NOTES ON WIENER INTEGRALS [1]

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(Communicated by T.Kawata)

1. Let C be the space of all continuous functions $x(t)$, $0 \le t \le 1$, $x(0)=0$. N.Wiener introduced into C a probability measure, and recently R.H.Cameron and W.T.Martin have developed its various aspects on point transformations, averages of certain functionals, unitary transformations, and orthogonal developments of arbitrary functionals. In this note we shall emphasize the fact that the Wiener measure can be equivalently transformed into a probability measure on a product space.[2] In the following we shall prove three theorems, of which the first two are known, showing how this measure can be used to simplify considerations based on the function space C. This point of view is really contained in Wiener's expression of the random function, and even more explicitly, in considerations by Cameron and Martin. In the following we use basic concepts and notations by these authors.

2. We will begin by proving a theorem by Cameron and Martin in a slightly generalized form, which proves to be essential for our later use.

Theorem 1. (Cameron and Martin)[3] Given an element $x_0(t) \in C$, for which $x_0'(t) \in L^2$, we consider a transformation T:

$$T(x(t)) = x(t) - x_0(t), \quad x(t) \in C.$$

Let Γ be a measurable subset of C and $F(x)$ a bounded measurable functional, then

$$(1) \quad \mathrm{meas}_w(T^{-1}\Gamma) = \exp\left(-\int_0^1 x_0'^2(t)\,dt\right) \times \int_\Gamma^w \exp\left(-2\int_0^1 x_0'(t)\,dx(t)\right) d_w x,$$

$$(2) \quad \int_C^w F(x - x_0)\, d_w x = \exp\left(-\int_0^1 x_0'^2(t)\,dt\right) \times \int_C^w F(x) \exp\left(-2\int_0^1 x_0'(t)\,dx(t)\right) d_w x.$$

Proof. Obviously (1) can be obtained from (2) by a special choice of the functional F. However (2) can be also deduced from (1) by making use of the usual approximation of a summable function by step functions. Hence we shall give here only a proof of (1). In the proof, since any measurable set can be approximated by "quasi-intervals", we have only to prove (1) with a specified Γ:

$$a_1 < x(t_1) < b_1,\ a_2 < x(t_2) < b_2, \cdots, a_n < x(t_n) < b_n,$$

where t_i, a_i, b_i, $i = 1, 2, \cdots n$, are real numbers. Consider the functions $\Phi_i(u)$ such that

$$\Phi_i(u) = 1 \quad \text{for} \quad a_i < u < b_i$$
$$= 0 \quad \text{otherwise},$$

then the characteristic functions of the set Γ can be expressed by $\prod_{i=1}^n \Phi_i(x(t_i))$. Define a function $f_i(t)$ by

$$f_i(t) = 1 \quad \text{for} \quad 0 < t < t_i$$
$$= 0 \quad \text{otherwise},$$

and consider its Fourier series with respect to the complete orthogonal set of functions[4] $\alpha_j(t) = \sqrt{2}\cos(2j-1)\pi t/2$, $j = 1, 2, \cdots$, $0 \le t \le 1$,

$$f_i(t) \sim \sum_j \frac{2}{(2j-1)\pi} \beta_j(t_i)\alpha_j(t)$$

where $\beta_j(t) = \sqrt{2}\sin(2j-1)\pi t/2$. Then, for almost every $x(t) \in C$, we have

$$(3) \quad x(t_i) = \int_0^1 f_i(t)\,dx(t) = \sum_{j=1}^\infty \frac{2}{(2j-1)\pi} \beta_j(t_i) u_j, \quad u_j = \int_0^1 \alpha_j(t)\,dx(t),$$

and

$$\Phi(u_1, u_2 \cdots) \equiv \prod_{i=1}^n \Phi_i(x(t_i)) \equiv \prod_{i=1}^n \Phi_i\left(\sum_{j=1}^\infty \frac{2}{(2j-1)\pi} \beta_j(t_i) u_j\right)$$

