ON HARMONIC FIELD IN RIEMANNIAN MANIFOLD

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Recently S. Bochner has treated the estimation of Betti-numbers of Riemannian manifold by means of locally defined curvature, [2] , [3] . Its radical idea was based on Hodge-de Rham's isomorphy theorem between harmonic integrals and Betti-numbers, but this shows an important aspect of the differential geometry which connects the grobal theory and local properties of the Riemannian manifold.

Now, in the present paper we shall clarify the characteristic property of Lapalcian operated to harmonic tensor fields, and then reproduce some of Bochner's results with some modifications and supplements. Using these results we shall give new viewpoints on T.Y.Thomas' theory on metric tensors, [6] , [7] . By the treatment, we can also show an important meaning of so called "index", defined by T.Y.Thomas.

1. Definitions.

$\mathfrak{M}$ be a compact or non-compact orientable Riemannian manifold with positive definite fundamental quadratic differential form

$$ds^2 = g_{\lambda\mu}\, dx^\lambda dx^\mu.$$

We assume that $\mathfrak{M}$ and $g_{\lambda\mu}$ belong to differentiable class u and u-1 (u $\geq$ 3), respectively. We consider, on $\mathfrak{M}$, the harmonic tensor fields with class u $\geq$ 2:

$$\xi_{\alpha_1\cdots\alpha_p} = \xi_{[\alpha_1\cdots\alpha_p]} \quad ,$$

$$(D\xi)_{\alpha_1\cdots\alpha_{p+1}} = \sum_{k=1}^{p+1}(-1)^{k-1}\xi_{\alpha_1\cdots\hat{\alpha}_k\cdots\alpha_{p+1};\alpha_k} = 0,$$

$$(D^*\xi)_{\alpha_1\cdots\alpha_{p-1}} = \xi_{\alpha_1\cdots\alpha_{p-1}\lambda;\mu}\, g^{\lambda\mu} = 0,$$

where " ; " denotes the covariant differentiation.

Definition. A tensor $T_{\alpha_1\cdots\alpha_p}$ on $\mathfrak{M}$ is called a restrained tensor, if either

$$\Delta\phi(P_0) < 0$$

at some point P_0 in $\mathfrak{M}$, or

$$\phi(P) = \text{const.}$$

throughout $\mathfrak{M}$, where

$$\phi = T_{\alpha_1\cdots\alpha_p} T_{\beta_1\cdots\beta_p}\, g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}$$

and

$$\Delta\phi = g^{\lambda\mu}\phi_{;\lambda;\mu}$$

Lemmas.

I. If $\mathfrak{M}$ is compact, tensor field is restrained, [1] .

II. A covariant constant tensor field is always restrained.

Defining the Laplacian for any skew symetric tensor field as follow:

$$\Delta\xi_{\alpha_1\cdots\alpha_p} = \xi_{\alpha_1\cdots\alpha_p;\lambda;\mu}\, g^{\lambda\mu},$$

we obtain the expression:

$$\begin{aligned}\Delta\xi_{\alpha_1\cdots\alpha_p} &= \sum_{(j<k)}^{1\cdots p}(-1)^{j+k} R^{\lambda\mu}{}_{\alpha_j\alpha_k}\,\xi_{\lambda\mu\alpha_1\cdots\hat{\alpha}_j\cdots\hat{\alpha}_k\cdots\alpha_p} \\ &\quad + \sum_{k=1}^{p}(-1)^k R^{\lambda}_{\cdot\alpha_k}\,\xi_{\lambda\alpha_1\cdots\hat{\alpha}_k\cdots\alpha_p} \qquad (1.1) \\ &\quad + (-1)^p (D^*D - DD^*)\,\xi_{\alpha_1\cdots\alpha_p} ,\end{aligned}$$

where $\sum_{(j<k)}^{1\cdots p}$ means the summation over all combination (j,k) ordered by j < k, and

$$R^{\lambda}_{\mu\nu\omega} = \frac{\partial\left\{{\lambda \atop \mu\nu}\right\}}{\partial x^\omega} - \frac{\partial\left\{{\lambda \atop \mu\omega}\right\}}{\partial x^\nu} + \left\{{\alpha \atop \mu\nu}\right\}\left\{{\lambda \atop \alpha\omega}\right\} - \left\{{\alpha \atop \mu\omega}\right\}\left\{{\lambda \atop \alpha\nu}\right\},$$

