

ON THE ORR-SOMMERFELD TYPE EQUATIONS, II CONNECTION FORMULAS

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§ 1. Introduction

In the previous paper [3], the author studied the asymptotic properties of solutions of the fourth order ordinary differential equation of the form

$$(1.1) \quad \varepsilon^2 \frac{d^4y}{dx^4} - \left\{ p_3(x, \varepsilon) \frac{d^2y}{dx^2} + p_2(x, \varepsilon) \frac{dy}{dx} + p_1(x, \varepsilon)y \right\} = 0$$

in a certain bounded region D_M :

$$D_M: |x| < M, \quad 0 < \varepsilon < \varepsilon_0 < 1,$$

where M and ε_0 are positive constants, and constructed the asymptotic expansions of fundamental systems of solutions of (1.1) in several subregions of D_M . The above equation is one of the generalized form of the Orr-Sommerfeld equation which appears in the analysis of the stability problems of viscous fluid, and by this connection we call the equation (1.1) Orr-Sommerfeld type equations.

To understand the asymptotic nature of a solution in the whole region D_M , we have to obtain the linear relation between two fundamental systems defined in different subregions of D_M , and this is the purpose of this paper.

We assume throughout this paper the following conditions.

(i) The functions $p_i(x, \varepsilon)$ have the asymptotic expansions in power series of ε with polynomial coefficients:

$$(1.2) \quad p_i(x, \varepsilon) \cong \sum_{\nu=0}^{\infty} p_{i\nu}(x) \varepsilon^\nu, \quad (i=1, 2, 3)$$

uniformly in $|x| < M$.

ii) All of the turning points are simple, that is, the order of zero of the polynomial $p_{s0}(x)$ are one, and are contained in D_M .

We call the following equation the reduced equation of (1.1):

$$(1.3) \quad p_{s0}(x) \frac{d^2y}{dx^2} + p_{20}(x) \frac{dy}{dx} + p_{10}(x)y = 0.$$

This is obtained from (1.1) by letting ε zero.

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In the section 2, we summarize the results of the previous paper [3] and especially introduce the notion of the regions of admissibility, which are subregions of D_M containing no turning point in their interiors. For each region of admissibility, the asymptotic expansion of a fundamental system is constructed and characterized. In the section 3, we construct a fundamental system of solutions in the direct neighborhood of a turning point, the so-called inner solutions. In the last section, the connection matrices between two fundamental systems are considered. There exist five types of the connection and leading terms of their connection matrices are calculated.

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§ 2. Solutions in a region containing no turning point.

As in the previous paper [3], the differential equation (1.1) is equivalent to the vector equation of the form :

$$(2.1) \quad \varepsilon \frac{dY}{dx} = P(x, \varepsilon)Y,$$

where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \\ \varepsilon y^{(3)} \end{pmatrix},$$

$$P(x, \varepsilon) = \begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ p_1(x, \varepsilon) & p_2(x, \varepsilon) & p_3(x, \varepsilon) & 0 \end{pmatrix}.$$

The regions of admissibility are conveniently introduced by using the notions of the canonical regions. The notion of the canonical regions was firstly introduced by Evgrafov and Fedoryuk [1] when they established the existence theorem of the $W-K-B$ approximation of the second order ordinary differential equations.

In the complex x -plane, we plot the turning points which we denote by a_1, a_2, \dots, a_m . From each turning point a_k , we describe the Stokes curves :

$$\operatorname{Re} \xi(a_k, x) = \operatorname{Re} \int_{a_k}^x \sqrt{p_{30}(x)} dx = 0.$$

Then the x -plane is divided by these Stokes curves into a finite number of

simply connected unbounded regions which are called Stokes regions by Wasow [5]. The canonical region is a union of an appropriate number of adjacent Stokes regions, which is bounded by the Stokes curves, and contains no turning point in its interior. The configuration of Stokes curves in the complex x -plane and the family of canonical regions are all determined by $p_{80}(x)$. Various properties of the canonical regions or examples are given in [1] and [5].

Let a be one of the turning point. Since the turning point is simple, three Stokes curves start from a and tend to infinity or another turning point. Let S be one of these Stokes curves, then there exist at least one canonical regions $C[D]$ such that the turning point a is on the boundary of $C[D]$ and the Stokes curve S is going into the interior of $C[D]$.

Now the region of admissibility with respect to $\{a, S\}$ is the region $D_M[x, a, r]$ which is defined in [3], section 4. We denote this region simply by $D_M[S, a]$ in this paper. This region is, roughly speaking, obtained by the following manner (as an example, see Fig. 1).

- (1) Let $D_M[C] \equiv D_M \cap C[D]$.
- (2) We delete from $D_M[C]$ the neighborhood of a : the domain of influence Na ,

$$Na = \{x : |x - a| \leq N\epsilon^{2/3}, 0 < \epsilon \leq \epsilon_0\} \quad (N, \text{ constant}).$$

- (3) If there exist another turning points on the boundary of $C[D]$, their small neighborhoods of the form $\{x : |x - a| \leq \rho, \rho \text{ const.}\}$ are deleted.

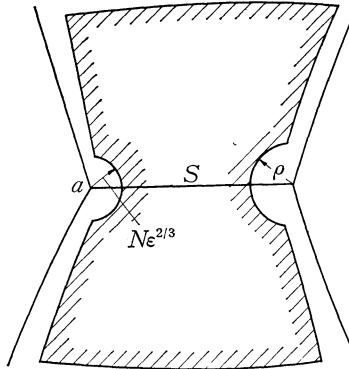


Fig. 1.

- (4) Lastly, certain neighborhoods of the Stokes curves that bound the canonical region $C[D]$ are deleted.

By this construction of the region of admissibility, we proved the following existence Theorem ([3], Theorem 5.2).

