where on the right side $x^j = f_1^j(t_1) + \cdots + f_n^j(t_n)$ .

Proof. The integral on the right side is

$$\int_{0 \leqq t_i \leqq 1} A(x^1, \cdots, x^n)\, \det \left| \frac{d f_i^j(t_i)}{d t_i} \right| dt_1 \cdots dt_n .$$

Let us substitute $f_i(t)$ by partially linear curves $g_i(t)$ whose corners are $g_i(k/N) = f_i(k/N)$, $k = 1, 2, \cdots, N$ . For an arbitrarily given positive number $\delta$ , we can choose $N$ sufficiently large such that $|f_i^j(t) - g_i^j(t)| < \varepsilon$ and $\left| \frac{d f_i^j(t)}{dt} - \frac{d g_i^j(t)}{dt} \right| < \varepsilon$ ; consequently we get

$$\left| A(x(f))\, \det \left| \frac{d f_i^j(t_i)}{d t_i} \right| - A(x(g))\, \det \left| \frac{d g_i^j(t_i)}{d t_i} \right| \right| < \delta$$

and error of the integral (2) and that substituted $g_i(t_i)$ thereinto is less than $\delta$ . Then, on account of (1) our result follows.

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# VECTOR-GROUP IN REAL EUCLIDEAN SPACE

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We shall describe in this paper an elementary proof of the theorem which has also been proved in this volume by Prof. Iwamura, Messrs. M. Kuranishi and T. Hayashida.

We denote "free vectors" in an n-dimensional real euclidean space $R_n$ by $x, y, z, \ldots, a, b, c, \ldots\ldots$, and the corresponding points in $R_n$ by the same symbols, i.e., "a point $x$" means the point which is located by the free vector $x$ starting from the original point $o$ previously determined in $R_n$ . The distance between any two points $x$ and $y$ is defined by the euclidean one, i.e., $|x - y|$. We shall prove in this paper the following Theorem and Corollary.

THEOREM. Let $M$ be a real euclidean vector-group in $R_n$ and contain a continuum $K$ . Then $M$ contains the whole straight-line through any two distinct points of $K$ .

COROLLARY. Let $M$ be a real euclidean vector-group in $R_n$ and let any two points of $M$ be connected by a continuum in $M$ . Then $M$ coincides with a real linear vector-group.

We shall prove the theorem by the induction with respect to the dimension-number $n$ of $R_n$ . If $n = 1$ , the theorem is evident. Suppose $n > 1$ .

LEMMA 1. Let $K$ be any continuum in $M$ . We define $K'$ as the aggregate of all the points $x - y + z$ , where $x, y$ and $z$ run throughout $K$ . Then $K'$ is also a continuum in $M$ and $K \subset K'$ .

The proof is immediate. We are going to prove that the straight-line segment joining any two distinct points $a$ and $b$ of $K$ is contained in $K'' = (K')'$ . As $K$ is connected, $a$ and $b$ can be connected for any positive $\varepsilon$ by an $\varepsilon$-chain with its points of joint all belonging to $K$ . This chain can be represented by

$$x(t) ; \quad 0 \leqq t \leqq 1 ,$$

where $x(t)$ is a continuous curve in $0 \leqq t \leqq 1$ , with its points of joint $x(t_i)$: $0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$ all belonging to $K$ and the parts $x(t)$ , $t_i \leqq t \leqq t_{i+1}$ , $i = 0, 1, 2, \ldots, m-1$ are all straight-line segments. Moreover $|x(t_{i+1}) - x(t_i)| < \varepsilon$ , for $i = 0, 1, 2, \ldots, m-1$.

Now let $R_{n-1}$ be an $(n-1)$-dimensional hyperplane in $R_n$ through $a$ and $b$, two distinct points of $K$. Then the $R_{n-1}$ separates $R_n$ into two closed convex point sets $H_1$ and $H_2$.

LEMMA 2. There exists a line-segments-$\varepsilon$-chain in $H_1$ connecting $a$ and $b$ with its points of joint all belonging to $K'$.

