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BOUNDEDNESS OF SOME SUBLINEAR OPERATORS IN WEIGHTED HERZ-TYPE SPACES

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Abstract

In this paper, the authors first establish the boundedness of sublinear operators on the weighted Herz space with general weights. At the extreme case, the authors show these operators are bounded from the weighted Herz space to the weighted weak Herz space. Moreover, the authors also discuss the boundedness of the local Calderón-Zygmund operator of the non-convolution type on the weighted Herz-type Hardy spaces and show that these operators map the weighted Herz-type Hardy space into the weighted weak Herz-type Hardy space at the extreme case.

1. Introduction

Let $B_k = B(0, 2^k) = \{x \in \boldsymbol{R}^n : |x| \le 2^k\}$ for any $k \in \boldsymbol{Z}$ and $R_k = B_k \backslash B_{k-1}$. The following weighted Herz space is introduced by Lu and Yang in [11].

DEFINITION 1. Let $\alpha \in \boldsymbol{R}$, $0 < p$, $q \le \infty$, ω_1 and ω_2 be any non-negative weight functions.

(a) The homogeneous weighted Herz space $\dot{K}_q^{\alpha, p}(\omega_1, \omega_2)$ is defined by $\dot{K}_q^{\alpha, p}(\omega_1, \omega_2) = \{f : f$ is a measurable function on $\boldsymbol{R}^n$ and $\|f\|_{\dot{K}_q^{\alpha, p}(\omega_1, \omega_2)} < \infty\}$, where

$$\|f\|_{\dot{K}_q^{\alpha, p}(\omega_1, \omega_2)} = \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \|f\chi_{R_k}\|_{L_{\omega_2}^q(\boldsymbol{R}^n)}^p \right\}^{1/p}$$

with the usual modifications when $p = \infty$ and/or $q = \infty$.

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(b) The non-homogeneous weighted Herz space $K_q^{\alpha,p}(\omega_1,\omega_2)$ is defined by $K_q^{\alpha,p}(\omega_1,\omega_2)=\{f : f$ is a measurable function on $\boldsymbol{R}^n$ and $\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}<\infty\}$, where

$$\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}=\left\{\|f\chi_{B_0}\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}+\sum_{k=1}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\|f\chi_{R_k}\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right\}^{1/p}$$

with the usual modifications when $p=\infty$ and/or $q=\infty$.

Here and in what follows, for any non-negative weight function ω, any measurable function f on $\boldsymbol{R}^n$ and any $q\in(0,\infty]$, we write

$$\|f\|_{L^q_\omega(\boldsymbol{R}^n)}=\left\{\int_{\boldsymbol{R}^n}|f(x)|^q\omega(x)\,dx\right\}^{1/q}$$

and

$$\|f\|_{WL^q_\omega(\boldsymbol{R}^n)}=\sup_{\lambda>0}\lambda[\omega(\{x\in\boldsymbol{R}^n:|f(x)|>\lambda\})]^{1/q}$$

with the usual modification when $q=\infty$. If $\omega(x)\equiv 1$, we will denote $L^q_\omega(\boldsymbol{R}^n)$ and $WL^q_\omega(\boldsymbol{R}^n)$ simply by $L^q(\boldsymbol{R}^n)$ and $WL^q(\boldsymbol{R}^n)$.

Obviously, if $\alpha=0$, then $\dot{K}_q^{0,q}(\omega_1,\omega_2)=K_q^{0,q}(\omega_1,\omega_2)=L^q_{\omega_2}(\boldsymbol{R}^n)$ for any $q\in(0,\infty]$. In what follows, if $\omega_1(x)\equiv\omega_2(x)\equiv 1$, we will denote $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and $K_q^{\alpha,p}(\omega_1,\omega_2)$ simply by $\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ and $K_q^{\alpha,p}(\boldsymbol{R}^n)$.

Let T be a sublinear operator satisfying that for any integrable function f with a compact support and $x\notin\operatorname{supp} f$,

$$|Tf(x)|\le c\int_{\boldsymbol{R}^n}\frac{|f(y)|}{|x-y|^n}\,dy, \tag{1}$$

where c is independent of f and x. In [11], [10] and [8], such a sublinear operator is proved to be bounded on $\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ and $K_q^{\alpha,p}(\boldsymbol{R}^n)$ provided T is bounded on $L^q(\boldsymbol{R}^n)$, $1<q<\infty$, $0<p\le\infty$ and $-n/q<\alpha<n(1-1/q)$. Some weighted version of this is also considered by Lu and Yang in [11]. The first target of this paper will extend the result in [11]. In other words, we will much relax the restriction on the weights; see the following Theorem 1.