THEOREM 2.1. *There exists a fundamental system of solutions $Y(x, \epsilon)$ of the equation (2.1) in the region $D_M[S, a]$ whose asymptotic expansion has a form*

$$Y(x, \varepsilon) \sim \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2} p_{30}(x)} \varepsilon^2 & \frac{-1}{\sqrt{2} p_{30}(x)} \varepsilon^2 \\ 0 & 1 & \frac{1}{\sqrt{2} p_{30}(x)} \varepsilon & \frac{1}{\sqrt{2} p_{30}(x)} \varepsilon \\ \frac{-p_{10}(x)}{p_{30}(x)} & \frac{-p_{20}(x)}{p_{30}(x)} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0(\varepsilon) & 0(\varepsilon) & \sqrt{\frac{p_{30}(x)}{2}} & \sqrt{\frac{p_{30}(x)}{2}} \end{pmatrix} \begin{pmatrix} U_0(x, a) & 0 \\ 0 & V_0(x, a, \varepsilon) \end{pmatrix}.$$

Here 2 by 2 matrix $U_0(x, a)$ is a fundamental system of solutions of the reduced equation (1.3) and $V_0(x, a, \varepsilon)$ is defined by

$$(2.3) \quad V_0(x, a, \varepsilon) = r(a) p_{30}(x)^{-\frac{1}{4}}$$

$$\times \begin{cases} \exp \left[\int_a^x \left\{ \frac{\sqrt{p_{30}(x)}}{\varepsilon} + \frac{p_{31}(x)}{2\sqrt{p_{30}(x)}} \right\} + \int_b^x \frac{p_{20}(x)}{2p_{30}(x)} dx \right], 0 \\ 0, \exp \left[- \int_a^x \left\{ \frac{\sqrt{p_{30}(x)}}{\varepsilon} + \frac{p_{31}(x)}{2\sqrt{p_{30}(x)}} \right\} dx + \int_b^x \frac{p_{20}(x)}{2p_{30}(x)} dx \right] \end{cases},$$

where $r(a)$ is some constant, and b is such that $p_{30}(b) \neq 0$.

To calculate the connection matrix between two fundamental systems of solutions defined in different regions of admissibility, we have to characterize more precisely the matrices $U_0(x, a)$ and $V_0(x, a, \varepsilon)$.

(1) $U_0(x, a)$ consists of two linearly independent solutions of (1.3) which have convergent expressions in a neighborhood of a of the form

$$(2.4) \quad \begin{aligned} y^{(1)} &= 1 + \sum_{i=1}^{\infty} d_i (x-a)^i, \\ y^{(2)} &= (x-a)^{1-\lambda} \{1 + \sum_{i=1}^{\infty} e_i (x-a)^i\}, \end{aligned}$$

where d_i and e_i are constants.

Here we assume that λ is not an integer in spite of the fact that for Orr-Sommerfeld equation λ becomes zero and for its adjoint equation λ becomes two.

(2) For $V_0(x, a, \varepsilon)$, the constant $r(a)$ is determined so that at $x=a$,

$$r(a) \exp \int_b^x \frac{p_{20}(x)}{2p_{30}(x)} dx = (x-a)^{\lambda/2} \{1 + o(x-a)\}.$$

Such a constant $r(a)$ can be obtained as follows. Let the rational function $p_{20}(x)/2p_{30}(x)$ be of the form

$$\frac{p_{20}(x)}{2p_{30}(x)} = q(x) + \sum_{i=1}^m \frac{A_i}{(x-a_i)},$$

where A_i are constants and $q(x)$ is a polynomials. Here we suppose that $a=a_1$ and $A_1=\lambda/2$. Then $r(a)$ is such that

$$r(a) = \{\exp \int_a^b q(x)dx\} \cdot (b-a)^{\lambda/2} \prod_{i=2}^m \left(\frac{b-a_i}{a-a_i} \right)^{A_i}.$$

(3) The branch of the function $\xi(a, x) = \int_a^x \sqrt{p_{30}(x)} dx$ is determined in $D_M[S, a]$ by the condition

$$I_m \int_a^x \sqrt{p_{30}(x)} dx > 0 \text{ on } S.$$

We do not specify the branches of other multivalued functions appeared in $U_0(x, a)$ or $V_0(x, a, \varepsilon)$, but are determined appropriately. We denote by $Y\{S, a, C[D]\}$ fundamental systems of solutions thus defined to emphasize that it is characterized by the turning point a , Stokes curve S and the canonical region $C[D]$.

§ 3. Solutions in the neighborhood of a turning point.

For each canonical region, there corresponds the region of admissibility where we constructed the asymptotic expansion of the fundamental system of solutions of (2.1). Now in this section, we obtain asymptotic solutions of (2.1) in the direct neighborhood of a turning point.

Let a be one of the turning points. By the stretching and shearing transformations

$$(3.1) \quad \begin{aligned} x-a &= s\varepsilon^{2/3}, \\ Y &= Q(\varepsilon)W, \quad Q(\varepsilon) = \text{diag}\{\varepsilon^{4/3}, \varepsilon^{2/3}, 1, \varepsilon^{1/3}\}, \end{aligned}$$

the equation (2.1) becomes

$$(3.2) \quad \frac{dw}{ds} = A(s, \varepsilon)W,$$

where

$$A(s, \varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p_1(x, \varepsilon)\varepsilon^{2/3} & p_2(x, \varepsilon) & p_3(x, \varepsilon)\varepsilon^{-2/3} & 0 \end{pmatrix},$$

$$x = s\varepsilon^{2/3} + a.$$

If we rewrite the functions $p_i(x, \varepsilon)$ in power series of ε with coefficients of polynomials of $(x-a)$ and put $x-a=s\varepsilon^{2/3}$, then the matrix $A(s, \varepsilon)$ can be expanded in power series of $\varepsilon^{1/3}$ with polynomial coefficients of s . We construct the solu-

tions of (3.2) in the form

$$(3.3) \quad W(s, \varepsilon) = \sum_{i=0}^{\infty} W_i(s) \varepsilon^{i/3}.$$