Let U_i, $i = 1, 2, \cdots$, be real lines, each with measures determined by the common density $\pi^{-1/2} e^{-u_i^2}$, and $U \equiv U_1 \times U_2 \times \cdots$, $U_{(k)} = U_1 \times \cdots \times U_k$, $U^{(k)} = U_{k+1} \times U_{k+2} \times \cdots$ be their product spaces with measures determined, from those of U_i, by the usual multiplicative definition. In computations of averages, it is convenient to consider the functionals u_j, $(u_1 u_2 \cdots)$, $(u_1 \cdots, u_k)$, $(u_{k+1}, u_{k+2} \cdots)$ at the same time as points on the product spaces just defined. With respect to these spaces the right-hand member of (1) will be transformed into

$$\exp\left(-\sum_1^\infty \xi_j^2\right) \int_U \Phi(u_1, u_2 \cdots) \exp\left(-2\sum_1^\infty u_j \xi_j\right) dU$$

$$(4) \quad = \exp\left(-\sum_{k+1}^\infty \xi_j^2\right) \int_{U^{(k)}} dU^{(k)} \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \pi^{-k/2} \times \exp\left(-2\sum_{k+1}^\infty u_j \xi_j - \sum_1^k u_j^2\right) \times \Phi(u_1 - \xi_1, \quad, u_k - \xi_k, u_{k+1}, \cdots) du_1 \cdots du_k$$

where $\int_V dV$, $\int_{V^{(k)}} dV^{(k)}$ denote integrals over V and $V^{(k)}$ with respect to their measures, and ξ_j is defined by

$$\xi_j = \int_0^1 \alpha_j(t)\, d\, x_0(t)$$

Convergence of the series

$$x_0(t_k) = \sum_{j=1}^{\infty} \frac{2}{(2j-1)\pi} \beta_j(t_k)\xi_j$$

yields

$$\lim_{k\to\infty} \Phi(u_1-\xi_1, \cdots, u_k-\xi_k, u_{k+1}, \cdots)$$
$$(5) \quad = \Phi(u_1-\xi_1, u_2-\xi_2, \cdots)$$
$$= \prod_{i=1}^{n} \Phi_i(x(t_i)-x_0(t_i))$$

almost everywhere on V, and easy calculations show that

$$\int_V [\exp(-2\sum_{k+1}^{\infty} u_j\xi_j)-1]^2 dV$$
$$(6) \quad = \exp(4\sum_{k+1}^{\infty}\xi_j^2)+1-2\exp(\sum_{k+1}^{\infty}\xi_j^2)$$
$$\to 0, \text{ as } k\to\infty.$$

Writing (4) in the form

$$(7) \quad \exp(-\sum_{k+1}^{\infty}\xi_j^2)(\int_V H_k G_k dV)$$

where

$$G_k = \Phi(u_1-\xi_1, \cdots, u_k-\xi_k, u_{k+1}, \cdots)$$
$$H_k = \exp(-2\sum_{k+1}^{\infty} u_j\xi_j)$$

and substituting (5) and (6) into (7) we finally obtain

$$\exp(-\int_0^1 x_0'^2(t)\,dt)\int_{\Gamma}^{w}\exp(-2\int_0^1 x_0'(t)\,dx(t))d_wx$$
$$= \int_V \Phi(u_1-\xi_1, u_2-\xi_2, \cdots)dV$$
$$= \int_{T^{-1}\Gamma}^{w} d_wx = \mathrm{meas}_w(T^{-1}\Gamma)$$

This proves the theorem.

3. We now pass on to proofs of two theorems on orthogonal developments of Wiener functionals, of which the first has been obtained by Cameron and Hatfield.[5]

Theorem 2. (Cameron and Hatfield) Let $\Psi_{m_1,\cdots,m_N}(x)$ be the Fourier-Hermite functionals constructed from the Hermite polynomials and the orthogonal system $\sqrt{2}\cos(2j-1)\pi t/2$, $0\le t\le 1$, and $F(x)$ be a bounded functional, $|F(x)|<M$, which is measurable over C and continuous in the Hilbert topology at $x_0(t)\in C$, then

$$(8) \quad F(x_0) = \lim_{N\to\infty,\lambda\to 1-0}\sum_{m_1\cdots m_N=0}^{\infty} A_{m_1,\cdots,m_N}\lambda^{m_1+\cdots+m_N} \times \Psi_{m_1,\cdots,m_N}(x_0),$$