Proof. To construct this new $\varepsilon$-chain, we replace every part of the chain $x(t)$, $t_{i_\nu} \le t \le t_{k_\nu}$, $i_\nu < k_\nu$, where $x(t_i)$ mean points of joint and

$$x(t_{i_\nu}) \in H_1 \;, \quad x(t_{k_\nu}) \in H_1 \;;$$

$$x(t_i) \in H_2 \,, \quad t_{i_\nu} < t_i < t_{k_\nu} \,,$$

by the following chains

$$y(t) = x(t_{k_\nu}) - x(t_{i_\nu} + t_{k_\nu} - t) + x(t_{i_\nu})$$

for $t_{i_\nu} < t < t_{k_\nu}$.

This new chain is evidently a straight-line-segments-$\varepsilon$-chain in $H_1$ connecting $a$ and $b$, and their points of joint all belong to $K'$. Hereafter we denote this chain by $y(t)$, of which points of joint are $y(t_i^0)$. Now let us displace this chain uniformly within its $\varepsilon$-neighbourhood so slightly into a new s.-l.-s.-$\varepsilon$-chain that it remains in $H_1$ and no two of its points of joint have the same height from the $R_{n-1}$ except the couple, $a$ and $b$ whose heights are both zero. Let us introduce into this chain a new parameter $t$ so that $t$ is proportional to its curve-length between two consecutive points of joint. Now let us represent this chain by $z(t)$ and let $z(t)$ have the maximum height from the $R_{n-1}$ at the unique point $z(t_p)$ : $0 < t_p < 1$

LEMMA 3. For the above mentioned chain $z(t)$ : $0 \le t \le 1$, $z(0) = a$, $z(1) = 1$ there exist two one-valued continuous real functions $t(u)$ and $s(u)$ in $0 \le u \le 1$ which satisfy .

$$t(0) = s(0) = t_p \;, \quad t(1) = 0, \quad s(1) = 0;$$

$$0 < t(u) < t_p \,, \; t_p < s(u) < 1 \quad \text{for} \quad 0 < u < 1,$$

and $z(t(u))$ and $z(s(u))$ have the same height from the $R_{n-1}$ for any $u$, $0 \le u \le 1$, and reciprocally. Let $F$ be the set of all the points $(t, s)$, where $0 \le t \le t_p$, $t_p \le s \le 1$ and $z(t)$ and $z(s)$ have the same height from the $R_{n-1}$. It can easily be seen that $F$ is a compact and locally connected set. In fact, any point of $F$, except $(0, 1)$ and $(t_p, t_p)$, has for its sufficiently small neighbourhood relative to $F$ two different straight-line segments, where the considered point of $F$ belongs simultaneously to one end of each segment. In other words, the relative neighbourhood of any point of $F$ except the two points, if sufficiently small, is homeomorphic to a line segment and the point is not on its end. As for the two exceptional points, their relative neighbourhoods are both straight-line segments but they lie on one end of each one. From the above mentioned properties we can easily conclude that $F$ has a simple chain (i.e., having no loop) joining $(0, 1)$ and $(t_p, t_p)$. ). If we represent this curve by

$$(t(u), s(u)) : \quad \begin{array}{l} t(0) = t_p \,, \;\; s(0) = t_p \\ t(1) = 0, \;\; s(1) = 1, \\ 0 \le u \le 1 . \end{array}$$

These $t(u)$ and $s(u)$ are the desired functions.

LEMMA 4. There exists a continuum in $K'' \cap R_{n-1}$ joining $a$ and $b$.

Proof. We can see easily that the continuous curve $x(u) = a + z(s(u)) - z(t(u))$ lies in the $R_{n-1}$ and there exists at least one point of $K''$ within $3\varepsilon$-neighbourhood of any point of the curve. If we tend $\varepsilon$ to zero, we can conclude that the $R_{n-1}$ contains a curve in the $K''$ connecting $a$ and $b$. q.e.d.

The Lemma 4 has reduced the problem to the case of $(n-1)$-dimensional space. This completes the induction. The theorem and the corollary are immediate consequences of Lemma 4.

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