Then each $W_i(s)$ must satisfy

$$(3.4) \quad \frac{dW_0}{ds} = A_0(s)W_0,$$

$$(3.4)' \quad \frac{dW_1}{ds} = A_0(s)W_1 + A_1(s)W_0, \quad \text{etc.,}$$

with

$$A_0(s) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & p_{20}(a) & p_{30}'(a)s & 0 \end{pmatrix},$$

$$A_1(s) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p_{10}(a) & p_{20}'(a)s & \frac{p_{30}''(a)s^2}{2} & 0 \end{pmatrix}.$$

The differential system (3.4) is equivalent to the fourth order differential equation.

$$(3.5) \quad \frac{d^4w}{ds^4} - \left\{ p_{30}'(a)s \frac{d^2w}{ds^2} + p_{20}(a) \frac{dw}{ds} \right\} = 0.$$

If we put

$$z = p_{30}'(a)^{1/3}s, \quad \lambda = \frac{p_{20}(a)}{p_{30}'(a)},$$

then (3.5) becomes

$$(3.6) \quad \frac{d^4w}{dz^4} - \left\{ z \frac{d^2w}{dz^2} + \lambda \frac{dw}{dz} \right\} = 0.$$

The differential equation (3.6) can be solved easily by the Laplace integral.

Let us define $w_j(z)$ by

$$(3.7) \quad w_j(z) = \frac{1}{2\pi i} \int_{C_j} t^{\lambda-2} \exp\left(zt - \frac{1}{3}t^3\right) dt, \quad (j=1, 2, \dots, 6)$$

Where the contour C_i are as in the Fig. 2. By the same analyses as for the well known Airy function, we can calculate their convergent expressions in the neighborhood of $z=0$ as well as the asymptotic expansions for large absolute values of z .

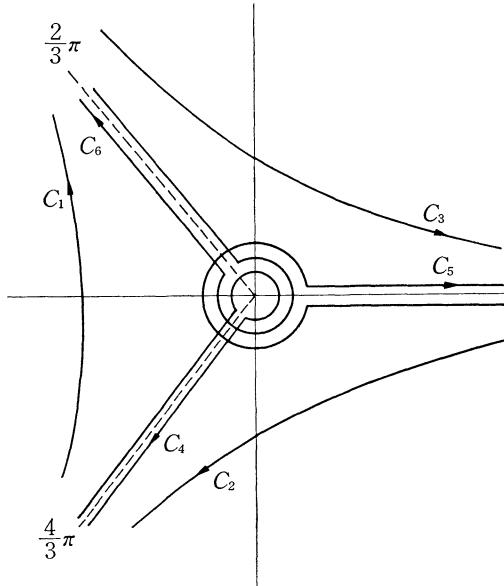


Fig. 2.

The asymptotic properties of $w_i(z)$ are as follows (for details, see Nishimoto [4]):

$$w_1(z) = -\frac{e^{\lambda\pi i}}{2\sqrt{\pi}} z^{\frac{\lambda}{2} - \frac{5}{4}} \exp\left\{-\frac{2}{3}z^{\frac{3}{2}}\right\} \left\{1 + O(z^{-\frac{3}{2}})\right\}, \quad |\arg z| < \pi,$$

$$w_2(z) = \frac{e^{-\lambda\pi i}}{2\sqrt{\pi}} z^{\frac{\lambda}{2} - \frac{5}{4}} \exp\left\{-\frac{2}{3}z^{\frac{3}{2}}\right\} \left\{1 + O(z^{-\frac{3}{2}})\right\}, \quad \frac{\pi}{3} < \arg z < \frac{7\pi}{3},$$

$$w_3(z) = \frac{i}{2\sqrt{\pi}} z^{\frac{\lambda}{2} - \frac{5}{4}} \exp\left\{\frac{2}{3}z^{\frac{3}{2}}\right\} \left\{1 + O(z^{-\frac{3}{2}})\right\}, \quad -\frac{\pi}{3} < \arg z < \frac{5\pi}{3},$$

$$w_4(z) = \frac{e^{\lambda\pi i} - e^{-\lambda\pi i}}{2\pi i} \Gamma(\lambda - 1) z^{1-\lambda} \left\{1 + O(z^{-3})\right\}, \quad -\pi < \arg z < \frac{\pi}{3},$$

$$w_5(z) = \frac{e^{\lambda\pi i} - e^{-\lambda\pi i}}{2\pi i} \Gamma(\lambda - 1) z^{1-\lambda} \left\{1 + O(z^{-3})\right\}, \quad \frac{\pi}{3} < \arg z < \frac{5\pi}{3},$$

$$w_6(z) = \frac{e^{\lambda\pi i} - e^{-\lambda\pi i}}{2\pi i} \Gamma(\lambda - 1) z^{1-\lambda} \left\{1 + O(z^{-3})\right\}, \quad -\frac{\pi}{3} < \arg z < \pi,$$

The sets of functions $\{1, w_6(z), w_3(z), w_1(z)\}$, $\{1, w_5(z), w_2(z), w_3(z)\}$ and $\{1, w_4(z), w_1(z), w_2(z)\}$ are fundamental systems of solutions of (3.6).