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda}\ , \qquad R^{\lambda\mu}{}_{\nu\omega} = g^{\mu\alpha}R^{\lambda}{}_{\alpha\nu\omega},$$

$$R^{\lambda}_{\cdot\mu} = g^{\lambda\alpha}R_{\alpha\mu}.$$

In culculation we have used the formulae:

$$\begin{aligned}A_{\alpha_1\cdots\alpha_p;\lambda;\mu} &- A_{\alpha_1\cdots\alpha_p;\mu;\lambda} \\ &= \sum_{k=1}^{p} R^{\alpha}{}_{\alpha_k\lambda\mu}\, A_{\alpha_1\cdots\alpha_{k-1}\alpha\alpha_{k+1}\cdots\alpha_p} ,\end{aligned}$$

$$R^{\lambda\cdot\cdot\mu}{}_{\alpha_j\cdot\alpha_k} - R^{\lambda\cdot\cdot\mu}{}_{\alpha_k\alpha_j\cdot} = -R^{\lambda\mu}{}_{\alpha_k\alpha_j}.$$

Especially, if $\xi_{\alpha_1\cdots\alpha_p}$ is harmonic, then (1.1) becomes

$$\begin{aligned}\Delta\xi_{\alpha_1\cdots\alpha_p} &= \sum_{(j<k)}^{1\cdots p}(-1)^{j+k} R^{\lambda\mu}{}_{\alpha_j\alpha_k}\,\xi_{\lambda\mu\alpha_1\cdots\hat{\alpha}_j\cdots\hat{\alpha}_k\cdots\alpha_p} \qquad (1.2) \\ &\quad + \sum_{k=1}^{p}(-1)^k R^{\lambda}_{\cdot\alpha_k}\,\xi_{\lambda\alpha_1\cdots\hat{\alpha}_k\cdots\alpha_p} .\end{aligned}$$

Introducing the notations:

$$(1.3)\quad \begin{aligned} &\xi_{\alpha_1\cdots\alpha_p}\eta_{\beta_1\cdots\beta_p}g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}=(\xi,\eta)\ ,\\ &\xi_{\alpha_1\cdots\alpha_p;\alpha_{p+1}}\eta_{\beta_1\cdots\beta_p;\beta_{p+1}}g^{\alpha_1\beta_1}\cdots g^{\alpha_p\beta_p}g^{\alpha_{p+1}\beta_{p+1}}\\ &\qquad =(\xi',\eta')\ , \end{aligned}$$

one can easily obtain the formula:

$$(1.4)\qquad \frac{1}{2}\Delta\phi=(\Delta\xi,\xi)+(\xi',\xi')\ ,$$

where

$$\phi=(\xi,\xi)\ .$$

If ξ is restrained, then

$$(1.5)\qquad \Delta\phi \underset{*}{\gtreqless} 0 \longrightarrow \Delta\phi=0\ ,$$

where $\underset{*}{\gtreqless}$ means that the strict inequality hold at least a point in the concerning domain, and the condition

$$(1.6)\qquad (\Delta\xi,\xi)\geqq 0$$

is sufficient for (1.5). Now, following [1], [2] and [3], we shall investigate the quantity appeared in the right hand member in (1.6).

