b_1+b_2.

By $\omega(z, b_1, D)$ we denote the function which is harmonic in D and is 1 on b_1 and is 0 on b_2. Let $\{r_n\}$ be a sequence of positive numbers such that

$$r_n < r_{n+1}, \quad \lim_{n\to\infty} r_n = 1.$$

Putting $u_n(z) = \omega(z, A(r_n), \Delta(r_n))$ and $v_n(z) = \omega(z, A(1,r_n), E(r_n))$, then, by Harnack's theorem, $u_n(z)$ and $v_n(z)$ converge uniformly to harmonic functions $u(z)$ and $v(z)$, respectively.

In § 2, we will prove that $u(z)=v(z)$.

If $u(z) \not\equiv 0$, we say that S is a positive-region, and otherwise we say that S is a 0-region. Myrberg (Über den fundamentale Bereich der automorphic Functionen, Saja Series A Math-Physica) considered particular cases.

In this note the following problem will be researched:

What is a necessary or sufficient condition that S should be a 0-region?

What properties has a harmonic function in a 0-region S?

§ 2. Let $\{r_i\}$ and $\{d_i\}$ be sequences of positive numbers such that $r_i < r_{i+1}$, $d_i < d_{i+1}$, $\lim_{i\to\infty} r_i = 1$, and $\lim_{i\to\infty} d_i = 1$.

Putting $u_n(z) = \omega(z, A(r_n), \Delta(r_n))$ and $U_n(z) = \omega(z, A(d_n), \Delta(d_n))$, then we can easily prove that, if $\lim_{n\to\infty} u_n(z) = u(z) \not\equiv 0$, $\lim_{n\to\infty} U_n(z) = u(z)$, and, if $\lim_{n\to\infty} u_n(z) \equiv 0$, $\lim_{n\to\infty} U_n(z) \equiv 0$. Thus the fact that S is a 0-region, does not depend upon the choice of the sequence $\{r_n\}$. By $E(r_i, r_n)$ we denote the region which is common to $E(r_n)$ and the circle $|z|<r_i$, and by $A(r_i, r_n)$ the arcs of $|z|=r_i$ contained in $E(r_n)$. Putting $v_n(z) = \omega(z, A(1,r_n), E(r_n))$ and $v_{n,i}(z) = \omega(z, A(r_i, r_n), E(r_i, r_n))$, then $v_n(z) = \lim_{i\to\infty} v_{n,i}(z)$. Since, for $i>n$, $v_{n,i}(z) > u_i(z)$ in $\Delta(r_n)$, we have $v_n(z) = \lim_{i\to\infty} v_{n,i}(z) \geqq \lim_{i\to\infty} u_i(z) = u(z)$. Hence $v(z) = \lim_{n\to\infty} v_n(z) \geqq u(z)$. Since, on the other hand, $u_n(z) \geqq v_n(z)$ in $\Delta(r_n)$, we have $u(z) = \lim_{n\to\infty} u_n(z) \geqq \lim_{n\to\infty} v_n(z) = v(z)$. Therefore we see that $u(z) = v(z)$.

§ 3. We put $A(r) = \sum_{i=1}^{N(r)} A_i(r)$, where $A_i(r)$ is an arc of $A(r)$ whose length is $r\theta_i(r)$ and $A_i(r)$ and $A_j(r)$ $(i \neq j)$ have not a common part. If the circle $|z|=r$ is entirely contained in S, then we put $\theta(r)=\infty$, otherwise we put $\theta(r) = \max_i \theta_i(r)$.

Theorem 1. Let $u(z)$ be a positive function which is harmonic in $\Delta(R)$ $(R<1)$ and is 0 on $B(r)$. We put

$$m(r) = \frac{1}{2\pi} \int_{A(r)} u(z)^2 \, d\theta, \quad \text{where } z = re^{i\theta},$$

and

$$D(r,u) = D(r) = \frac{1}{\pi} \iint_{\Delta(r)} \left[\left(\frac{\partial u(z)}{\partial \log r} \right)^2 + \left(\frac{\partial u(z)}{\partial \theta} \right)^2 \right] d\log r \, d\theta.$$

Then

$$D(r) \geqq D(r_0) \exp \int_{r_0}^{r} \frac{2\pi}{r\,\theta(r)} \, dr,$$

and

$$m(r)-m(r_0) \geq D(r_0)\int_{r_0}^{r}\frac{dt}{t}\exp\left[\int_{r_0}^{t}\frac{2\pi}{s\,\theta(s)}ds\right].$$

This result was reported by the author at the spring meeting of the Mathematical Society of Japan in 1949 and will appear in the Journal of the Math. Soc. of Japan, so we omit the proof of this theorem.