We construct $W_0^{(1)}(s)$ from $\{1, w_6(z), w_3(z), w_1(z)\}$ by

$$W_0^{(1)}(s) = \begin{pmatrix} 1 & w_6 & w_3 & w_1 \\ 0 & w_6' & w_3' & w_1' \\ 0 & w_6'' & w_3'' & w_1'' \\ 0 & w_6^{(3)} & w_3^{(3)} & w_1^{(3)} \end{pmatrix},$$

and analogously $W_0^{(2)}$, $W_0^{(3)}$ from $\{1, w_5(z), w_2(z), w_3(z)\}$, $\{1, w_4(z), w_1(z), w_2(z)\}$ respectively. Clearly $W_0^{(i)}(s)$ ($i=1, 2, 3$) are fundamental systems of differential system (3.4). Let Π_1 , Π_2 and Π_3 be connection matrices defined by

$$(3.8) \quad W_0^{(1)}(s) = W_0^{(2)}(s)\Pi_1, \quad W_0^{(2)}(s) = W_0^{(3)}(s)\Pi_2,$$

$$W_0^{(3)}(s) = W_0^{(1)}(s)\Pi_3,$$

then we have

$$(3.9) \quad \begin{aligned} \Pi_1 &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega^{3\lambda} \\ 0 & 0 & 0 & -\omega^{3\lambda} \\ 0 & 1-\omega^{-3\lambda} & 1 & -1 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^{-3\lambda} & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1-\omega^{-3\lambda} & 1 & -1 \end{pmatrix}, \\ \Pi_3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 1-\omega^{-3\lambda} & 1 & -\omega^{-3\lambda} \end{pmatrix}. \end{aligned}$$

Once the global solutions of the homogeneous equation (3.4) are obtained as above, the nonhomogeneous equation (3.4)' can be globally analyzed, and, in particular, we can obtain the growth order of solutions as s tends to infinity. By using these results, we can construct, as for more general equation in Nishimoto [2], the inner solutions in a certain region of s -plane, which overlaps in the x -plane with a region of admissibility $D_M[S, a]$ for all sufficiently small ε . We can prove the following theorem as in [2].

THEOREM 3.1. *Let $\Sigma^{(i)}$ ($i=1, 2, 3$) be the sectors defined by*

$$\Sigma^{(1)}: -\frac{\pi}{3} < \arg(p_{30}'(a)^{1/3}s) < \pi,$$

$$\Sigma^{(2)} : \frac{\pi}{3} < \arg(p_{s_0}'(a)^{1/3}s) < \frac{5\pi}{3},$$

$$\Sigma^{(3)} : -\pi < \arg(p_{s_0}'(a)^{1/3}s) < \frac{\pi}{3}.$$

Then there exist fundamental systems of solutions $W^{(i)}(s, \varepsilon)$ of the differential system (3.2) such that $W^{(i)}(s, \varepsilon)$ have the asymptotic expansions in the regions $D[\Sigma^{(i)}, a]$ depending on ε (Fig. 3):

$$D[\Sigma^{(i)}, a] = \{s : |s| \leq s_0\} \{s : s_0 \leq |s| < \delta\varepsilon^{-1/6}, \arg s \in \Sigma^{(i)},$$

(δ , positive constant),

such that

$$(3.10) \quad W^{(i)}(s, \varepsilon) = \begin{cases} W_0^{(i)}\{E + O(\varepsilon)\} & \text{for } |s| \leq s_0, \\ W_0^{(i)}\{E + O(s^2\varepsilon^{1/3})\} & \text{for } s_0 \leq |s| < \delta\varepsilon^{-1/6}. \end{cases}$$

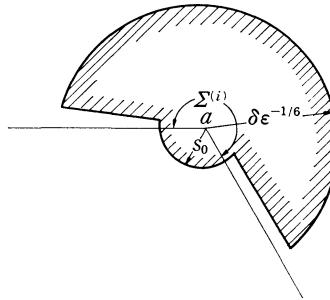


Fig. 3. $D[\Sigma^{(i)}, a]$

The region $D[\Sigma^{(i)}, a]$ are neighborhood of $x=a$ in the x -plane which shrink to a as ε tends to zero, but overlap with some $D_M[s, a]$. We denote the solution $Y(x, \varepsilon) = \Omega(\varepsilon)W^{(i)}(s, \varepsilon)$ of (2.1) by $Y\{\Sigma^{(i)}, a\}$.

§ 4. Connection formulas

In this section, we calculate the leading terms of connection matrices between two fundamental systems defined in the section 2 and 3. According to Evgrafov and Fedoryuk [1], it is sufficient if we can obtain the following four types of connection matrices to know asymptotic behavior of $Y\{S, a, C[D]\}$ in the region D_M except neighborhoods of turning points:

$$(1) \quad Y\{S, a_2, C[D]\} = Y\{S, a_1, C[D]\}\Omega_1,$$

$$(2) \quad Y\{S_2, a_2, C[D]\} = Y\{S_1, a_1, C[D]\}\Omega_2,$$

where $\xi(a_1, x)$ and $\xi(a_2, x)$ have the same branch in the canonical region $C[D]$,

$$(3) \quad Y\{S, a, C^{(2)}[D]\} = Y\{S, a, C^{(1)}[D]\} Q_3,$$

$$(4) \quad Y\{S_2, a, C^{(2)}[D]\} = Y\{S_1, a, C^{(1)}[D]\} Q_4$$

(lateral connection matrix).

We add to these the central connection matrix in order to know asymptotic expansions at turning points

$$(5) \quad Y\{S, a, C[D]\} = Y\{\Sigma^{(i)}, a\} A \quad \text{for some } i.$$

These connection matrices do not depend on x but may depend on ε . The lateral connection matrices are determined after the calculation of the central connection matrices.

Let the 2 by 2 matrix $C(a_1, a_2)$ be the connection matrix between two fundamental systems of solutions of the reduced equation (1.3) defined by (2.4) at regular singular points a_1 and a_2 . And we introduce the quantity $\eta(a, x, \varepsilon)$ by

$$\eta(a, x, \varepsilon) = \int_a^x \left\{ \frac{\sqrt{p_{30}(x)}}{\varepsilon} + \frac{p_{31}(x)}{2\sqrt{p_{30}(x)}} \right\} dx.$$

Then the calculations of the connection matrices Q_1 , Q_2 and Q_3 are straightforward by using asymptotic expressions of fundamental systems.