From (1.2) we obtain

$$(1.7)\quad (\Delta\xi,\xi)=-\frac{p}{2}{}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu\alpha_3\cdots\alpha_p}\xi^{\nu\omega}{}_{\alpha_3\cdots\alpha_p},$$

where

$$\begin{aligned}{}_{(p)}H_{\lambda\mu\nu\omega}=&(p-1)R_{\lambda\mu\nu\omega}\\ &+\frac{1}{2}\left(R_{\lambda\nu}g_{\mu\omega}-R_{\lambda\omega}g_{\mu\nu}+g_{\lambda\nu}R_{\mu\omega}-g_{\lambda\omega}R_{\mu\nu}\right).\end{aligned}$$

2. Fundamental Theorem.

Now, we can conclude imediately

Theorem 1. For any non zero tensor $\xi^{\lambda\mu}=\xi^{[\lambda\mu]}$ or ξ^λ, if the conditions

$$(2.1)\quad \begin{aligned} &{}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu}\xi^{\nu\omega}\underset{*}{\leqq}0 \qquad \text{for } 1<p\leqq n,\\ &R_{\lambda\mu}\xi^\lambda\xi^\mu \underset{*}{\leqq} 0 \qquad \text{for } p=1\end{aligned}$$

hold on $\mathfrak{M}$, there exist no restrained harmonic p-fields.

Theorem 2. For any non zero tensor $\xi^{\lambda\mu}=\xi^{[\lambda\mu]}$ or ξ^λ, if the conditions

$$(2.2)\quad \begin{aligned} &{}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu}\xi^{\nu\omega}\underset{*}{\geqq}0 \qquad \text{for } 1<p\leqq n,\\ &R_{\lambda\mu}\xi^\lambda\xi^\mu \underset{*}{\geqq} 0 \qquad \text{for } p=1\end{aligned}$$

hold on $\mathfrak{M}$, there exists no restrained p-field such that

$$(2.3)\qquad \xi_{\alpha_1\cdots\alpha_p;\alpha_{p+1}}=\xi_{[\alpha_1\cdots\alpha_p;\alpha_{p+1}]}.$$

Proof. We use (1.1). If we had (2.3), then $D^*\xi=0$, and

$$(D\xi)_{\alpha_1\cdots\alpha_{p+1}}=(-1)^p(p+1)\xi_{\alpha_1\cdots\alpha_p;\alpha_{p+1}}\ ,$$

$$\therefore (D^*D-DD^*)\xi_{\alpha_1\cdots\alpha_p}=(-1)^p(p+1)\Delta\xi_{\alpha_1\cdots\alpha_p},$$

$$\begin{aligned}\therefore (\Delta\xi,\xi)=&(p+1)(\Delta\xi,\xi)\\ &-\frac{p}{2}{}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu\alpha_3\cdots\alpha_p}\xi^{\nu\omega}{}_{\alpha_3\cdots\alpha_p},\end{aligned}$$

$$\therefore (\Delta\xi,\xi)=\frac{p}{2}{}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu\alpha_3\cdots\alpha_p}\xi^{\nu\omega}{}_{\alpha_3\cdots\alpha_p}.$$

Hence, from (1.4), we can conclude the theorem, as theorem 1.

For p=1, (2.3) is reduced to

$$\xi_{\lambda;\mu}+\xi_{\mu;\lambda}=0\ ,$$

therefore, as Bochner has shown, we have

Corollary 1. For any vector ξ^λ, if the condition

$$R_{\lambda\mu}\xi^\lambda\xi^\mu \underset{*}{\geqq} 0$$

holds, $\mathfrak{M}$ can not admit the restrained one parameter group of motion, [1].

Definition. In the domain $\mathcal{D}$, for any non zero tensor field $\xi^{\lambda\mu}=\xi^{[\lambda\mu]}$ or ξ^λ, if the condition (2.2) holds, then we say that ${}_{(p)}H_{\lambda\mu\nu\omega}$ or $R_{\lambda\mu}$ has the positive condition in $\mathcal{D}$, and if (2.1), then negative condition. If at least one of them holds, then we say that they have definite condition.