Let $f(z)$ be an integral function, then applying this theorem to $u(z)=\log^{+}|f(z)|$ we have the same result as Pfluger's.

We put

$$u_n(z) = \omega(z, A(r_n), \Delta(r_n))$$

and

$$m_n(r) = \frac{1}{2\pi}\int_{A(r)} u_n(z)^2\, d\theta,$$

where $z = re^{i\theta}$, $0 \leqq r \leqq r_n$.

Since $0 \leqq u_n(z) \leqq 1$, it becomes $0 \leqq m_n(r) \leqq 1$. From Theorem 1, we have

$$1 > 1-m_n(r_0) > m_n(r)-m_n(r_0) \geqq D(r_0,u_n)\int_{r_0}^{r}\frac{dt}{t}\exp\left[\int_{r_0}^{t}\frac{2\pi}{s\,\theta(s)}ds\right].$$

If $\lim_{n\to\infty}\int_{r_0}^{r_n}\frac{dt}{t}\exp\left[\int_{r_0}^{t}\frac{2\pi}{s\,\theta(s)}ds\right]$, we have $\lim_{n\to\infty} D(r_0,u_n)=0$ and hence $\lim_{n\to\infty} u_n(z)=0$ in $\Delta(r_0)$. Therefore we have the next theorem:

Theorem 2. If $\int_0^1\frac{dt}{t}\exp\left[\int_0^t\frac{2\pi}{s\,\theta(s)}ds\right]$, S is a 0-region.

Now,

$$\frac{d\,m_n(r)}{d\log r} = \frac{1}{\pi}\int_{A(r)} u_n(z)\frac{\partial u_n(z)}{\partial \log r}d\theta$$
$$= \frac{1}{\pi}\iint_{\Delta(r)}\left[\left(\frac{\partial u_n(z)}{\partial\log r}\right)^2+\left(\frac{\partial u_n(z)}{\partial\theta}\right)^2\right]d\log r\, d\theta$$
$$\geqq \frac{1}{\pi}\iint_{\Delta(r)}\frac{1}{r}\left(\frac{\partial u_n(\theta)}{\partial\theta}\right)^2 dr\,d\theta \geqq 0$$

$$\int_{A(r)}\left(\frac{\partial u_n(z)}{\partial\theta}\right)^2 d\theta \geqq \sum_{i=1}^{N(r)}\frac{\pi^2}{\theta_i(r)^2}\int_{A_i(r)} u_n(z)^2 d\theta$$
$$\geqq \frac{\pi^2}{\theta(r)^2}\int_{A(r)} u_n(z)^2 d\theta.$$

Since $\frac{d\,m_n(r)}{d\log r} \geqq 0$, $m_n(r) > m_n(r_0)$ for $r_0 < r < r_n$. Hence $\int_{A(r)}\left(\frac{\partial u_n(z)}{\partial\theta}\right)^2 d\theta \geqq \frac{2\pi^3}{\theta(r)^2}m_n(r_0)$. Therefore

$$\frac{d\,m_n(r)}{d\log r} \geqq \frac{1}{\pi}\int_{r_0}^{r}\frac{2\pi^3}{\theta(r)^2}m_n(r_0)\,dr = 2\pi^2 m_n(r_0)\int_{r_0}^{r}\frac{ds}{\theta(s)^2}.$$

By integrating, we have

$$m_n(r)-m_n(r_0) \geqq 2\pi^2 m_n(r_0)\int_{r_0}^{r}dt\int_{r_0}^{t}\frac{ds}{\theta(s)^2}$$
$$= 2\pi^2 m_n(r_0)\int_{r_0}^{r}\frac{r-s}{\theta(s)^2}ds.$$

Since $m_n(r_n) \leqq 1$,

$$\frac{1}{m_n(r_0)}-1 = \frac{1-m_n(r_0)}{m_n(r_0)} \geqq \frac{m_n(r_n)-m_n(r_0)}{m_n(r_0)}$$
$$\geqq 2\pi^2\int_{r_0}^{r}\frac{r_n-s}{\theta(s)^2}ds.$$

If $\lim_{n\to\infty}\int_{r_0}^{r_n}\frac{r_n-s}{\theta(s)^2}ds = \infty$, then $m_n(r_0)\to 0$ as $n\to\infty$. Hence $\lim_{n\to\infty}u_n(z)=0$ on $A(r_0)$.