Also, a sublinear operator satisfying (1) and being bounded on $L^q(\boldsymbol{R}^n)$ maybe is not bounded on $\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ or $K_q^{\alpha,p}(\boldsymbol{R}^n)$ for $0<p\le\infty$, $1<q<\infty$ and $\alpha=-n/q$ or $\alpha=n(1-1/q)$; see [11], [8] and [10]. However, in [6] and [7], Hu, Lu and Yang introduced the weak Herz space and proved that such an operator is indeed bounded from $\dot{K}_q^{n(1-1/q),p}(\boldsymbol{R}^n)$ to $W\dot{K}_q^{n(1-1/q),p}(\boldsymbol{R}^n)$ or from $K_q^{n(1-1/q),p}(\boldsymbol{R}^n)$ to $WK_q^{n(1-1/q),p}(\boldsymbol{R}^n)$ if $0<p\le 1$ and $1<q<\infty$. But, this is not true for $\alpha=-n/q$ or $p>1$; see [7] for some counter-examples. The second purpose of this paper is to establish the weighted versions of these results in these extreme cases. First, we introduce the following weighted weak Herz space.

Let ω_2 be any non-negative weight function. For $k \in \boldsymbol{Z}$, $\sigma > 0$ and any measurable function f on $\boldsymbol{R}^n$, we define

$$D_{k,\omega_2}(\sigma, f) = \omega_2(\{x \in R_k : |f(x)| > \sigma\});$$

for $k \in \boldsymbol{N}$, let $\tilde{D}_{k,\omega_2}(\sigma, f) = D_{k,\omega_2}(\sigma, f)$ and

$$\tilde{D}_{0,\omega_2}(\sigma, f) = \omega_2(\{x \in B_0 : |f(x)| > \sigma\}).$$

Definition 2. Let $\alpha \in \boldsymbol{R}$, $0 < q < \infty$, $0 < p \le \infty$ and ω_1, ω_2 be any non-negative weight functions.

(i) A measurable function f on $\boldsymbol{R}^n$ is said to belong to the homogeneous weighted weak Herz space $W\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ if

$$\|f\|_{W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda, f)]^{p/q} \right\}^{1/p} < \infty$$

with the usual modification made when $p = \infty$.

(ii) A measurable function f on $\boldsymbol{R}^n$ is said to belong to the non-homogeneous weighted weak Herz space $WK_q^{\alpha,p}(\omega_1, \omega_2)$ if

$$\|f\|_{WK_q^{\alpha,p}(\omega_1,\omega_2)} = \sup_{\lambda>0} \lambda \left\{ \sum_{k=0}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [\tilde{D}_{k,\omega_2}(\lambda, f)]^{p/q} \right\}^{1/p} < \infty$$

with the usual modification made when $p = \infty$.

If $\omega_1(x) \equiv \omega_2(x) \equiv 1$, we will denote $W\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ and $WK_q^{\alpha,p}(\omega_1, \omega_2)$ simply by $W\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ and $WK_q^{\alpha,p}(\boldsymbol{R}^n)$ which are introduced by Hu, Lu and Yang in [6] and [7]. Also, $W\dot{K}_q^{0,q}(\omega_1, \omega_2) = WK_q^{0,q}(\omega_1, \omega_2) = WL_{\omega_2}^q(\boldsymbol{R}^n)$ for any $q \in (0, \infty)$.

The third purpose of this paper is to relax the restriction on the weight of the weighted Herz-type Hardy spaces studied in [12]. That is, we shall establish the atomic decomposition for the weighted Herz-type Hardy space with more general weights. Using this atomic decomposition, we shall establish the boundedness of local Caldrón-Zygmund operators of non-convolutional type from these weighted Herz-type Hardy sapces into weighted Herz spaces or into weighted weak Herz spaces at the extreme cases. Moreover, if we further suppose that these operators satisfy a vanishing moment condition, we then shall show that they are indeed bounded on the weighted Herz-type Hardy space or, in the extreme case, from the weighted Herz-type Hardy space into the weighted weak Herz-type Hardy space whose definition will be given later. Our results of this part extend the corresponding results in [7] to both non-convolutional types and weighted versions.

Finally, we recall the definition of the weight as follows. Let $1 < p < \infty$. Following [5], a weight $\omega \ge 0$ is a Muckenhoupt $A_p(\boldsymbol{R}^n)$ weight if for any ball B

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B [\omega(x)]^{-1/(p-1)}\,dx\right)^{p-1} \le c$$

with c a constant independent of the ball B. The class $A_1(\boldsymbol{R}^n)$ is defined by letting $p \to 1$, namely,

$$\frac{1}{|B|}\int_B \omega(x)\,dx \le c\,\operatorname{essinf}_{x\in B}\omega(x)$$

with c independent of B. The smallest value of c satisfying the above inequalities is called the $A_p(\boldsymbol{R}^n)$-constant of ω. The following properties for $A_p(\boldsymbol{R}^n)$ weights will be repeatedly used in this paper; see [5], [15] for their proofs.