Theorem 3. On $\mathfrak{M}$, if ${}_{(p)}H_{\lambda\mu\nu\omega}$ (for $1<p\leqq n$) or $R_{\lambda\mu}$ (for p=1) has the definite condition, there exists no harmonic field such that

$$(2.4)\qquad \Delta\xi_{\alpha_1\cdots\alpha_p}=0\ ,$$

Proof. If harmonic field $\xi_{\alpha_1\cdots\alpha_p}$ has the condition (2.4), then

$$(\Delta\xi,\xi)=0\ ,$$

which implies

$${}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu\alpha_3\cdots\alpha_p}\xi^{\nu\omega}{}_{\alpha_3\cdots\alpha_p}=0$$

or

$$R_{\lambda\mu}\xi^\lambda\xi^\mu=0$$

If ${}_{(p)}H_{\lambda\mu\nu\omega}$ and $R_{\lambda\mu}$ have the definite condition, we have

$$\xi_{\alpha_1\cdots\alpha_p}=0\ ,$$

which proves the theorem.

Remark. In this statement we have not concerned the condition that the fields ξ's are restrained. But, if ξ's are restrained, then evidently

$$(*)\quad \Delta\xi_{\alpha_1\cdots\alpha_p}=0 \rightleftarrows \xi_{\alpha_1\cdots\alpha_p;\alpha_{p+1}}=0.$$

Therefore, Theorem 3 shows the condition for the non existence of covariant constant p-field.

Iwamoto [5] has studied the differential form

$$\xi_{\alpha_1\cdots\alpha_p}dx^{\alpha_1}\cdots dx^{\alpha_p}\qquad \text{(p-form)}$$

whose coefficients are covariantly constant. These forms constitute an important class of p-forms which are invariant by the group of holonomy.

Now, if $\xi_{\alpha_1\cdots\alpha_p;\omega}=0$, then evidently $\xi_{\alpha_1\cdots\alpha_p}$ is restrained, and therefore we have $(*)$. Hence, we can rewrite the Theorem 3 as follows:

Theorem 3$'$. On $\mathfrak{M}$, if ${}_{(p)}H_{\lambda\mu\nu\omega}$ (for $1<p\leq n$) or $R_{\lambda\mu}$ (for p=1) have the definite condition, there exists no no p-form which is invariant by the group of holonomy.

Now, we shall proceed to some special cases, which are locally conformally flat, of constant curvature and Einstein space. First we take the case where the space is locally conformally flat, that is

$$R^{\lambda}{}_{\mu\nu\omega}=\frac{1}{n-2}\left(R_{\mu\nu}\delta^{\lambda}_{\omega}-R_{\mu\omega}\delta^{\lambda}_{\nu}+g_{\mu\nu}R^{\lambda}_{\omega}-g_{\mu\omega}R^{\lambda}_{\nu}\right)+\frac{R}{(n-1)(n-2)}\left(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu}\right).$$

In this case, we have

$$\begin{aligned}{}_{(p)}H_{\lambda\mu\nu\omega}&=\left(\frac{1}{2}-\frac{p-1}{n-2}\right)\left(R_{\lambda\nu}g_{\mu\omega}-R_{\lambda\omega}g_{\mu\nu}+g_{\lambda\nu}R_{\mu\omega}-g_{\lambda\omega}R_{\mu\nu}\right)\\&+\frac{(p-1)R}{(n-1)(n-2)}\left(g_{\lambda\nu}g_{\mu\omega}-g_{\lambda\omega}g_{\mu\nu}\right).\end{aligned}$$

$$\begin{aligned}\therefore\ {}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu}\xi^{\nu\omega}&=4\left(\frac{1}{2}-\frac{p-1}{n-2}\right)R_{\lambda\mu}\xi^{\lambda\alpha}\xi^{\mu}{}_{\alpha}+\frac{2(p-1)R}{(n-1)(n-2)}\xi_{\lambda\mu}\xi^{\lambda\mu}\\&=\frac{2(n-2p)}{n-2}R_{\lambda\mu}\xi^{\lambda\alpha}\xi^{\mu}{}_{\alpha}+\frac{2(p-1)R}{(n-1)(n-2)}\xi_{\lambda\mu}\xi^{\lambda\mu}.\end{aligned}$$

From this last expression, we can conclude that, if $n\geqq 2p$, the positive condition or negative conditions of ${}_{(p)}H_{\lambda\mu\nu\omega}$ reduce to those of $R_{\lambda\mu}$. Thus we have

Theorem 4. On a locally conformally flat $\mathfrak{M}$, there can not exist:

(1) the restrained p-field ($p\leqq n/2$), if $R_{\lambda\mu}$ have the negative condition;

(2) the restrained p-field ($p\leqq n/2$) which satisfies the condition (2.3), if $R_{\lambda\mu}$ have the positive condition;

(3) the harmonic p-field ($p\leqq n/2$) which satisfies the condition (2.4), if $R_{\lambda\mu}$ have the definite condition.