Theorem 3. If $\int_0^1\frac{1-s}{\theta(s)^2}ds = \infty$, S is a 0-region.

Let $\{p_n\}$ and $\{q_n\}$ be increasing sequences of positive integers such that $\lim_{n\to\infty}p_n=\infty$ and $\lim_{n\to\infty}q_n=\infty$. By $C_{n,k}$ we denote the segment connecting $z=r_n e^{i\theta_{n,k}}$ and $z=e^{i\theta_{n,k}}$, where $r_n = 1-\frac{1}{q_n}$ and $\theta_{n,k}=\frac{2\pi k}{p_n}$ $(k=0,1,\cdots,p_n-1)$. Let S be a region bounded by $\sum C_{n,k}$ and by the unit circle. Then

$$\int_{1-\frac{1}{q_1}}^{1}\frac{1-r}{\theta(r)^2}dr = \sum_{n=1}^{\infty}p_n^2\left(\frac{1}{q_n^2}-\frac{1}{q_{n+1}^2}\right).$$

By Theorem 3, we have the next theorem:

Theorem 4. If $\sum_{n=1}^{\infty}p_n^2\left(\frac{1}{q_n^2}-\frac{1}{q_{n+1}^2}\right)=\infty$, S is a 0-region.

Corollary (Myrberg's theorem). If $\lim_{n\to\infty}\frac{p_n}{q_n}=\infty$, S is a 0-region.

§ 4. By $\beta(r)$ we denote the length of $A(1,r)$. Since $\beta(r_n)$ is a monotone decreasing function of n, we put $\lim_{n\to\infty}\beta(r_n)=\beta$.

Theorem 5. If $\beta=0$, S is a 0-region.

Proof. Without loss of generality, we may assume that for sufficiently small r_0, the circle $|z|=r_0$ is contained in the complementary set of S. Putting $v_k(z)=\omega(z,A(1,r_k),E(r_k))$ and applying Green's formula, we have

$$\sum_{i=1}^{n(r_k)}\int_{C_i}\log\frac{1}{|z|}\frac{\partial v_k(z)}{\partial n}ds = \int_{A(1,r_k)}\frac{\partial\log\frac{1}{|z|}}{\partial n}ds$$
$$= \beta(r_k).$$

Given $\mathcal{E}$, for sufficiently large k_0, $\beta(r_k) < \mathcal{E}$ for $k \geq k_0$. We have

$$\sum_{i=1}^{n(r_k)} \int_{C_i} \log \frac{1}{|z|} \frac{\partial v_k(z)}{\partial n} ds < \mathcal{E};$$

$$\log \frac{1}{|z|} > 0 \text{ and } \frac{\partial v_k(z)}{\partial n} \geq 0 \text{ for } |z| < 1$$

Since C_i is an analytic curve, $\lim_{k\to\infty} \frac{\partial v_k(z)}{\partial n} = 0$ on C_i, where $|z| < \rho < 1$ and ρ is a fixed number. Hence $\lim_{k\to\infty} v_k(z) = 0$ in S. Thus the theorem has been proved.

Let P_i be a region bounded by C_i and by the unit circle $|z| = 1$. And let α_i be the boundary arcs of P_i on the unit circle. We put

$$\overline{v}_i(z) = \omega(z, \alpha_i, P_i).$$

Theorem 6. Suppose that $\beta > 0$ and $\int_{C_i} \frac{\partial \overline{v}_i(z)}{\partial n} ds < K$ for all i, where K is a constant. If $\sum_{i=1}^{\infty} \log \frac{1}{\rho_i}$ is finite, S is a positive-region.