(1) Connection of type 1 (for an example, see Fig. 4):

$$(4.1) \quad Y\{S, a_2, C[D]\} = Y\{S, a_1, C[D]\} Q_1.$$

Here both of the regions of admissibility $D_M[S, a_1]$ and $D_M[S, a_2]$ are almost the same, but different a little bit in the neighborhoods of a_1 and a_2 (Fig. 4).

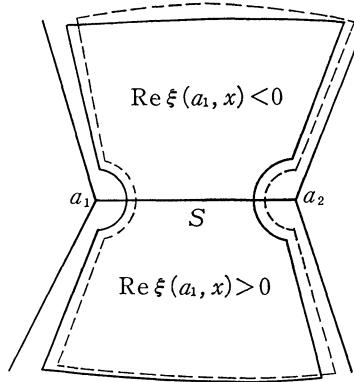


Fig. 4. Real curve $D_M[S, a_1]$, dotted curve $D_M[S, a_2]$

Now we can assume that there exist positive constants σ_- and σ_+ such that

$$-\sigma_- < \operatorname{Re} \xi(a_1, x) < \sigma_+$$

for all x in $D_M[S, a_1] \cap D_M[S, a_2]$, and let σ be $\min\{\sigma_-, \sigma_+\}$. From the definition of $Y\{S, a, C[D]\}$, the branches of $\xi(a_1, x)$ and $\xi(a_2, x)$ differ in sign. Then if we change the branch of $\xi(a_2, x)$, the third column and the fourth column of $Y\{S, a_2, C[D]\}$ must be exchanged, or it must be rewritten as

$$Y\{S, a_2, C[D]\}\Gamma \quad \text{with} \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If the branches of other multivalued functions in both fundamental systems are taken identical, we have

$$\begin{aligned} \mathcal{Q}_1 &= Y\{S, a_1, C[D]\}^{-1} Y\{S, a_2, C[D]\} \Gamma \\ &= \begin{bmatrix} V_0(x, a_1)^{-1} & 0 \\ 0 & V_0(x, a_1, \varepsilon)^{-1} \end{bmatrix} (E + O(\varepsilon)) \begin{bmatrix} U_0(x, a_2) & 0 \\ 0 & V_0(x, a_2, \varepsilon) \end{bmatrix} \Gamma \\ &= \begin{bmatrix} C(a_1, a_2)(E_2 + O(\varepsilon)) & U_0^{-1}(x, a_1)V_0(x, a_2, \varepsilon)O(\varepsilon) \\ V_0(x, a_1, \varepsilon)^{-1}U_0(x, a_2)O(\varepsilon) & V_0(x, a_1, \varepsilon)^{-1}(E_2 + O(\varepsilon))V_0(x, a_2, \varepsilon) \end{bmatrix} \Gamma, \end{aligned}$$

where E is the 4 by 4 unit matrix and E_2 is the 2 by 2 unit matrix, and more over

$$\begin{aligned} V_0(x, a_1, \varepsilon)^{-1}(E_2 + O(\varepsilon))V_0(x, a_2, \varepsilon) &= r(a_1)^{-1}r(a_2) \\ &\times \begin{bmatrix} \exp[-\eta(a_1, a_2, \varepsilon)](1+O(\varepsilon)) & O(\varepsilon) \exp\{-\eta(a_1, x, \varepsilon) - \eta(a_2, x, \varepsilon)\} \\ O(\varepsilon) \exp\{\eta(a_1, x, \varepsilon) + \eta(a_2, x, \varepsilon)\} & \exp[\eta(a_1, a_2, \varepsilon)](1+O(\varepsilon)) \end{bmatrix}. \end{aligned}$$

In this case we have $\eta(a_1, x, \varepsilon) = \eta(a_2, x, \varepsilon)$ since the branches of $\xi(a_1, x)$ and $\xi(a_2, x)$ were made identical. Since \mathcal{Q}_1 does not depend on x , we can take as x in the above expressions an appropriate point of the intersection $D_M[S, a_1] \cap D_M[S, a_2]$. Thus we have established

$$(4.2) \quad \mathcal{Q}_1 = \begin{bmatrix} C_1(a_1, a_2)(E_2 + O_2(\varepsilon)) & O_2\left[\varepsilon \exp\left(-\frac{\sigma}{\varepsilon}\right)\right] \\ O_2\left[\varepsilon \exp\left(-\frac{\sigma}{\varepsilon}\right)\right] & \Gamma_1(a_1, a_2, \varepsilon) \end{bmatrix},$$

where

$$\begin{aligned} \Gamma_1(a_1, a_2, \varepsilon) = & r(a_1)^{-1} r(a_1) \\ & \left(\begin{array}{cc} O\left[\varepsilon \exp\left(-\frac{2\sigma}{\varepsilon}\right)\right] & \{\exp[-\eta(a_1, a_2, \varepsilon)]\}(1+O(\varepsilon)) \\ \{\exp[\eta(a_1, a_2, \varepsilon)]\}(1+O(\varepsilon)) & O\left[\varepsilon \exp\left(-\frac{2\sigma}{\varepsilon}\right)\right] \end{array} \right). \end{aligned}$$

Here and hereafter $O_2[g(\varepsilon)]$ means 2 by 2 matrix whose entries are of the order $g(\varepsilon)$ as ε tends to zero.

