ON DISCRETE SUBGROUPS OF A LIE GROUP

By Hiraku TOYAMA.

The 1-dimensional commutative Lie group G, whether a vector group or a circle, can be approximated by discrete subgroups. Or in other words, we can construct an infinite sequence $g_1, g_2, g_3, \ldots, g_n, \ldots$ of discrete subgroups, so that for any given open subset U of G almost all g_i except a finite number of them have a non-empty intersection with U.

This fact holds clearly for n-dimensional commutative Lie groups, because they can be decomposed into vector group and circles.[1]

In this paper the author intended to discuss this problem of approximation by discrete groups for non-commutative Lie groups. The main result obtained is the following

Theorem 1. Let G be a connected Lie group with the discrete center, then it cannot be approximated by a sequence of discrete subgroups.

Proof. In G we construct a canonical system of coordinates of the first kind,[2] where any 1-parameter subgroup can be written as follows: $g(t)=at\ (|t|<\alpha)$ In accordance with it, there exists an open subset U, every point of which lies on one and only one 1-parameter group.

Let the commutator $xyx^{-1}y^{-1}$ be written in a power series

$$(xyx^{-1}y^{-1})^i = C^i_{\kappa\ell}x^\kappa y^\ell + \cdots\cdots$$

and for a sufficiently small sphere S it can be written [3]

$$(xyx^{-1}y^{-1})^i = C^i_{\kappa\ell}x^\kappa y^\ell + O(r^3)\quad r=\max(|x|,|y|),$$

and therefore

$$|xyx^{-1}y^{-1}| < \min(|x|,|y|).$$

At first we assume that such a sequence really exists.

Let a be an arbitrary point in S and $U(a)$ be some neighborhood of a. We designate by C_i one of elements of g_i whose distance from e is the smallest. Then for any $x \in S \cap g_i$

$$|xC_ix^{-1}C_i^{-1}| < \min(|x|,|C_i|) = |C_i|,$$

hence $xC_ix^{-1}C_i^{-1} = e$ that is, C_i commutes with every element of $S \cap g_i$.

We denote the 1-parameter subgroup $C_i t\ (|t|<\alpha)$ by h_i, then h_i cuts a boundary B of S in a point C_i'. Because B is compact, $\{C_i'\}$ has at least one limiting point C, or we can choose a subsequence $\{C'_{i_n}\}$ which converges to C.

Let g be an arbitrary point in S and $U(a)$ be an arbitrary neighborhood of g, then by the above assumption $U(g)$ has a non-empty intersection with g_i $(i=N,N+1,\cdots)$. Let g_i be one of g_i which lies in $U(g)$. As C_{i_n} commutes with g_{i_n}, $C_{i_n}^{k_n}$ commutes with g_{i_n},

$$g_{i_n}C_{i_n}^{k_n} = C_{i_n}^{k_n}g_{i_n}$$

We can suppose

$$\lim_{n\to\infty} g_{i_n} = g$$

for a suitably chosen element g_{i_n} of g_{i_n} and for every point p on $Ct\ (|t|<\alpha)$

$$\lim_{n\to\infty} C_{i_n}^{k_n} = p$$

because $|C_{i_n}| \to 0$, then by the continuity of multiplication

$$gp = pg$$

Hence we can conclude that p commutes with every element of S and by the connectedness of G, with every element of G.

Thus G must have a non-discrete center. This conclusion contradicts our assumption of the discrete center.

q.e.d.

The above problem of approximation remains unsolved for Lie groups with non-discrete center. In the following we will show two examples of non-commutative Lie groups with non-discrete center, the first of them being approximable by discrete subgroups and the second being not approximable.

The first example. Let G be a 3-dimensional group of matrices:

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

where a_{12}, a_{13}, a_{23} are any real numbers. We denote by g_λ a subgroup of matrices:

$$\begin{pmatrix} 1 & \lambda b_{12} & \lambda^2 b_{13} \\ 0 & 1 & \lambda b_{23} \\ 0 & 0 & 1 \end{pmatrix},$$

where b_{12}, b_{13}, b_{23} are integers and λ a real number. Then G can be approximated by a sequence g_λ, when $\lambda \longrightarrow 0$.

The second example: G consists of matrices:

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

where a_{11}, a_{33} are real positive and a_{12} real. Such a matrix is denoted simply by (a_{11}, a_{12}, a_{33}). If (a_{11}, a_{12}, a_{33}) $(a_{11} \neq 1)$ belongs to a discrete subgroup g, then a suitable conjugate subgroup cgc^{-1} contains $a=(a_{11}, 0, a_{33})$. If cgc^{-1} contains an elements $b=(b_{11}, b_{12}, b_{33})(b_{12}\neq 0)$ we make a commutator of b and a^n,

$$b a^n b^{-1} a^{-n} = (1, b_{12}(1-a_{11}^n), 1)$$

Let n tend to $-\infty$ in the case $a_{11} > 1$, and to $+\infty$ in the case $a_{11} < 1$, then it converges to $(1, b_{12}, 1)$. Hence cgc^{-1} is not discrete and the same for g. Therefore g does not contain such a element b $(b_{12} \neq 0)$, and is commutative, which cannot approximate G.

However, our problem is completely solved for compact Lie groups:

Theorem 2. Every non-commutative compact Lie group is not approximable by finite subgroups.

Proof is easily established, if we consider the Levi decomposition of Lie groups, and the commutativity of solvable compact Lie groups.

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(1) L.Pontrjagin, Topological groups, p.170.
(2) loc.cit.p.187.
(3) loc.cit.p.236.

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AN ELEMENTARY METHOD TO DERIVE THE NORMAL FORM OF N-DIMENSIONAL REAL EUCLIDEAN ROTATION

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It is a well-known theorem that n-dimensional real orthogonal matrix A can be transformed into a direct-product of several 2-dimensional rotations and, if it exists, one reflection. In this paper an elementary geometric method is explained. The essential point of this method is to find the fixed planes of the rotation.

Let $x, y, z, \ldots, a, b, c, \ldots$ be real vectors in n-dimensional real euclidean space R_n, and let $A, B, C, \ldots$ be real orthogonal (n,n)-matrices, while small Greek letters mean real numbers. We use the ordinary symbols of matrix-calculus, i.e., a' and A' mean the transposed ones, A^{-1} is its inverse and E is the unit-matrix.

THEOREM. Let A be any real orthogonal matrix, i.e., $A' = A^{-1}$. Suppose that $x'Ax$ with $|x| = 1$, attains the maximum value λ for $x = a$, where $|a| = 1$. Then $|\lambda| \leq 1$ and

a. if $\lambda = 1$, $Aa = a$;

b. if $\lambda = -1$, $A = -E$;

c. if $-1 < \lambda < 1$, $A^2a - 2\lambda Aa + a = 0$,

i.e., a and Aa span a fixed plane of A.

Proof. Put

$$(1) \qquad \lambda = \max_{|x|=1} x'Ax.$$

Then it is evident that $|\lambda| = 1$. Since the unit-sphere S^n in R_n is compact, there exists at least one vector a with $|a| = 1$, where

$$(2) \qquad \lambda = a'Aa.$$

We know that $|x|\cdot|y| = |xy|$ if and only if x and y are linearly dependent. Therefore if $\lambda = 1$, $Aa = a$, i.e., a is a fixed point. If $\lambda = -1$, we have $x'Ax = -1$ for any x, $|x| = 1$, i.e., $A = -E$. Finally if $-1 < \lambda < 1$, two vectors a and Aa are linearly