Proof. Without loss of generality, we may assume that, for sufficiently small r_0, the circle $|z| = r_0$ is contained in the complementary set of S. We put $v_k(z) = \omega(z, A(1, r_k), E(r_k))$. By Green's formula, we get

$$\sum_{i=1}^{n(r_k)} \int_{C_i} \log \frac{1}{|z|} \frac{\partial v_k(z)}{\partial n} ds = \int_{A(1,r_k)} \frac{\partial \log \frac{1}{|z|}}{\partial n} ds$$

$$= \beta(r_k) \geq \beta > 0.$$

We assume that $\lim_{k\to\infty} v_k(z) \equiv 0$, and will prove that this assumption is absurd. Now,

$$\lim_{n\to\infty} \rho_n = 1 \quad \text{and} \quad \rho_n \leq \rho_{n+1}.$$

Given $\mathcal{E}$, for sufficiently large N_0, $\sum_{i=N_0}^{\infty} \log 1/\rho_i < \mathcal{E}$. If k is sufficiently large, we have for $i \leq N_0$,

$$\int_{C_i} \frac{\partial v_k(z)}{\partial n} ds < \mathcal{E}.$$

Otherwise at a point z on C_i ($i \leq N_0$) $\lim_{k\to\infty} \frac{\partial v_k(z)}{\partial n} \neq 0$, hence $\lim_{k\to\infty} v_k(z) \not\equiv 0$. This is inconsistent with our assumption $\lim_{k\to\infty} v_k(z) \equiv 0$. Now,

$$\sum_{i=1}^{N_0} \int_{C_i} \log \frac{1}{|z|} \frac{\partial v_k(z)}{\partial n} ds < \mathcal{E} \sum_{i=1}^{N_0} \log 1/\rho_i < \mathcal{E} L,$$

where

$$\sum_{i=1}^{\infty} \log 1/\rho_i = L,$$

and

$$\sum_{i=N_0+1}^{n(r_k)} \int_{C_i} \log \frac{1}{|z|} \frac{\partial v_k(z)}{\partial n} ds$$

$$\leq \sum_{i=N_0+1}^{n(r_k)} \log 1/\rho_i \int_{C_i} \frac{\partial v_k(z)}{\partial n} ds.$$

Since $v_k(z) \leq \overline{v}_k(z)$, we get $\int_{C_i} \frac{\partial v_k(z)}{\partial n} ds \leq \int_{C_i} \frac{\partial \overline{v}_k(z)}{\partial n} ds < K$. Hence $\sum_{i=N_0+1}^{n(r_k)} \int_{C_i} \log \frac{1}{|z|} \frac{\partial v_k(z)}{\partial n} ds \leq K \sum_{i=N_0+1}^{\infty} \log 1/\rho_i < \mathcal{E} K$.

Thus $\mathcal{E}K + \mathcal{E}L > \beta > 0$. But, since $\mathcal{E}$ is an arbitrary positive number, this is absurd.

Corollary. If $\int_{C_i} \frac{\partial \overline{v}_i(z)}{\partial n} ds < K$, $\beta > 0$ and $\sum_{i=1}^{\infty} (1 - \rho_i) < \infty$, then S is a positive-region.

§ 5. In this paragraph we will give some examples.

Exp.1. Let $S_i(z)$ ($i = 0, 1, 2, \cdots$) be a linear transformation of a Fuchsoid group G and let C be a circle contained in fundamental region. By $S_i(E)$ we denote the set into which E is transformed by $S_i(z)$. We put $S_i(C) = C_i$ and $S_i(0) = a_i$. Let S be the region bounded by $\sum_{i=1}^{\infty} C_i$ and by the unit circle, then a well known theorem is stated as follows.

Theorem (I). If $\sum_{i=0}^{\infty} (1 - |a_i|) = \infty$, then S is a 0-region. (II) If $\sum_{i=0}^{\infty} (1 - |a_i|) < \infty$, then S is a positive-region.

The second part of the theorem is a simple result derived from the preceding corollary. On the basis of this theorem we will say that in case (I) G is a Fuchsoid group of O-type and in case (II) G is a Fuchsoid group of P-type.