(2) Connection of type 2 (Fig. 5):

$$(4.3) \quad Y\{S_2, a_2, C[D]\} = Y\{S_1, a_1, C[D]\} \Omega_2.$$

Here the branches $\xi(a_1, x)$ and $\xi(a_2, x)$ are the same (Fig. 5). Let σ be the positive constant defined by

$$\sigma = \min \{\sigma_+, \sigma_-\}$$

where σ_+ and σ_- are such that

$$-\sigma_- < \operatorname{Re} \xi(a_1, x), \quad \operatorname{Re} \xi(a_2, x) < \sigma_+$$

for all x in $D_M[S_1, a_1] \cap D_M[S_2, a_2]$.

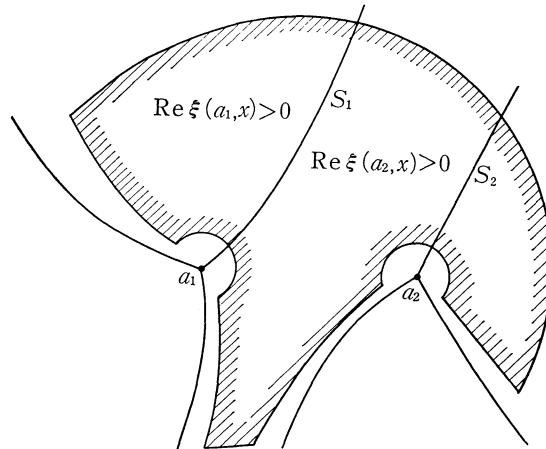


Fig. 5. $D_M[S_1, a_1] \cap D_M[S_2, a_2]$

Then we have from the same calculation as we did for Ω_1 ,

$$(4.4) \quad \Omega_2 = \begin{pmatrix} C(a_1, a_2)(E_2 + O(\varepsilon)) & O_2\left[\varepsilon \exp\left(-\frac{\sigma}{\varepsilon}\right)\right] \\ O_2\left[\varepsilon \exp\left(-\frac{\sigma}{\varepsilon}\right)\right] & I_2(a_1, a_2, \varepsilon) \end{pmatrix},$$

where

$$\begin{aligned} \Gamma_2(a_1, a_2, \varepsilon) = & r(a_1)^{-1} r(a_2) \\ & \times \begin{cases} \{\exp[-\eta(a_1, a_2, \varepsilon)]\}(1+O(\varepsilon)), & O\left[\varepsilon \exp\left(-\frac{2\sigma}{\varepsilon}\right)\right] \\ O\left[\varepsilon \exp\left(-\frac{2\sigma}{\varepsilon}\right)\right], & \{\exp[\eta(a_1, a_2, \varepsilon)]\}(1+O(\varepsilon)) \end{cases}. \end{aligned}$$

(3) Connection of type 3 (Fig. 6)

$$(4.5) \quad Y\{S, a, C^{(2)}[D]\} = Y\{S, a, C^{(1)}[D]\} \Omega_3.$$

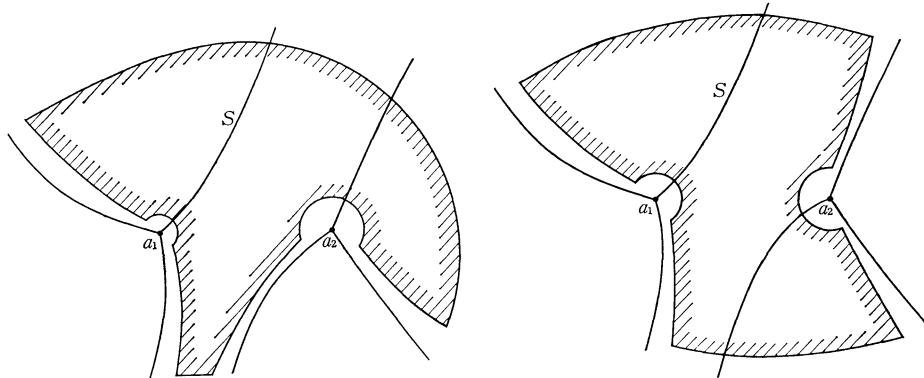


Fig. 6-1. $D_M^{(1)}[S, a]$

Fig. 6-2. $D_M^{(2)}[S, a]$

We define positive constant σ as before

$$\sigma = \min(\sigma_+, \sigma_-)$$

where σ_+ and σ_- are constants such that

$$-\sigma_- < \operatorname{Re} \xi(a, x) < \sigma_+$$

for all x in $D_M^{(1)}[S, a] \cap D_M^{(2)}[S, a]$. Then we have

$$(4.6) \quad \Omega_3 = \begin{pmatrix} E_2 + O_2[\varepsilon] & O_2\left[\varepsilon \exp\left(-\frac{\sigma}{\varepsilon}\right)\right] \\ O_2\left[\varepsilon \exp\left(-\frac{\sigma}{\varepsilon}\right)\right] & \Gamma_3(\varepsilon) \end{pmatrix},$$

where

$$\Gamma_3(\varepsilon) = \begin{pmatrix} 1+O(\varepsilon) & O\left[\varepsilon \exp\left(-\frac{2\sigma}{\varepsilon}\right)\right] \\ O\left[\varepsilon \exp\left(-\frac{2\sigma}{\varepsilon}\right)\right] & 1+O(\varepsilon) \end{pmatrix}.$$

(4) Central connection :

$$(4.7) \quad Y\{S_i, a, C^{(i)}[D]\} = Y\{\Sigma^{(i)}, a\} A^{(i)} \quad (i=1, 2, 3),$$

where the Stokes curves S_1, S_2 and S_3 start from a in the direction $\arg z \equiv \arg p_{30}'(a)^{1/3}(x-a) = \pi/3, \pi$ and $-\pi/3$ respectively (Fig. 7). Since the two regions $D_M^{(i)}[S_i, a]$ and $D[\Sigma^{(i)}, a]$ are overlapped for all sufficiently small ε , we can apply the matching method to obtain $A^{(i)}(i=1, 2, 3)$ (Fig. 8).