Next, we shall proceed to the case of constant curvature space, which is characterized by the condition

$$R^{\lambda}{}_{\mu\nu\omega}=-\frac{R}{n(n-1)}\left(\delta^{\lambda}_{\nu}g_{\mu\omega}-\delta^{\lambda}_{\omega}g_{\mu\nu}\right).$$

In this case, we have

$${}_{(p)}H_{\lambda\mu\nu\omega}=\frac{(n-p)R}{n(n-1)}\left(g_{\lambda\nu}g_{\mu\omega}-g_{\lambda\omega}g_{\mu\nu}\right).$$

$$\therefore\ {}_{(p)}H_{\lambda\mu\nu\omega}\xi^{\lambda\mu}\xi^{\nu\omega}=\frac{2(n-p)}{n(n-1)}R\,\xi_{\lambda\mu}\xi^{\lambda\mu}.$$

From the last expression we can conclude that the positive or negative conditions of ${}_{(p)}H_{\lambda\mu\nu\omega}$ reduce to those of R, and therefore we have the following:

Theorem 5. On a space of locally constant curvature $\mathfrak{M}$, there can not exist

(1) the restrained harmonic p-field, if $R\lessgtr 0$, [2];

(2) the restrained p-field which satisfies the condition (2.3), if $R\gtrless 0$;

(3) the harmonic p-field which satisfies (2.4), if $R\lessgtr 0$ or $R\gtrless 0$.

On the other hand, we have broader case which contains conformally flat spaces and spaces of constant curvature, that is Einstein spaces which are characterized by the condition

$$R_{\lambda\mu}=\frac{1}{n}Rg_{\lambda\mu}.$$

In this case, the definite conditions on $R_{\lambda\mu}$ will be reduced to the conditions on R, therefore we have

Theorem 6. On a locally Einstein $\mathfrak{M}$, there can not exist

(1) the restrained harmonic 1-field, if $R\lessgtr 0$;

(2) the restrained 1-field which satisfies the condition (2.3), if $R \gtrless 0$;

(3) the harmonic 1-field which satisfies the condition (2.4), if $R \gtrless 0$ or $R \lesseqgtr 0$.

Now, from the Theorem 3′, we can conclude the

Corollary 1. If $\mathfrak{M}$ is a space of locally constant curvature, there can not exist the covariant constant p-field, in other words, on such a space $\mathfrak{M}$, there is no p-form which is invariant by group of holonomy.

3. Estimation of Betti-numbers of the compact Riemannian manifold.

In case where $\mathfrak{M}$ is compact, by means of Hodge-de Rham's isomorphy theorem between harmonic integrals and Betti-numbers, we can use the Theorem 1 for estimation of the Betti-numbers of manifold $\mathfrak{M}$. (see [2]), that is

Theorem 5. On a compact $\mathfrak{M}$, if ${}_{(p)}H_{\lambda\mu\nu\omega}$ has the negative condition,

$$B_p = 0 \qquad (1 < p < n-1),$$

where Bp denotes the p-th Betti-numbers of $\mathfrak{M}$, and if $R_{\lambda\mu}$ has the negative condition,

$$B_1 = B_{n-1} = 0 \qquad .$$

Corollary 1. On a locally conformally flat $\mathfrak{M}$, if $R_{\lambda\mu}$ has the negative condition,

$$B_p = 0 \quad , \quad 0 < p < n.$$

Corollary 2. On $\mathfrak{M}$ of locally constant curvature, if $R \lesseqgtr 0$, then

$$B_p = 0 \quad , \quad 0 < p < n.$$

Corollary 3. On a locally Einstein space $\mathfrak{M}$, if $R \lesseqgtr 0$, then

$$B_1 = 0 \quad .$$

Theorem 6. If a compact $\mathfrak{M}$ is simply connected, then there can not exist any parallel vector field, [8] .