Exp.2. Let $\{a_k = r_k e^{i\theta_k}\}$ be a sequence of complex numbers such that $r_k \leq r_{k+1}$, $\lim_{k\to\infty} r_k = 1$, and let C_i be a segment connecting a_i and $e^{i\theta_k}$. Denoting by S the region bounded by $\sum_{i=1}^{\infty} C_i$ and by the unit circle $|z| = 1$ then we have the following theorem:

Theorem (Myrberg's theorem). If $\sum_{i=1}^{\infty} (1 - |a_i|) < \infty$, S is a positive-region.

If G is a Fuchsoid group of the P-type, S is a positive-region. But the author has not yet solved the following problem:

If G is a Fuchsoid group of O-type, is S a 0-region ?

We denote by $S(r, \alpha, \beta)$ the sector in which $\alpha < \arg z < \beta$ and $|z| \leq r$, and by $n(r, \alpha, \beta)$ the number of a_i which are contained in $S(r, \alpha, \beta)$. Then we can easily prove the next theorem:

If S is a 0-region, $\int_0^1 n(r, \alpha, \beta) dr = \infty$ for any pair (α, β).

Exp. 3. Let L_i be a circle which is orthogonal to the unit circle $|z|=1$, and let C_i be a subarc of L_i contained in $|z|<1$. We suppose that C_i does not intersect C_j $(i \neq j)$ in the unit circle $|z|<1$. Then we get next theorem:

Theorem (Myrberg's theorem). If $\beta=0$, S is a 0-region and, if $\beta>0$, S is a positive-region.

§ 6. We will consider the particular case such that all C_n are closed analytic curves. If the following conditions are satisfied, we will say that a point P on the unit circle is a normal point:

1. $\angle OPA = \angle OPB \neq 0$, where O is the origin;

2. there are finite number of C_n which are entirely contained in $\angle APB$.

3. PA and PB intersect finite number of C_n;

4. $PA \perp OA$ and $PB \perp OB$.

Theorem 7. Let B be a set of normal points. If linear measure (E) is positive, S is a positive-region.

Proof. Let E be a set of normal points. By $N(r,\alpha,\theta)$ we denote a number of C_n which are contained in $\angle APB$ or intersect AP or BP in the circle $|z|<r$, where $P: z=e^{i\theta}$ and $APB=\alpha$. Since for fixed r and α, $N(r,\alpha,\theta)$ is a measurable function of θ and $N(r,\alpha,\theta)$ is a monotone increasing function of r, we see that $\lim_{r\to 1} N(r,\alpha,\theta) = N(\alpha,\theta)$ is a measurable function of θ. By $E(m,n)$ we denote the subset of E such that at $P(z=e^{i\theta}) \in E(m,n)$, $N(\frac{1}{m},\theta)=n$. Then $E(m,n)$ is a measurable set and $\sum_{m,n} E(m,n) = E$. Since $m(E)\neq 0$, there exists a set $E(m_0,n_0)$ such that, at $P\in E(m_0,n_0)$, $N(1/m_0,\theta)=n_0$ and $m(E(m_0,n_0))\neq 0$. We denote by $E(m_0,n_0,k)$ a subset of $E(m_0,n_0)$ such that, if $P\in E(m_0,n_0,k)$, there exists no C_n in the annulus $1-1/k \leq |z| < 1$ which is contained in APB or intersect AP or BP. For fixed k, we denote by $n(1/m_0,k,r,\theta)$ the number of C_n in the annulus $1-1/k \leq |z| < r < 1$ which are entirely contained in $\angle APB$ or intersect AP or BP, where $\angle APB = 1/m_0$ $P: z=e^{i\theta}$. Since for fixed r and k, $n(1/m_0,k,r,\theta)$ is a measurable function of θ and is a monotone increasing function of r, we see that $\lim_{r\to 1} n(1/m_0,k,r,\theta) = n(1/m_0,k,\theta)$ is a measurable function. Therefore, for a fixed k, $E(m_0,n_0,k)$ is a measurable set of $e^{i\theta}$ and $\sum_{k=1} E(m_0,n_0,k) = E(m_0,n_0)$. Hence there exists a set $E(m_0,n_0,k_0)$ such that at $P\in E(m_0,n_0,k_0)$, $N(1/m_0,\theta)=n_0$ and there is no C_n in the annulus $1-1/k_0 \leq |z| < 1$ which is entirely contained in $\angle APB$ or intersects AP or BP and $m(E(m_0,n_0,k_0))\neq 0$.