LEMMA 1. *Let $\omega \in A_p(\boldsymbol{R}^n)$ for some $p \in [1, \infty)$ and B be any ball. Then* (i) *for any measurable function f on B,*

$$|B|^{-1}\int_B |f(x)|\,dx \le c[\omega(B)]^{-1/p}\left(\int_B |f(x)|^p\omega(x)\,dx\right)^{1/p},$$

where c is independent of f and B;

(ii) *if E is a measurable subset of B, then*

$$\frac{\omega(B)}{\omega(E)} \le c\left(\frac{|B|}{|E|}\right)^p,$$

where c is independent of B and E;

(iii) *there exists a $\delta > 0$ such that if E is a measurable subset of B, then*

$$\frac{\omega(E)}{\omega(B)} \le c\left(\frac{|E|}{|B|}\right)^\delta,$$

where c is independent of B and E.

Throughout this paper, c always denotes a constant which is independent of the main parameters, but may vary from line to line.

We also remark that there is a similar result on the non-homogeneous Herz-type space for any of our result on the homogeneous Herz-type space. For simplicity, we only state our results in the homogeneous Herz-type version.

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2. Boundedness on weighted Herz spaces

We begin with the boundedness on the weighted Herz space $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ for the sublinear operator satisfying certain "size" conditions.

THEOREM 1. *Let $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$, $0 < p \le \infty$ and $1 < q < \infty$. If a sublinear operator T is bounded on $L^q_{\omega_2}(\boldsymbol{R}^n)$ and satisfies* (1), *then T is also bounded on $\dot{K}_q^{\alpha,p}(\omega_1, \omega_2)$ provided that ω_1 and ω_2 satisfy either of the following*

(i) $\omega_1 = \omega_2$, $1 \le q_{\omega_1} \le q$ *and* $-nq_{\omega_1}/q < \alpha q_{\omega_1} < n(1 - q_{\omega_1}/q)$;
(ii) $1 \le q_{\omega_1} < \infty$, $1 \le q_{\omega_2} \le q$ *and* $0 < \alpha q_{\omega_1} < n(1 - q_{\omega_2}/q)$.

Proof. In what follows, let $\chi_k = \chi_{R_k}$ for any $k \in \boldsymbol{Z}$. We write

$$\begin{aligned}
\|Tf\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} &= \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \|(Tf)\chi_k\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}^p \right\}^{1/p} \\
&\le c \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \left(\sum_{l=-\infty}^{k-3} \|\chi_k(Tf\chi_l)\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \right)^p \right\}^{1/p} \\
&\quad + c \left\{ \sum_{k=-\infty}^{\infty} \cdots \left(\sum_{l=k-2}^{k+2} \cdots \right)^p \right\}^{1/p} + \left\{ \sum_{k=-\infty}^{\infty} \cdots \left(\sum_{l=k+3}^{\infty} \cdots \right)^p \right\}^{1/p} \\
&\equiv E_1 + E_2 + E_3.
\end{aligned}$$

By the $L^q_{\omega_2}(\boldsymbol{R}^n)$-boundedness of T, we are easy to obtain a desirable estimate for E_2.

For E_1, when $x \in R_k$ and $l \le k-3$, by (1) and Hölder's inequality, we have

$$\begin{aligned}
|T(f\chi_l)(x)| &\le c \int_{R_l} \frac{|f(y)|}{|x-y|^n}\, dy \\
&\le \frac{c}{2^{kn}} \int_{R_l} |f(y)|\, dy \\
&\le \frac{c}{2^{kn}} \|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \left(\int_{B_l} [\omega_2(x)]^{-1/(q-1)}\, dx \right)^{1/q'} \\
&\le c 2^{(l-k)n} \|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \frac{1}{[\omega_2(B_l)]^{1/q}},
\end{aligned}$$

since $\omega_2 \in A_q(\boldsymbol{R}^n)$. Thus, by $\alpha q_{\omega_1} < n(1 - q_{\omega_2}/q)$, we have

$$\begin{aligned}
E_1 &\le c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-3} [\omega_1(B_l)]^{\alpha/n} \|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} 2^{(l-k)n} \times \left[\frac{\omega_1(B_k)}{\omega_1(B_l)}\right]^{\alpha/n} \left[\frac{\omega_2(B_k)}{\omega_2(B_l)}\right]^{1/q} \right)^p \right\}^{1/p} \\
&\le c \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{l=-\infty}^{k-3} [\omega_1(B_l)]^{\alpha/n} \|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \times 2^{(k-l)(\alpha q_{\omega_1} + nq_{\omega_2}/q - n)} \right)^p \right\}^{1/p}
\end{aligned}$$

$$
\leq \begin{cases} c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-3}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right.\right. \\ \quad \left.\left.\times\, 2^{(k-l)(\alpha q_{\omega_1}+nq_{\omega_2}/q-n)p}\right)\right\}^{1/p}, \quad \text{if } 0<p\leq 1 \\ c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-3}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\times 2^{(k-l)(\alpha q_{\omega_1}+nq_{\omega_2}/q-n)p/2}\right)\right. \\ \quad \left.\times\left(\sum_{l=-\infty}^{k-3}2^{(k-l)(\alpha q_{\omega_1}+nq_{\omega_2}/q-n)p'/2}\right)^{p/p'}\right\}^{1/p}, \quad \text{if } 1<p<\infty \end{cases}
$$