Proof. Let $\pi_1(\mathfrak{M})$ be the fundamental group (Poincaré group) of $\mathfrak{M}$, and C be its commutator subgroup. We have a well known relation

$$\pi_1(\mathfrak{M})/C \cong \mathfrak{B}_1(\mathfrak{M}),$$

where $\mathfrak{B}_1(\mathfrak{M})$ denotes the 1-dimensional integral homology group of $\mathfrak{M}$. If $\pi_1 = 0$ (simply connected), then we have $\mathfrak{B}_1 = 0$ also, and therefore we can not have any harmonic one-field on $\mathfrak{M}$ On the other hand, the parallel vector field is also harmonic. Thus, we can conclude the result.

Now, we can generalize Theorem 6 as follows:

Theorem 7. Let $\pi_k(\mathfrak{M})$ be k-th homotopy group of $\mathfrak{M}$. If

$$(3.1) \quad \pi_k(\mathfrak{M}) = 0 \qquad \text{for} \quad 1 \leqq k \leqq p ,$$

then there can not exist any harmonic field of k-th order ($1 \leqq k \leqq p$). And then, there can exist no k ($1 \leqq k \leqq p$)-. form which is invariant by group of holonomy.

Proof. Let $\mathfrak{B}_p$ be k-th integral homology groups of $\mathfrak{M}$. Being $\pi_1 = \cdots = \cdots \pi_{k-1} = 0$, we have, by Hurewicz's theorem,

$$\pi_k \cong \mathfrak{B}_k .$$

Therefore, (3.1) implies

$$\mathfrak{B}_k = 0 \qquad \text{for} \quad 1 \leqq k \leqq p$$

and

$$B_k = 0 \qquad \text{for} \quad 1 \leqq k \leqq p .$$

On the other hand, since the covariant constant tensor field is also harmonic, under the condition (3.1) there exists no k ($1 \leqq k \leqq p$)-. form which is invariant by group of holonomy.

Remark. We could easily reproduce all the Bochner's results in [2] [3] , but we have only stated the main part, and we investigate some other topics. They may be considered as the applications of Bochner's estimation theorem, and one can recognize the importance of this method, by those examples.

Now, we shall draw our attention to the important fact that for any harmonic tensor fields $\Delta\xi_{\alpha_1 \cdots \alpha_p}$ does not vanish, and this relation contains $\xi_{\alpha_1 \cdots \alpha_p}$ itself and local curvature quantities of $\mathfrak{M}$, but not contain its own derivatives. This is a very simple fact, but is an important property of harmonic tensors. Our present method is based just on this property.

4. Metric tensors in $\mathfrak{M}$. [6] , [7] .

Let the symmetric tensor $T_{\lambda\mu}$ of order two be a solution of the differential equations

$$(4.1) \qquad T_{\lambda\mu;\alpha} = 0 .$$

We call it the metric tensor on $\mathfrak{M}$. Index I of $\mathfrak{M}$ denotes the maximal numbers of the metric tensors of $\mathfrak{M}$ which are algebraically independent with constant real coefficients. Let I_p be the maximal number of metric tensors which have the rank p and essentially differ from each other, i.e. none of them is constant multiple of any other.

Thus

$$(4.2) \qquad \sum I_p = I \quad .$$

Existence of a metric tensor with rank p is equivalent to existence of parallel vector fields V_p and V_{n-p} which are perpendicular to each other. Therefore, we remark that

$$(4.3) \qquad I_p = I_{n-p}$$

Normalizing the basis $\overset{\wedge}{\xi}_1 \cdots\cdots \overset{\lambda}{\xi}_p$ of V_p, we consider the current Plücker coordinates

$$\pi_{\alpha_1 \cdots \alpha_p} = \begin{vmatrix} \xi_{1\alpha_1} & \cdots\cdots & \xi_{1\alpha_p} \\ \cdot & & \vdots \\ \dot{\xi}_{p\alpha_1} & \cdots\cdots & \xi_{p\alpha_p} \end{vmatrix} ,$$

