# ON A CHARACTERIZATION OF MULTIPLE NORMAL DISTRIBUTIONS

By Yukiyosi KAWADA

(Communicated by T.Kawata)

Let the n-dimensional multiple distribution defined by n random variables $x_1$, ..., $x_n$ be the normal distribution with the probability density

$$(1)\quad f(x_1,\dots,x_n) = (2\pi)^{-\frac{n}{2}}\sigma^{-n} \cdot \exp\left(-\frac{\sigma^2}{2}\sum_{k=1}^{n}(x_k - m_k)^2\right).$$

Then the two random variables $y$, $z$ defined by the linear forms:

$$(2)\qquad y = \sum_{k=1}^{n}\alpha_k x_k,\quad z = \sum_{k=1}^{n}\beta_k x_k$$

are independent if and only if

$$(3)\qquad \sum_{k=1}^{n}\alpha_k\beta_k = 0$$

holds. Now we consider the converse of this property.

Theorem. Let $x_1,\dots,x_n$ be n random variables. If any two random variables $y$, $z$ defined as the linear forms of $x_1,\dots,x_n$ by (2) are independent whenever the relation (3) holds, then the multiple distribution of $x_1,\dots,x_n$ is the normal distribution with the probability density (1).

Proof. Let $\varphi_k(s) = E(\exp(isx_k))$ ($k = 1, \dots, n$) be the characteristic function of $x_k$. Since $y = x_1\cos\theta + x_2\sin\theta$ and $z = -x_1\sin\theta + x_2\cos\theta$ are independent by our hypothesis, we have

$$(4)\quad E(\exp(isy + itz)) = E(\exp(isy))\,E(\exp(itz)).$$

Since $x_1$ and $x_2$ are independent by our hypothesis, we can represent the both side of (4) by $\varphi_1$ and $\varphi_2$:

$$(5)\quad \varphi_1(s\cos\theta - t\sin\theta)\,\varphi_2(s\sin\theta + t\cos\theta) = \varphi_1(s\cos\theta)\,\varphi_1(-t\cdot\sin\theta)\,\varphi_2(s\sin\theta)\,\varphi_2(t\cos\theta).$$

Putting $\theta = \pi/4$ and $\sqrt{2}\,s$, $\sqrt{2}\,t$ instead of s, t in (5) we have

$$(6)\quad \varphi_1(s-t)\,\varphi_2(s+t) = \varphi_1(s)\,\varphi_1(-t)\,\varphi_2(s)\,\varphi_2(t).$$

Taking $s = t$ or $s = -t$ in (6), we have especially the relations:

$$(7)\quad \varphi_2(2s) = |\varphi_1(s)|^2\cdot\varphi_2(s)^2,\qquad \varphi_1(2s) = \varphi_1(s)^2\cdot|\varphi_2(s)|^2.$$

Now follows from (7)

$$(8)\qquad |\varphi_1(s)| = |\varphi_2(s)|$$

Hence we have also from (7) $|\varphi_k(2t)| = |\varphi_k(t)|^4$ ($k=1,2$). Now put $s=rt$ ($r = 1,2,\dots$) in (6) we have then $|\varphi_k((r-1)t)|\cdot|\varphi_k((r+1)t)| = |\varphi_k(rt)|^2\cdot|\varphi_k(t)|^2$. Thus we can prove by the mathematical induction the relation $|\varphi_k(rt)| = |\varphi_k(t)|^{r^2}$ ($r = 1,2,\dots$). Take then $t = s/p$, we have

$$(9)\qquad |\varphi_k(\lambda s)| = |\varphi_k(s)|^{\lambda^2}$$

for $\lambda = r/p$. This relation holds also for any positive number $\lambda$ by the continuity of $\varphi_k$.

Let us put s=1 in (9), we have $|\varphi_k(\lambda)| = |\varphi_k(1)|^{\lambda^2} = \exp(\alpha_k\lambda^2)$, $\alpha_k \leqq 0$. By the relation (8) we have $\alpha_1 = \alpha_2 = \alpha$. Hence $\varphi_k$ has the functional form:

$$\varphi_k(t) = \exp(\alpha t^2 + 2\pi i\,\theta_k(t)).$$

From $\varphi_k(-t) = \overline{\varphi_k(t)}$ follows $\theta_k(-t) \equiv \theta_k(t)$ (mod.1). Since we have the relation $\varphi_k(2t) = \varphi_k(t)^3\,\varphi_k(-t)$ from (7), (8), $\theta_k(t)$ must satisfy $\theta_k(2t) \equiv 2\,\theta_k(t)$ (mod.1). From this follows also $\theta_k(2^r t) \equiv 2^r\theta_k(t)$ (mod.1). Now choose an irrational number $\omega$ and put $\theta_k(\omega) \equiv m_k\omega$ (mod.1). Then we have

$$(10)\qquad \theta_k(\lambda) \equiv m_k\lambda \quad (\text{mod.}1)$$

for $\lambda = 2^r\omega$. Since $\theta_k(t)$ is continuous and the relation (10) holds for a dense subset $\{2^r\omega\ ;\ r = 1,2,\dots\}$, (10) holds also for every value $\lambda$.