$$
\leq \begin{cases} c\left\{\sum_{l=-\infty}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right. \\ \quad \left.\times\left(\sum_{k=l+3}^{\infty}2^{(k-l)(\alpha q_{\omega_1}+nq_{\omega_2}/q-n)p}\right)\right\}^{1/p}, \quad \text{if } 0<p\leq 1 \\ c\left\{\sum_{l=-\infty}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right. \\ \quad \left.\times\left(\sum_{k=l+3}^{\infty}2^{(k-l)(\alpha q_{\omega_1}+nq_{\omega_2}/q-n)p/2}\right)\right\}^{1/p}, \quad \text{if } 1<p<\infty \end{cases}
$$

$$
\leq c\left\{\sum_{l=-\infty}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right\}^{1/p}
$$

$$
= c\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)},
$$

where, and in what follows, $1/p+1/p'=1$ and we used the fact that $\alpha+n/q_1>0$ when $\omega_1=\omega_2$ and $\alpha>0$ when $\omega_1\neq\omega_2$. So far, we have obtained a desirable estimate for E_1.

For the estimate of E_3, when $x\in R_k$ and $l\geq k+3$, by (1), we have

$$
\begin{aligned}
|T(f\chi_l)(x)| &\leq c\int_{R_l}\frac{|f(y)|}{|x-y|^n}\,dy \\
&\leq \frac{c}{2^{ln}}\int_{R_l}|f(y)|\,dy \\
&\leq \frac{c}{2^{ln}}\|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\left(\int_{B_l}[\omega_2(x)]^{-1/(q-1)}\,dx\right)^{1/q'} \\
&\leq c\frac{\|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}}{[\omega_2(B_l)]^{1/q}},
\end{aligned}
$$

since $\omega_2 \in A_q(\boldsymbol{R}^n)$. Thus, we have

$$E_3 \le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+3}^{\infty}[\omega_1(B_l)]^{\alpha/n}\|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\left[\frac{\omega_1(B_k)}{\omega_1(B_l)}\right]^{\alpha/n}\left[\frac{\omega_2(B_k)}{\omega_2(B_l)}\right]^{1/q}\right)^p\right\}^{1/p}$$

$$\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+3}^{\infty}[\omega_1(B_l)]^{\alpha/n}\|f\chi_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}2^{(k-l)n(\delta_1\alpha/n+\delta_2/q)}\right)^p\right\}^{1/p}$$

$$\le \begin{cases} c\left\{\displaystyle\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+3}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\times 2^{(k-l)(\delta_1\alpha+\delta_2 n/q)p}\right)\right\}^{1/p}, & \text{if } 0<p\le 1 \\ c\left\{\displaystyle\sum_{k=-\infty}^{\infty}\left(\sum_{l=k+3}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\times 2^{(k-l)(\delta_1\alpha+\delta_2 n/q)p/2}\left(\sum_{l=k+3}^{\infty}2^{(k-l)(\delta_1\alpha+\delta_2 n/q)p'/2}\right)^{p/p'}\right)\right\}^{1/p}, & \text{if } 1<p<\infty \end{cases}$$

$$\le \begin{cases} c\left\{\displaystyle\sum_{l=-\infty}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\times\left(\sum_{k=-\infty}^{l-3}2^{(k-l)(\delta_1\alpha+\delta_2 n/q)p}\right)\right\}^{1/p}, & \text{if } 0<p\le 1 \\ c\left\{\displaystyle\sum_{l=-\infty}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\times\left(\sum_{k=-\infty}^{l-3}2^{(k-l)(\delta_1\alpha+\delta_2 n/q)p/2}\right)\right\}^{1/p}, & \text{if } 1<p<\infty \end{cases}$$

$$\le c\left\{\sum_{l=-\infty}^{\infty}[\omega_1(B_l)]^{\alpha p/n}\|f\chi_l\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right\}^{1/p}$$

$$= c\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)},$$

where $\delta_1 > 0$, $\delta_2 > 0$ depend only on n and the $A_{q_{\omega_1}}(\boldsymbol{R}^n)$-constant and the $A_{q_{\omega_2}}(\boldsymbol{R}^n)$-constant, and $\delta_1 = \delta_2$ when $\omega_1 = \omega_2$.

We leave the case $p = \infty$ to the reader.

This finishes the proof of Theorem 1. □

We remark that the condition (1) can be replaced by more general conditions; see [10].

On the end cases of Theorem 1, we have the following conclusion, which can be regarded as a weighted version of Theorem 1 in [7].