then we have

$$\pi_{\alpha_1 \cdots\cdots \alpha_p} = \pi_{[\alpha_1 \cdots\cdots \alpha_p]} \quad ,$$

$$\pi_{\alpha_1 \cdots\cdots \alpha_p ; \beta} = 0 \quad ,$$

thus Theorem 3 implies

Theorem 8. On $\mathfrak{M}$, if ${}_{(p)}H_{\lambda\mu\nu\omega}$ (for $1<p\leq n$) or $R_{\lambda\mu}$ (for p=1) has the definite condition,

$$I_p = 0 \qquad \text{for} \qquad 0 < p < n \quad .$$

Now, we consider the compact case. Let B_p be the Betti-number of $\mathfrak{M}$ and B'_p be the maximal number of linearly independent covariant constant skew symmetric covariant tensor fields which is maximal number of linearly independent differential forms, invariant by group of holonomy of this space. Then, we can conclude the

Theorem 9. On a compact $\mathfrak{M}$, we have

$$B_p \geq B'_p \geq I_p \qquad \text{for} \qquad n > p > 0.$$

Corollary 1. On a compact $\mathfrak{M}$, if

$$\pi_k(\mathfrak{M}) = 0 \ \text{for} \quad 1 \leq k \leq p \quad ,$$

then

$$I_k = 0 \quad \text{for} \quad 1 \leq k \leq p \quad .$$

Corollary 2. On a compact $\mathfrak{M}$, if

$$B_p = 0 \qquad \text{for} \qquad 0 < p < n \quad ,$$

then there can not exist any metric tensor which essentially differs from the fundamental tensor of $\mathfrak{M}$.

Theorem 10. On a compact $\mathfrak{M}$ (connected), we have

$$B_0 = B'_0 = I_0 = I_n = B'_n = B_n = 1 \quad .$$

Proof. We have immediately $B_n = B_0 = 1$, and, on the other hand, $I_0 = I_n \geq 1$, for fundamental tensor itself is metric tensor in our sense.

Theorem 11. On any $\mathfrak{M}$, $B'_{n-1} = B'_1 = I_{n-1} = I_1$ and especially, if a compact $\mathfrak{M}$ is simply connected, then we have

$$B'_{n-1} = B'_1 = I_{n-1} = I_1 = 0 \quad .$$

Proof. $T_{\alpha_1 \cdots \alpha_{n-1}} = T_{[\alpha_1 \cdots \alpha_{n-1}]}$ are always simple. Therefore, if $T_{\alpha_1 \cdots \alpha_{n-1};\beta} = 0$, then their bases span the parallel vector space. Thus we can conclude the first half of the theorem. The second half can be directly concluded from Theorem 9, corollary 1.

Theorem 12. If we have $B'_p \geq 1$ for some p, and

$$T_{\alpha\beta} = \pi_{\lambda_1 \cdots \lambda_{p-1}\alpha}\, \pi^{\lambda_1 \cdots \lambda_{p-1}}{}_{\beta} \neq \phi\, g_{\alpha\beta} \ ,$$

$$\text{Rank}\,(T_{\alpha\beta}) = q \neq 0 \quad ,$$

then

$$I_q \geq 1.$$

Remark. If $\mathfrak{M}$ is compact or on general $\mathfrak{M}$ has a suitable boundary condition, then we can characterize the metric tensors as the solutions of the differential equation

$$\Delta T_{\lambda\mu} = 0 \ .$$

(*) Received June 30, 1950.

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Addendum.
The Theorem 5 is essent[illegible] identical Prof. A. Lichnerowicz'[illegible] result [Courbure et nombre de Betti d une variete riemanniene compacte, C.R. Paris 226 (1948) pp. 1678-1680].

The summary of our results has been presented at the annual meeting of Mathematical Society of Japan in May 1950, and the present author then received a kind letter from Prof. A. Lichnerowicz with his fine results in July.

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