Hence the characteristic function $\varphi_K(t)$ of $x_K$ is given by

(11) $$\varphi_K(t) = \exp(im_K t + \alpha t^2)$$
$$(\alpha \leqq 0) \quad (k = 1,2).$$

Thus we have the following result: $x_K$ ($k = 1,\ldots,n$) has the normal distribution with the mean value $m_K$ and with the same variance $\sigma^2 = -2\alpha$. Any two $x_K$ and $x_j$ are independent.

Now take two random variables $\tilde{x}_1 = \sum_{K=1}^{n} \lambda_K x_K$ and $\tilde{x}_2 = \sum_{K=1}^{n} \mu_K x_K$ such that

$$\sum_{K=1}^{n} \lambda_K^2 = \sum_{K=1}^{n} \mu_K^2, \quad \sum_{K=1}^{n} \lambda_K \mu_K = 0.$$

Let us put $y = \tilde{x}_1 \cos\theta + \tilde{x}_2 \sin\theta = \sum \alpha_K x_K$, $z = -\tilde{x}_1 \cdot \sin\theta + \tilde{x}_2 \cdot \cos\theta$

$= \sum \beta_K x_K$ ($\alpha_K = \lambda_K \cos\theta + \mu_K \sin\theta$, $\beta_K = -\lambda_K \sin\theta + \mu_K \cos\theta$)

then $\sum \alpha_K \beta_K = 0$ holds in this case. Hence we can apply the results obtained above and the characteristic function of $\tilde{x}_1$ is given by $E(\exp(it\,\tilde{x}_1)) = \exp(i\tilde{m}t + \tilde{\alpha} t^2)$, where $\tilde{m} = E(\tilde{x}_1) = \sum \lambda_K m_K$ and $\tilde{\alpha} = -\frac{1}{2}E((\tilde{x}_1 - \tilde{m})^2) = -\frac{1}{2}\sum_{K,j}\lambda_K\lambda_j E((x_K - m_K)(x_j - m_j)) = -\frac{\sigma^2}{2}(\sum_{K=1}^{n} \lambda_K^2)$. Putting $t=1$ and $\lambda_K = t_K$ in it we have

(12) $$E(\exp(\sum_{K=1}^{n} i\, t_K x_K)) = \exp(it \sum_{K=1}^{n} m_K t_K - \frac{\sigma^2}{2}\sum_{K=1}^{n} t_K^2).$$

Hence the multiple distribution of $(x_1, \ldots, x_n)$ is the normal distribution with the probability density (1), q.e.d.

Corollary 1. Let $x_1, \ldots, x_n$ be n random variables. If any two random variables $q_1$, $q_2$ defined by $q_1 = (\mathscr{C}A, \mathscr{C})$, $q_2 = (\mathscr{C}B, \mathscr{C})$ (A and B are symmetric matrices and $\mathscr{C} = (x_1, \ldots, x_n)$) are independent whenever AB=0, then the multiple distribution of $(x_1, \ldots, x_n)$ is the normal distribution with the probability density (1).

Proof. Take $q_1 = y^2$, $q_2 = z^2$ for $y, z$ in (2), then AB=0 means the relation (3) for $y, z$. $q_1$ and $q_2$ are independent if and only if $y$ and $z$ are independent. Hence we have Cor.1 from our Theorem, q.e.d.

Corollary 2. Let $x_1, \ldots, x_n$ be n random variables with means and with finite variances. If any two random variables $y, z$ defined by (2) are independent whenever the correlation coefficient of $y$ and $z$ is 0, then the multiple distribution of $(x_1, \ldots, x_n)$ is the normal distribution.

Proof. By a suitable linear transformation $(\alpha_{Kj})$ we can take

(13) $$\tilde{x}_K = \sum_{j=1}^{n} \alpha_{Kj} x_j$$

so that the variance matrix of $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is the unit matrix. Then the correlation coefficient of $y = \sum_K \alpha_K \tilde{x}_K$ and $z = \sum_K \beta_K \tilde{x}_K$ is 0 if and only if $\sum \alpha_K \beta_K = 0$. Hence we can apply our Theorem and the multiple distribution of $(x_1, \ldots, x_n)$ is the normal distribution of the form (1). Since $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is defined by (13), the multiple distribution of $(x_1, \ldots, x_n)$ is also the normal distribution, q.e.d.

Remark. A characterization of the n-dimensional normal distribution whose variance matrix V is proportional with the given positive definite non-degenerate matrix $\Lambda = (\lambda_{ij})$ is given by changing the condition (3) of the independence of $y$ and $z$ to

(3 ) $$\sum_{i,j} \lambda_{ij} \alpha_i \beta_j = 0.$$

(*) Received June 4; 1949.
Tokyo Bunrika Daigaku.