THEOREM 2. *Let* $\omega_1, \omega_2 \in A_1(\boldsymbol{R}^n)$, $0 < p \le 1$, $1 \le q < \infty$ *and* $\alpha = n(1 - 1/q)$. *If a sublinear operator* T *is bounded from* $L^q_{\omega_2}(\boldsymbol{R}^n)$ *into* $WL^q_{\omega_2}(\boldsymbol{R}^n)$ *and satisfies* (1), *then* T *is also bounded from* $\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$ *into* $W\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$.

Proof. Let $f \in \dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$ and for any $k \in \boldsymbol{Z}$, we write

$$f(x) = f(x)\chi_{\{|x| \le 2^{k-3}\}}(x) + f(x)\chi_{\{2^{k-3}<|x| \le 2^{k+2}\}}(x) + f(x)\chi_{\{|x|>2^{k+2}\}}(x)$$
$$\equiv f_1^k(x) + f_2^k(x) + f_3^k(x).$$

Then $|Tf(x)| \le |Tf_1^k(x)| + |Tf_2^k(x)| + |Tf_3^k(x)|$, and

$$\begin{aligned}
\|Tf\|_{W\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} &= \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda, Tf)]^{p/q} \right\}^{1/p} \\
&= c \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda/3, |Tf_1^k|)]^{p/q} \right\}^{1/p} \\
&\quad + c \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda/3, |Tf_2^k|)]^{p/q} \right\}^{1/p} \\
&\quad + c \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda/3, |Tf_3^k|)]^{p/q} \right\}^{1/p} \\
&\equiv F_1 + F_2 + F_3.
\end{aligned}$$

By the fact that T is bounded from $L^q_{\omega_2}(\boldsymbol{R}^n)$ into $WL^q_{\omega_2}(\boldsymbol{R}^n)$, we easily obtain a desirable estimate for F_2.

To estimate F_1, for $x \in R_k$, by (1) and Minkowski's inequality, we have

$$\begin{aligned}
|Tf_1^k(x)| &\le c \int_{\boldsymbol{R}^n} \frac{|f_1^k(y)|}{|x-y|^n}\, dy \\
&\le \begin{cases} \dfrac{c}{\omega_2(B_k)} \|f_1^k\|_{L^1_{\omega_2}(\boldsymbol{R}^n)}, & q = 1 \\[2ex] c2^{-kn} \displaystyle\sum_{j=-\infty}^{k-3} \|f\chi_j\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \left\{ \int_{B_j} [\omega_2(x)]^{-1/(q-1)}\, dx \right\}^{1/q'}, & q > 1 \end{cases} \\
&\le \begin{cases} \dfrac{c}{\omega_2(B_k)} \|f_1^k\|_{L^1_{\omega_2}(\boldsymbol{R}^n)}, & q = 1 \\[2ex] \dfrac{c}{2^{kn}} \displaystyle\sum_{j=-\infty}^{k-3} \frac{2^{jn}}{[\omega_2(B_j)]^{1/q}} \|f\chi_j\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}, & q > 1 \end{cases}
\end{aligned}$$

$$\le \frac{c_1}{[\omega_1(B_k)]^{\alpha/n}[\omega_2(B_k)]^{1/q}} \sum_{j=-\infty}^{k-3} [\omega_1(B_j)]^{\alpha/n} \|f\chi_j\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}$$

$$\le \frac{c_1}{[\omega_1(B_k)]^{\alpha/n}[\omega_2(B_k)]^{1/q}} \|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}$$

since $\omega_2 \in A_q(\mathbb{R}^n)$, $p \le 1$ and $\alpha = n(1-1/q)$. Now, for any given $\lambda > 0$, let k_λ be the greatest integer satisfying

$$\lambda/3 < \frac{c_1}{[\omega_1(B_{k_\lambda})]^{\alpha/n}[\omega_2(B_{k_\lambda})]^{1/q}} \|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}.$$

Then,

$$\begin{aligned}
F_1 &\le c \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{k_\lambda} [\omega_1(B_k)]^{\alpha p/n}[\omega_2(B_k)]^{p/q} \right\}^{1/p} \\
&\le c\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \sup_{\lambda>0} \left\{ \sum_{k=-\infty}^{k_\lambda} \left[\frac{\omega_1(B_k)}{\omega_1(B_{k_\lambda})}\right]^{\alpha p/n} \left[\frac{\omega_2(B_k)}{\omega_2(B_{k_\lambda})}\right]^{p/q} \right\}^{1/p} \\
&\le c\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)} \sup_{\lambda>0} \left\{ \sum_{k=-\infty}^{k_\lambda} 2^{(k-k_\lambda)(\delta_1\alpha+\delta_2 n/q)p} \right\}^{1/p} \\
&\le c\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)},
\end{aligned}$$

where $\delta_1 > 0$ and $\delta_2 > 0$ depend on n and the $A_1(\boldsymbol{R}^n)$-constants of ω_1 and ω_2; see Lemma 1. This is a desirable estimate for F_1.