where

$$A_{m_1\cdots m_N} = \int_C^w F(x)\Psi_{m_1\cdots m_N}(x)\,d_wx.$$

Proof. Define u_j and ξ_j as before, and put

$$x_{0,N} = \sum_{j=1}^{N}\beta_j(t)\int_0^1\beta_j(t)x_0(t)\,dt,$$
$$A(x_0,\lambda,N) = \sum_{m_1\cdots m_N=0}^{\infty} A_{m_1\cdots m_N}\lambda^{m_1+\cdots+m_N} \times \Psi_{m_1\cdots m_N}(x_0),$$

then

$$\int_0^1\alpha_j(t)\,d(x(t)+x_{0,N}(t)) = u_j+\lambda\xi_j, \quad 1\le j\le N$$
$$= u_j \qquad j>N$$
$$(9) \quad \sum_{j=1}^{N}\xi_j^2 = \sum_{j=1}^{N}(\int_0^1\alpha_j(t)x_{0N}'(t)dt)^2 = \int_0^1(x_{0N}'(t))^2dt$$
$$\sum_{j=1}^{N}u_j\xi_j = \sum_{j=1}^{N}(\int_0^1\alpha_j(t)dx(t))(\frac{2j-1}{2}\pi\int_0^1\beta_j(t)x_{0,N}(t)dt)$$
$$= \int_0^1(\frac{d}{dt}\sum_{j=1}^{N}\beta_j(t)\int_0^1\beta_j(s)x_{0,N}(s)ds)\,dx(t)$$
$$= \int_0^1 x_{0,N}'(t)\,dx(t).$$

Hence by a formula by Cameron and Hatfield[6] and Theorem 1

$$A(x_0,\lambda,N)-F(x_0) = C_\lambda\int_C^w[F(x)-F(x_0)] \times\exp(\sum_1^N\frac{2\lambda u_j\xi_j-\lambda^2(u_j^2+\xi_j^2)}{1-\lambda^2})d_wx$$
$$= C_\lambda\int_C^w[F(x+\lambda x_{0,N})-F(x_0)]$$
$$\times\exp[-\lambda^2\int_0^1(x_{0,N}'(t))^2dt-2\lambda\int_0^1x_{0,N}'(t)dx(t)]$$
$$\times\exp[\sum_{j=1}^{N}(2\lambda u_j\xi_j+2\lambda^2\xi^2-\lambda^2u_j^2-2\lambda^3u_j\xi_j-\lambda^4\xi_j^2-\lambda^2\xi_j^2) \times\frac{1}{(1-\lambda^2)}]d_wx$$
$$C_\lambda = (1-\lambda^2)^{-N/2},$$

which, on applying (9), reduces to

$$(10) \quad C_\lambda\int_C^w[F(x+\lambda x_{0,N})-F(x_0)]\exp[-\lambda^2\sum_1^N u_j^2/(1-\lambda^2)]d_wx$$

Consider now the function $g(t)$:

$$g(t) = 1 \quad\text{for}\quad 4|u|/\pi^2 > \delta^2$$
$$= 0 \quad\text{otherwise},$$

then, since

$$\|x\|^2=\int_0^1 (x(t))^2 dt = \frac{4}{\pi^2}\sum_1^{\infty}\frac{u_j^2}{(2j-1)^2}$$

we have, by means of (10), the in-equality

$$|A(x_0,\lambda,N)-F(x_0)|$$
$$\le C_\lambda \int_{\|x\|<\delta}^{W} |F(x+\lambda x_{0,N})-F(x_0)|$$
$$\times \exp\left[-\lambda^2\sum_1^N u_j^2/(1-\lambda^2)\right] d_w x$$
$$+2M\int_C^W C_\lambda g\left(\sum_1^{\infty}\frac{u_j^2}{(2j-1)^2}\right)\exp\left[-\lambda^2\sum_1^N u_j^2/(1-\lambda^2)\right] d_w x$$