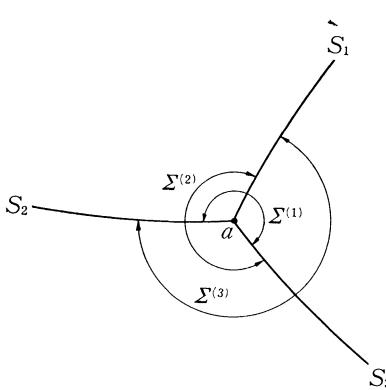


Fig. 7.

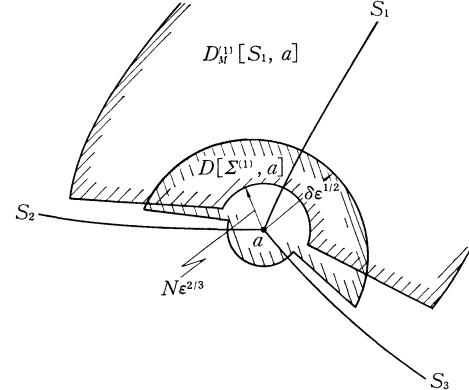


Fig. 8.

Let us denote the entries of the first row of $Y\{S_1, a, C^{(1)}[D]\}$ by $y^{(1)}, y^{(2)}$, $y^{(3)}$ and $y^{(4)}$. From the asymptotic expansions of $Y\{S_1, a, C^{(1)}[D]\}$ defined in the section 2, these functions have expressions in the neighborhood of $x=a$,

$$\begin{aligned} y^{(1)} &= 1 + O(x-a) + O[(x-a)^{-3/2}\varepsilon], \\ y^{(2)} &= (x-a)^{1-\lambda} \{1 + O(x-a) + O[(x-a)^{-3/2}\varepsilon]\}, \\ y^{(3)} &= \frac{\varepsilon^2}{\sqrt{2}} p_{30}'(a)^{-\frac{5}{4}} (x-a)^{\frac{\lambda}{2}-\frac{5}{4}} \exp \int_a^x \frac{\sqrt{p_{30}(x)}}{\varepsilon} dx \\ &\quad \times \{1 + O(x-a) + O[(x-a)^{-3/2}\varepsilon]\}, \\ y^{(4)} &= -\frac{\varepsilon^2}{\sqrt{2}} p_{30}'(a)^{-\frac{5}{4}} (x-a)^{\frac{\lambda}{2}-\frac{5}{4}} \exp \left(- \int_a^x \frac{\sqrt{p_{30}(x)}}{\varepsilon} dx \right) \\ &\quad \times \{1 + O(x-a) + O[(x-a)^{-3/2}\varepsilon]\}. \end{aligned}$$

On the other hand, from the Theorem 3.1, the entries of the first row of $Y[\Sigma^{(1)}, a]$ which we denote by $\tilde{y}^{(1)}, \tilde{y}^{(2)}, \tilde{y}^{(3)}$ and $\tilde{y}^{(4)}$ have asymptotic expressions for large absolute values of z

$$\begin{aligned}\tilde{y}^{(1)} &= \varepsilon^{\frac{4}{3}} \{1 + O(s^2 \varepsilon^{1/3})\}, \\ \tilde{y}^{(2)} &= \varepsilon^{\frac{4}{3}} w_6(z) \{1 + O(s^2 \varepsilon^{1/3})\} \\ &= \varepsilon^{\frac{4}{3}} \frac{e^{\lambda\pi i} - e^{-\lambda\pi i}}{2\pi i} \Gamma(\lambda-1) z^{1-\lambda} \{1 + O(z^{-3}) + O(s^2 \varepsilon^{1/3})\}, \\ \tilde{y}^{(3)} &= \varepsilon^{\frac{4}{3}} w_5(z) \{1 + O(s^2 \varepsilon^{1/3})\} \\ &= \varepsilon^{\frac{4}{3}} \frac{i}{2\sqrt{\pi}} z^{\frac{\lambda}{2} - \frac{5}{4}} \exp\left(\frac{2}{3} z^{\frac{3}{2}}\right) \{1 + O(z^{-3/2}) + O(s^2 \varepsilon^{1/3})\}, \\ \tilde{y}^{(4)} &= \varepsilon^{\frac{4}{3}} w_1(z) \{1 + O(s^2 \varepsilon^{1/3})\} \\ &= -\varepsilon^{\frac{4}{3}} \frac{e^{\lambda\pi i}}{2\sqrt{\pi}} z^{\frac{\lambda}{2} - \frac{5}{4}} \exp\left(-\frac{3}{2} z^{\frac{3}{2}}\right) \{1 + O(z^{-3/2}) + O(s^2 \varepsilon^{1/3})\},\end{aligned}$$

where $z = [p_{30}'(a)]^{-\frac{1}{3}} s$.