We now estimate F_3. For $x \in R_k$, by (1) we have

$$\begin{aligned}
|Tf_3^k(x)| &\le c \int_{\boldsymbol{R}^n} \frac{|f_3^k(y)|}{|x-y|^n}\, dy \\
&\le c \sum_{j=k+3}^{\infty} 2^{-jn} \|f\chi_j\|_{L^1(\boldsymbol{R}^n)} \\
&\le \begin{cases} c \displaystyle\sum_{j=k+3}^{\infty} \frac{1}{\omega_2(B_j)} \|f\chi_j\|_{L^1_{\omega_2}(\boldsymbol{R}^n)}, & q = 1 \\ c \displaystyle\sum_{j=k+3}^{\infty} 2^{-jn} \|f\chi_j\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \left(\int_{B_j} [\omega_2(x)]^{-1/(q-1)}\, dx \right)^{1/q'}, & q > 1 \end{cases} \\
&\le c \sum_{j=k+3}^{\infty} \frac{1}{[\omega_2(B_j)]^{1/q}} \|f\chi_j\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}
\end{aligned}$$

$$\leq \frac{c}{[\omega_1(B_k)]^{\alpha/n}[\omega_2(B_k)]^{1/q}} \sum_{j=k+3}^{\infty} [\omega_1(B_j)]^{\alpha/n} \|f\chi_j\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}$$

$$\leq \frac{c_2}{[\omega_1(B_k)]^{\alpha/n}[\omega_2(B_k)]^{1/q}} \|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)},$$

since $\omega_2 \in A_q(\boldsymbol{R}^n)$, $p \leq 1$ and $\alpha \geq 0$.

Now, similar to the estimate for F_1, we can show that

$$F_3 \leq c\|f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}.$$

This finishes the proof of Theorem 2. □

3. Boundedness on weighted Herz-type Hardy spaces

Now we turn to consider the behaviour of local Calderón-Zygmund type operators on the weighted Herz-type Hardy spaces. We begin with recalling some definitions.

In what follows, for $s \in \boldsymbol{R}$, let $[s]$ denote the greatest integer $\leq s$ if $s \geq 0$ or 0 if $s < 0$. Define

$$\mathscr{A}_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(\boldsymbol{R}^n) = \Bigg\{\phi \in \mathscr{S}(\boldsymbol{R}^n) : \sup_{|\beta| \leq [q_{\omega_1}\alpha + n(q_{\omega_2}/q-1)]+1} (1+|x|)^{[q_{\omega_1}\alpha+n(q_{\omega_2}/q-1)]+n+1}|D^\beta\phi(x)| \leq 1\Bigg\},$$

where $\mathscr{S}(\boldsymbol{R}^n)$ is the space of the Schwartz functions, $\beta = (\beta_1, \ldots, \beta_n) \in (\boldsymbol{N} \cup \{0\})^n$ and $D^\beta = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$. Moreover, we define

$$\|\phi\|_{\mathscr{A}_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(\boldsymbol{R}^n)} \equiv \sup_{x \in \boldsymbol{R}^n} \sup_{|\beta| \leq [q_{\omega_1}\alpha+n(q_{\omega_2}/q-1)]+1} (1+|x|)^{[q_{\omega_1}\alpha+n(q_{\omega_2}/q-1)]+n+1}|D^\beta\phi(x)|.$$

Let $\mathscr{S}'(\boldsymbol{R}^n)$ be the space of Schwartz distributions. For $f \in \mathscr{S}'(\boldsymbol{R}^n)$, we define

$$G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}f(x) = \sup_{\phi \in \mathscr{A}_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(\boldsymbol{R}^n)} \sup_{|x-y|<t} |(f * \phi_t)(y)|,$$

where $\phi_t(y) = t^{-n}\phi(y/t)$ for any $t > 0$. $G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}f$ is usually called to be the grand maximal function of f; see ([15], p. 90).

Now, we can give the definition of the weighted Herz-type Hardy space.

Definition 3. Let $\alpha \in \boldsymbol{R}$, $0 < p$, $q \leq \infty$, $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$ and $1 \leq q_{\omega_1}, q_{\omega_2} < \infty$. The homogeneous weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ on $\boldsymbol{R}^n$ is defined by

$$H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \{f \in \mathscr{S}'(\boldsymbol{R}^n) : G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}f \in \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)\},$$

and

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

The non-homogeneous weighted Herz-type Hardy space $HK_q^{\alpha,p}(\omega_1,\omega_2)$ on $\boldsymbol{R}^n$ is defined by

$$HK_q^{\alpha,p}(\omega_1,\omega_2) = \{f \in \mathscr{S}'(\boldsymbol{R}^n) : G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f \in K_q^{\alpha,p}(\omega_1,\omega_2)\},$$

and

$$\|f\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)} = \|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f\|_{K_q^{\alpha,p}(\omega_1,\omega_2)}.$$

If $\omega_1 \equiv \omega_2 \equiv 1$, we will denote $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and $HK_q^{\alpha,p}(\omega_1,\omega_2)$ simply by $H\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ and $HK_q^{\alpha,p}(\boldsymbol{R}^n)$ which are studied by [2], [4], [13] and so on when α and p take some special values. If $\omega_1,\omega_2 \in A_1(\boldsymbol{R}^n)$, the above Hardy spaces are studied by Lu and Yang in [12].