The first term in the right-hand side of (11) is not greater than

$$\operatorname{Max}_{\|x\|<\delta} |F(x+\lambda x_{0,N})-F(x_0)|,$$

which, by the continuity of $F(x)$, can be made as small as we please, if only we let δ and $1-\lambda$ be small and N large. On the other hand, the second integral of (11) can be written as

$$\int_{V^{(k)}} dV^{(k)}\,\pi^{-N/2} C_\lambda \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g\left(\sum_1^{\infty}\frac{u_j^2}{(2j-1)^2}\right)$$
$$\times \exp\left[-\sum_1^N \frac{u_j^2}{1-\lambda^2}-\sum_{N+1}^{k} u_j^2\right]$$
$$\times du_1\cdots du_k$$
$$=\int_{V^{(k)}} dV^{(k)}\int_{V_{(k)}} f\left((1-\lambda^2)\sum_1^N\frac{u_j^2}{(2j-1)^2}+\sum_{N+1}^{\infty}\frac{u_j^2}{(2j-1)^2}\right)dV_{(k)}$$

$$=\int_V f\left((1-\lambda^2)\sum_1^N\frac{u_j^2}{(2j-1)^2}+\sum_{N+1}^{\infty}\frac{u_j^2}{(2j-1)^2}\right)dV$$

In the last integral, if we let $\lambda\to 1-0$, $N\to\infty$, the integrand boundedly converges to zero almost everywhere on V, and hence the integral also tends to zero. Thus we have completely proved the theorem.

In the above theorem we could not see whether our Fourier-Hermite series is summable almost everywhere by the Abel method, even when the functional is bounded, whereas almost everywhere summability is true for ordinary Fourier series. In this connection the following theorem may be of interest.

<u>Theorem 3.</u> If $F(x)$ is any bounded measurable functional defined over C, we have

$$\lim_{N\to\infty}\lim_{\lambda\to 1-0} A(x_0,\lambda,N)=F(x_0)$$

almost everywhere in C.

To prove the theorem we require the following lemma.

<u>Lemma.</u> If $G(u_1\cdots u_N)$ is a bounded measurable function of the variables $u_1,\cdots,u_N$, then

$$\pi^{-N/2}C_\lambda\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G(u_1+\xi_1,\cdots u_N+\xi_N)$$
$$\times\exp\left[-\sum_1^N u_j^2/(1-\lambda^2)\right]du_1\cdots du_N$$
$$\to G(\xi_1,\cdots\xi_N),\quad \text{as } \lambda\to 1-0,$$

for almost every $\xi=(\xi_1,\cdots,\xi_N)$.

Proof. Let dS be the surface element of the spherical surface K, defined by $u_1^2+\cdots+u_N^2=1$, and $e_1(S),\cdots,e_N(S)$ be the coordinates of points on K, then the coordinates of points on the surface $u_1^2+\cdots+u_N^2=r^2$ are given by $re_1(S)\cdots re_N(S)$ and except for a constant depending only on N, the volume element of the N-dimensional u-space can be put in the form $r^{N-1}dS\,dr$. The Lebesgue theory gives

$$(12)\quad h^{-N}\int_{0\le r\le h}\int_K |\Phi_\xi(re_1(S),\cdots,re_N(S))|\, r^{N-1}dS\,dr\to 0\quad \text{as } h\to 0.$$

for almost every value of ξ, where

$$\Phi_\xi = G(u_1+\xi_1,\cdots,u_N+\xi_N)-G(\xi_1,\cdots\xi_N)$$

In other words, we have

$$(13)\quad \rho^{-N}\int_K J_\xi(\rho,S)dS\to 0\quad \text{as } \rho\to 0,$$
$$J_\xi\equiv\int_{0\le r\le\rho}|\Phi_\xi|\,r^{N-1}dr$$

for almost every ξ.

Thus prepared the proof of the lemma is immediate by a standard procedure of evaluating singular integrals. Let us put

$$\pi^{-N/2}C_\lambda\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G(u_1+\xi_1,\cdots,u_N+\xi_N)$$
$$\times\exp\left[-\sum_1^N u_j^2/(1-\lambda^2)\right]-G(\xi_1\cdots\xi_N)$$
$$(14)\quad =\pi^{-N/2}C_\lambda\Big(\int_K\int_{0\le r\le(1-\lambda^2)^{1/2}}+\int_K\int_{(1-\lambda^2)^{1/2}\le r\le\eta}$$
$$+\int_K\int_{\eta<r}\Big)\times\Phi_\xi(re_1(S),\cdots)$$
$$\exp\left[-r^2/(1-\lambda^2)\right]r^{N-1}dS\,dr$$
$$=I_1(\lambda)+I_2(\lambda)+I_3(\lambda),$$

where η is a small positive number. Then by (12) we have immediately

$$I_1(\lambda)\to 0,\quad \text{as } \lambda\to 1-0$$

To evaluate I_2, we observe that

$$\left[\frac{r}{(1-\lambda^2)^{1/2}}\right]^{N+1}\exp\left[-r^2/(1-\lambda^2)\right]\le C$$