From the general theory, we know that $A^{(1)}$ is asymptotically diagonal. Then putting $x-a=\eta\varepsilon^{7/12}$ or $s=\eta\varepsilon^{-1/12}$, which belongs to both regions $D[S_1, a]$ and $D[\Sigma^{(1)}, a]$, into the above expressions and comparing $\tilde{y}^{(i)}$ with $y^{(i)}$, we have

$$(4.8)_1 \quad A^{(1)} = \text{diag} \{ \varepsilon^{-\frac{4}{3}}, \varepsilon^{-\frac{2}{3}(\lambda+1)} C_{22}^{(1)}, \varepsilon^{\frac{\lambda}{3} - \frac{1}{6}} C_{33}^{(1)}, \varepsilon^{\frac{\lambda}{3} - \frac{1}{6}} C_{44}^{(1)} \} \times \{E + O(\varepsilon^{1/8})\},$$

where

$$\begin{aligned}C_{22}^{(1)} &= \frac{2\pi i p_{30}'(a)^{-\frac{1}{3}(1-\lambda)}}{(e^{\lambda\pi i} - e^{-\lambda\pi i}) \Gamma(\lambda-1)}, \quad C_{33}^{(1)} = \frac{\sqrt{2\pi}}{i} p_{30}'(a)^{-\frac{(\lambda+5)}{6}}, \\ C_{44}^{(1)} &= \sqrt{2\pi} e^{-\lambda\pi i} p_{30}'(a)^{-\frac{(\lambda+5)}{6}}.\end{aligned}$$

Analogously we have

$$(4.8)_i \quad A^{(i)} = \text{diag} \{ \varepsilon^{-\frac{4}{3}}, \varepsilon^{-\frac{2}{3}(\lambda+1)} C_{22}^{(i)}, \varepsilon^{\frac{\lambda}{3} - \frac{1}{6}} C_{33}^{(i)}, \varepsilon^{\frac{\lambda}{3} - \frac{1}{6}} C_{44}^{(i)} \} \times \{E + O(\varepsilon^{1/8})\}, \quad (i=2, 3)$$

where

$$C_{22}^{(2)} = C_{22}^{(1)}, \quad C_{33}^{(2)} = \sqrt{2\pi} e^{\lambda\pi i} p_{30}'(a)^{-\frac{(\lambda+5)}{6}},$$

$$C_{44}^{(2)} = \sqrt{2\pi} i p_{30}'(a)^{-\frac{(\lambda+5)}{6}},$$

$$C_{22}^{(3)} = C_{22}^{(1)}, \quad C_{33}^{(3)} = \sqrt{2\pi} e^{-\lambda\pi i} p_{30}'(a)^{-\frac{(\lambda+5)}{6}},$$

$$C_{44}^{(3)} = -\sqrt{2\pi} e^{\lambda\pi i} p_{30}'(a)^{-\frac{(\lambda+5)}{6}}.$$

We remark that the terms $O(\varepsilon^{1/8})$ in the above expressions comes from the terms $O[(x-a)^{-3/2}\varepsilon]$ in the expressions of $y^{(i)}$, but more delicate analyses may give us as in [2] these terms are $O(\varepsilon^{1/8})$ in place of $O(\varepsilon^{1/8})$.

(5) Lateral connection at a simple turning point :

$$\begin{aligned} Y\{S_1, a, C^{(1)}[D]\} &= Y\{S_2, a, C^{(2)}[D]\} Q_4^{(1)}, \\ (4.9) \quad Y\{S_2, a, C^{(2)}[D]\} &= Y\{S_3, a, C^{(3)}[D]\} Q_4^{(2)}, \\ Y\{S_3, a, C^{(3)}[D]\} &= Y\{S_1, a, C^{(1)}[D]\} Q_4^{(3)}. \end{aligned}$$

Since we have from (4.7) and (3.10)

$$\begin{aligned} Y\{S_i, a, C^{(i)}[D]\} &= Y\{\Sigma^{(i)}, a\} A^{(i)} = Q(\varepsilon) W^{(i)}(s, \varepsilon) A^{(i)} \\ &= Q(\varepsilon) \{E + O(\varepsilon^{1/8})\} W_0^{(i)} A^{(i)} \quad (i=1, 2, 3), \end{aligned}$$

then

$$\{E + O(\varepsilon^{1/8})\} W_0^{(1)} A^{(1)} = \{E + O(\varepsilon^{1/8})\} W_0^{(2)} A^{(2)} Q_4^{(1)},$$

but this equation and (3.8) give us

$$Q_4^{(1)} = A^{(2)-1} \Pi^{(1)} A^{(1)} \{E + O(\varepsilon^{1/8})\}.$$

Similarly we have

$$Q_4^{(2)} = A^{(3)-1} \Pi^{(2)} A^{(2)} \{E + O(\varepsilon^{1/8})\},$$

$$Q_4^{(3)} = A^{(1)-1} \Pi^{(3)} A^{(1)} \{E + O(\varepsilon^{1/8})\}.$$

From the formulas (3.9) and (4.8), and since $Y\{S_i, a, C^{(i)}[D]\}$ have asymptotic expansions in power series of ε , we have following formulas for the lateral connection matrices :

$$Q_4^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \varepsilon^{\lambda+1/2} i(1-e^{2\lambda\pi i}) \beta^{-1} \\ 0 & 0 & 0 & -1 \\ 0 & \varepsilon^{-\lambda-1/2} e^{-\lambda\pi i} \beta & -1 & ie^{-\lambda\pi i} \end{pmatrix} \{E + O(\varepsilon)\},$$

$$(4.10) \quad Q_4^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-2\lambda\pi i} & 0 & \varepsilon^{\lambda+1/2}(e^{\lambda\pi i} - e^{-\lambda\pi i})\beta^{-1} \\ 0 & 0 & 0 & -ie^{\lambda\pi i} \\ 0 & -\varepsilon^{-\lambda-1/2}ie^{-2\lambda\pi i}\beta & ie^{-\lambda\pi i} & ie^{-\lambda\pi i} \end{pmatrix} \{E + O(\varepsilon)\},$$

$$Q_4^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \varepsilon^{\lambda+1/2}(e^{2\lambda\pi i} - 1)\beta^{-1} \\ 0 & 0 & 0 & ie^{\lambda\pi i} \\ 0 & \varepsilon^{-\lambda-1/2}i\beta & 1 & 1 \end{pmatrix} \{E + O(\varepsilon)\},$$

where $\beta = \sqrt{2\pi} p_{s0}'(a)^{(\lambda+1)/2} \{\Gamma(\lambda-1)\}^{-1}$.

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