Applying Theorem 1, we can obtain the following relation between the weighted Herz space and the weighted Herz-type Hardy space. We omit the details.

THEOREM 3. *Let* $0 < p \le \infty$, $1 < q < \infty$, $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$ *and* $1 \le q_{\omega_1} < \infty$. *Then* $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ *and* $HK_q^{\alpha,p}(\omega_1,\omega_2) = K_q^{\alpha,p}(\omega_1,\omega_2)$ *provided either of the following holds*:

(i) $\omega_1 = \omega_2$, $1 \le q_{\omega_1} \le q$ *and* $-nq_{\omega_1}/q < \alpha q_{\omega_1} < n(1 - q_{\omega_1}/q)$;

(ii) $1 \le q_{\omega_2} \le q$ *and* $0 < \alpha q_{\omega_1} < n(1 - q_{\omega_2}/q)$.

Thus, the interesting case of the Herz-type Hardy space is $\alpha \ge (1 - q_{\omega_2}/q)/q_{\omega_1}$. For these spaces, we can establish their atomic decomposition.

DEFINITION 4. Let $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$, $1 \le q_{\omega_1}, q_{\omega_2} < \infty$, $1 < q < \infty$, $n(1-q_{\omega_2}/q) \le \alpha q_{\omega_1} < \infty$ and the non-negative integer $s = [\alpha q_{\omega_1} + n(q_{\omega_2}/q - 1)]$. A function $a(x)$ is said to be a central $(\alpha, q; \omega_1, \omega_2)$-atom, if it satisfies

(i) $\operatorname{supp} a \subset B(0,r) \equiv \{x \in \boldsymbol{R}^n : |x| \le r\}$ for some $r > 0$,

(ii) $\|a\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \le [\omega_1(B(0,r))]^{-\alpha/n}$,

(iii) $\int_{\boldsymbol{R}^n} a(x)x^\beta\, dx = 0$, $|\beta| \le s$.

When $\omega_1(x) \equiv \omega_2(x) \equiv 1$, we will denote the central $(\alpha, q; \omega_1, \omega_2)$-atom simly by (α, q)-atom.

Then by a similar proof to that of Theorem 1 in [12], we can show the following atomic decomposition; see also [4] and [13].

THEOREM 4. *Let* $0 < p \le \infty$, $1 < q < \infty$, $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$, $1 \le q_{\omega_1} < \infty$, $1 \le q_{\omega_2} \le q$ *and* $n(1 - q_{\omega_2}/q) \le \alpha q_{\omega_1} < \infty$. *Then* $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ (*or* $f \in HK_q^{\alpha,p}(\omega_1,\omega_2)$) *if and only if* $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$ (*or* $f = \sum_{k=0}^{\infty} \lambda_k a_k$) *holds in* $\mathscr{S}'(\boldsymbol{R}^n)$, *where* a_k *is the central* $(\alpha, q; \omega_1, \omega_2)$-*atom supported in* B_k *and*

$\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ (*or* $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$). *Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} \sim \inf\left\{\left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right)^{1/p}\right\}$$

$$\left(or\ \|f\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)} \sim \inf\left\{\left(\sum_{k=0}^{\infty} |\lambda_k|^p\right)^{1/p}\right\}\right),$$

where the infimum is taken over all the above decompositions of f.

We remark that by the proof, Theorem 4 is still true if $\alpha > 0$. Also, if $0 < p \le 1$, the central atom a_k appearing in Theorem 1 does not necessarily support in B_k and can support in any ball with the center at the origin.

Now, we have the following boundedness theorem on the local Calderón-Zygmund operator.

THEOREM 5. *Let $T : \mathscr{S}(\boldsymbol{R}^n) \to \mathscr{S}'(\boldsymbol{R}^n)$ be a linear and continuous operator. Suppose that the distribution kernel of T coincides in the complement of the diagonal with a locally integrable function $k(x, y)$ satisfying*

$$|k(x,y) - k(x,0)| \le c\frac{|y|^\delta}{|x|^{n+\delta}} \tag{2}$$

when $2|y| < |x|$ for some $\delta \in (0,1]$. Let $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$, $1 \le q_{\omega_1} < \infty$, $1 \le q_{\omega_2} \le q < \infty$ and $n(1 - q_{\omega_2}/q) \le \alpha q_{\omega_1} < n(1 - q_{\omega_2}/q) + \delta$. If T is bounded on $L^q_{\omega_2}(\boldsymbol{R}^n)$, then T is also bounded from $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ into $\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ for any $p \in (0,\infty]$.