C a constant depending only on N.

Substituting this into the integrand of I_2, we get

$$|I_2| \leq \pi^{-N/2} c_\lambda \int_\Sigma dS \int_{(1-\lambda^2)^{1/2} \leq r \leq \eta} r^{-(N+1)} (1-\lambda^2)^{(N+1)/2} \times |\Phi_\xi(re_1(S), \cdots)| r^{N-1} dr$$

$$\leq \pi^{-N/2} (1-\lambda^2)^{1/2} \int_\Sigma dS \{ T(\eta, S) \eta^{-(N+1)} + (N+1) \int_{(1-\lambda^2)^{1/2}}^{\eta} T(r, S) r^{-(N+2)} dr \}$$

$$= O((1-\lambda^2)^{1/2} \eta^{-1}) + o(1),$$

$$\lambda \to 1-0.$$

Finally it is obvious that $I_3(\lambda) \to 0$ as $\lambda \to 1-0$. Combining these results we get the proof of the lemma.

<u>Proof of Theorem 3.</u> First we observe that we can write

$$F(x + \lambda x_{0N})$$

$$= G(u_1 + \lambda \xi_1, \cdots, u_N + \lambda \xi_N, u_{N+1}, \cdots)$$

with a suitable function G defined over V. Then, by (10) we get

$$A(x_0, \lambda, N) = c_\lambda \int_C^w F(x + \lambda x_{0,N}) \exp[-\lambda^2 \sum_1^N u_j^2/(1-\lambda^2)] d_w x$$

$$= \pi^{-N/2} c_\lambda \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(u_1 + \lambda \xi_1, \cdots, u_N + \lambda \xi_N) \exp[-\lambda^2 \sum_1^N u_j^2/(1-\lambda^2)] du_1 \cdots du_N$$

$$= \pi^{-N/2} c_\lambda \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G(u_1 + \xi_1, \cdots u_N + \xi_N) \exp[-\sum_1^N u_j^2/(1-\lambda^2)] du_1 \cdots du_N$$

$$+ o(1), \quad \lambda \to 1-0$$

where

$$G(u_1, \cdots u_N) = \int_{V^{(N)}} G(u_1, u_2, \cdots) \, dV^{(N)}$$

Hence by the lemma

$$\lim_{\lambda \to 1-0} A(x_0, \lambda, N) = G(\xi_1, \cdots \xi_N)$$

for almost every $\xi = (\xi_1, \xi_2, \cdots)$, and finally on applying Jessen's theorem we obtain

$$\lim_{N \to \infty} \lim_{\lambda \to 1-0} A(x_0, \lambda, N) = \lim_{N \to \infty} G(\xi_1 \cdots \xi_N)$$

$$= G(\xi_1, \xi_2, \cdots)$$

$$= F(x_0)$$

for almost every x_0. This proves Theorem 3.

(*) Received May 11, 1950.

(1) Read before the meeting of the Kyusyu Section of the Mathematical Society of Japan, February 4, 1950.
(2) This point has been stressed also by G.Sunouchi, The Monthly of Real Analysis, Vol.3, No.8, 1950 (Japanese).
(3) R.H.Cameron and W.T.Martin: Transformations of Wiener integrals under translations, Annals of Math., Vol.45, 1944. The same form of generalization has been obtained by G.Sunouchi, op.cit.
(4) Cameron and Hatfield: On the summability of certain orthogonal developments of nonlinear functionals, Bull. Amer. Math. Soc., Vol.55, No.2, 1949.
(5) Cameron and Hatfield: op.cit.
(6) Cameron and Hatfield: op.cit.

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