We put $E(m_0,n_0,k_0)=F$. Constructing an angle $APB=1/m_0$ at each point of F, we get the region D_F bounded by sides of these angles, we see that there exist a finite number of C_n which are entirely contained in D_F or intersect the boundary. Let D_G be the region bounded by these C_n and by the boundary of D_F. Then $U(z) = \omega(z,F,D_G)\neq 0$, and if $r_k > 1-1/k_0$, we have $u_k(z) > U(z)$. Hence S is a positive-region.

Theorem (Myrberg-Tuji-Yûjôbô). Let G be a Fuchsoid group of 0-type. Let z^* be an equivalent point of z which lies in the fundamental region D. As z moves along the radius $(0,e^{i\theta})$, z^* moves on the set $M(\theta)$ in D. Then $M(\theta)$ is everywhere dense in D, except for θ belonging to a set of measure 0.

Proof. Let Q be any rational point in D, and let $C(Q,m)$ be a non-Euclidean circle with center at Q and radius m. We suppose that $C(Q,m)$ is entirely contained in D for a rational number m. Let $F(\theta)$ be a set of θ such that $M(\theta)$ and $C(Q,m)$ have no common part. We will prove that $m(F(\theta))=0$. Assuming that $m(F(\theta))\neq 0$ we will show that this assumption is absurd. Let S be the region bounded by $\sum_n S_n(C(Q,m))$ and by the unit circle, then, by Theorem 6, S is a 0-region. If $P(z=e^{i\theta})$ belongs to $F(\theta)$, we construct non-Euclidean circle with radius $\frac{1}{2}m$ and center at each point on OP. The region swept by these circles contains the angle APB such that there is no C_n contained in APB or intersected by AP or BP, and $\angle APB = k(m) \neq 0$, where $k(m)$ is a constant depending upon m. By Theorem 7 S is a positive-region. This is absurd. Thus $m(F(\theta))=0$. Hence Q is a limiting point of $M(\theta)$ for every θ except for some θ belonging to a set of measure 0. Thus our theorem has been proved.

Chapter II .

§ 1. In this chapter we suppose that S is a O-region. Let $V(z)$ be a bounded function: $|V(z)| < L$, which is harmonic in S and is 0 on Γ. Putting $u_n(z) = \omega(z, A(r_n), \Delta(r_n))$ and $w_n(z) = \frac{V(z)+L}{L} - (1-u_n(z))$, $w_n(z)$ is harmonic in $\Delta(r_n)$, and $w_n(z) = 0$ on $B(r_n)$, and $w_n(z) \geq 0$ on $A(r_n)$. Hence $w_n(z) \geq 0$ in $\Delta(r_n)$.

$\frac{V(z)+L}{L} \geq 1 - u_n(z)$ in $\Delta(r_n)$. Since $\lim_{n\to\infty} u_n(z) = 0$, we have $\frac{V(z)+L}{L} \geq 1$ in S, and therefore $V(z) \geq 0$. Replacing $V(z)$ by $-V(z)$, we get $1-V(z) \geq 0$. Hence $V(z) = 0$. Thus we have the following theorem.

Theorem 8. If S is a O-region, there exists no non-zero bounded harmonic function in S which is O on Γ.

Let $U(z)$ be a bounded harmonic function in S; $|U(z)| < L$. We suppose that $U(z)$ is continuous on Γ. Let $V_n(z)$ be a harmonic function in $\Delta(r_n)$ such that $V_n(z) = U(z)$ on $B(r_n)$ and is 0 on $A(r_n)$. Putting $W_{n+p,n}(z) = V_{n+p}(z) - V_n(z)$, $W_{n+p,n}(z)$ is harmonic in $\Delta(r_n)$ and is O on $B(r_n)$. Now,

$$|W_{n+p,n}(z)| = |V_{n+p}(z) - V_n(z)| < 2L.$$

Putting $U_{n+p,n}(z) = \frac{W_{n+p,p}(z) + 2L}{2L} - (1-u_n(z))$, then $U_{n+p,n}(z)$ is harmonic in $\Delta(r_n)$ and is 0 on $B(r_n)$ and $U_{n+p,n}(z) \geq 0$ on $A(r_n)$. Hence $U_{n+p,n}(z) \geq 0$ in $\Delta(r_n)$. Since $\lim_{n\to\infty} u_n(z) = 0$, we get $\lim_{n\to\infty} W_{n+p,n}(z) \geq 0$. Applying the same argument to $-W_{n+p,n}(z)$, we get

$$\lim_{n\to\infty} -W_{n+p,n}(z) \geq 0, \ \overline{\lim_{n\to\infty}} W_{n+p,n}(z) \leqq 0.$$

Hence $\lim_{n\to\infty} W_{n+p,n}(z) = 0$. Therefore, $V_n(z)$ converges to a harmonic function $V(z)$. Putting $W(z) = U(z) - V(z)$, $w(z)$ is a bounded harmonic function in S and is O on Γ. By Theorem 8, we have $U(z) = V(z)$. Thus we have the following theorem:

Theorem 9. Let $U(z)$ be a bounded harmonic function in S. If $U(z)$ is continuous on Γ, $U(z)$ is uniquely determined by boundary values given on Γ.

Theorem 10. If G be a Fuchsoid group of O-type, then there exists no non-constant, bounded, harmonic and automorphic function.

Proof. We suppose that this proposition is false. Let $u(z)$ be a function stated in theorem. Without loss of generality, we may assume that

$$m = \text{least upper bound of } u(z),$$
$$-m = \text{greatest lower bound of } u(z),$$
$$\text{and} \quad m > 0.$$

Let z_0 be a point in the fundamental region D such that $u(z_0) > 0$. Then there exists a non-Euclidean circle C with center z_0 such that C is entirely contained in D and $u(z) > 0$ on C. Let S be the region bounded by $\sum_n S_n(C)$ and by the unit circle, then S is a 0-region. By the same argument as theorem 8 $u(z) > 0$ in S. Since $u(z) > 0$ in C and $u(z)$ is an automorphic function, $u(z) > 0$ in $S_n(C)$. Thus $u(z) > 0$ in the unit circle. This is inconsistent with our assumption $0 > -m =$ greatest lower bound; the proposition has thus been proved.

Corollary (Myrberg's theorem). There exists no non-constant, bounded harmonic function on the "Null berandet" Riemann Surface.

§ 2. Theorem 11. We suppose that S is a 0-region, and $w = f(z) = u(z) + iv(z)$ is a meromorphic function in S. If $u(z)$ is 0 on Γ and $u(z) > 0$ in S, then $w = f(z)$ takes every value w ($R(w) > 0$), except possibly some values of w belonging to a set of capacity 0.

Proof. Putting $\xi = F(z) = \frac{1-f(z)}{1+f(z)}$, $|\xi| < 1$. We will assume that $f(z)$ does not take values w belonging to a set E_w contained in the half plane $R(w) > 0$ and E_w is of positive capapcity. Then $F(z)$ does not take values ξ belonging to a set E contained in the circle $|\xi| < 1$ and E is of positive capacity. Without loss of generality, we may assume that E is entirely contained in the circle $|\xi| < \rho < 1$. Let $G(\xi, \eta)$ be a Green's function of the unit circle with a pole at η. There exists a distribution of positive mass $d\mu(\eta)$ on E so that its potential $g(\xi) = \int_E G(\xi,\eta)\, d\mu(\eta)$ is bounded and non-constant. Putting $K(z) = g(F(z))$, $K(z)$ is a bounded harmonic function and is 0 on Γ. Hence we have, by Theorem 8, $K(z) \equiv 0$. This is absurd. Thus our theorem has been proved.

Theorem 12. Let G be a Fuchsoid group of O-type, and let $w = f(z)$ be an automorphic function with respect to G. Then $f(z)$ takes all values w, except some values of w belonging to a set of capacity 0.

As this theorem can be easily proved, we omit its proof.

Corollary. If f is a meromorphic function on the Null berandet Riemann Surface, f takes every value except some values belonging to a set of capacity 0.

By Theorem 1 we have next theorem:

Theorem 13. If $\int_0^1 \frac{1}{r\theta(r)} dr = \infty$, then there exists no non-constant harmonic function such that it is 0 on Γ and its Dirichlet's integral is finite.

(*) Received October 10, 1950.

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