Proof. Let $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$. By Theorem 4, we have $f = \sum_{k=-\infty}^{\infty} \lambda_k a_k$, where a_k is the central $(\alpha, q; \omega_1, \omega_2)$-atom supported in B_k and

$$\left\{\sum_{k=-\infty}^{\infty} |\lambda_k|^p\right\}^{1/p} \le c\|f\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

Write

$$\begin{aligned}
\|Tf\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} &= \left\{\sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \|\chi_k Tf\|^p_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right\}^{1/p} \\
&\le c\left\{\sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n}\left(\sum_{l=-\infty}^{k-4} |\lambda_l|\, \|\chi_k Ta_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right)^p\right\}^{1/p} \\
&\quad + c\left\{\sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n}\left(\sum_{l=k-3}^{\infty} |\lambda_l|\, \|\chi_k Ta_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right)^p\right\}^{1/p} \\
&\equiv H_1 + H_2.
\end{aligned}$$

For H_2, by the $L^q_{\omega_2}(\boldsymbol{R}^n)$-boundedness of T and $\alpha > 0$, we have

$$\begin{aligned}
H_2 &\le c\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{l=k-3}^{\infty}|\lambda_l|\,\|a_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\right)^p\right\}^{1/p}\\
&\le c\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{l=k-3}^{\infty}|\lambda_l|[\omega_1(B_l)]^{-\alpha/n}\right)^p\right\}^{1/p}\\
&\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=k-3}^{\infty}|\lambda_l|2^{(k-l)}\delta_\alpha\right)^p\right\}^{1/p}\\
&\le c\left\{\sum_{l=\infty}^{\infty}|\lambda_l|^p\right\}^{1/p}\\
&\le c\|f\|_{HK^{\alpha\, p}_q(\omega_1,\omega_2)},
\end{aligned}$$

where $\delta > 0$ depends only on n and $A_{q_{\omega_1}}(\boldsymbol{R}^n)$-constant of ω_1, and we have omitted some similar computational techniques to those used in the estimates for E_1 and E_3.

To estimate H_1, for $x \in R_k$ and $l \le k-4$, we have

$$\begin{aligned}
|Ta_l(x)| &\le \int_{B_l}|k(x,y)-k(x,0)||a_l(y)|\,dy \qquad (3)\\
&\le c\int_{B_l}\frac{|y|^\delta}{|x|^{n+\delta}}|a_l(y)|\,dy\\
&\le c2^{-k(n+\delta)+l\delta}\int_{B_l}|a_l(y)|\,dy\\
&\le c2^{-k(n+\delta)+l\delta}\|a_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\left(\int_{B_l}[\omega_2(x)]^{-1/(q-1)}\,dx\right)^{1/q'}\\
&\le c2^{(l-k)(n+\delta)}[\omega_1(B_l)]^{-\alpha/n}[\omega_2(B_l)]^{-1/q}.
\end{aligned}$$

Thus, by $\alpha > 0$, $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$ and $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$, we have

$$\begin{aligned}
H_1 &\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-4}|\lambda_l|\left[\frac{\omega_1(B_k)}{\omega_1(B_l)}\right]^{\alpha/n}\left[\frac{\omega_2(B_k)}{\omega_2(B_l)}\right]^{1/q}2^{(l-k)(n+\delta)}\right)^p\right\}^{1/p}\\
&\le c\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-4}|\lambda_l|2^{(k-l)(q_{\omega_1}\alpha+nq_{\omega_2}/q-n-\delta)}\right)^p\right\}^{1/p}\\
&\le c\left\{\sum_{l=-\infty}^{\infty}|\lambda_l|^p\right\}^{1/p}\\
&\le c\|f\|_{HK^{\alpha,\,p}_q(\omega_1,\omega_2)}
\end{aligned}$$

since $\alpha q_{\omega_1} + nq_{\omega_2}/q - n - \delta < 0$, where we have omitted some similar computational techniques to the before.

This finishes the proof of Theorem 5. □

If $q_{\omega_1} = q_{\omega_2} = 1$ and $\alpha q_{\omega_1} = n(1 - q_{\omega_2}/q) + \delta$, then we have the following weak boundedness theorem which can be regarded as the weighted version of Theorem 3 in [7].

THEOREM 6. *Let T and k be the same as in Theorem 5. Let $0 < p \le 1$, $\omega_1, \omega_2 \in A_1(\boldsymbol{R}^n)$, $1 < q < \infty$ and $\alpha = n(1 - 1/q) + \delta$. If T is bounded from $L^q_{\omega_2}(\boldsymbol{R}^n)$ into $WL^q_{\omega_2}(\boldsymbol{R}^n)$, then T is also bounded from $H\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$ into $W\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$.*

Proof. Let $f \in H\dot{K}^{\alpha,p}_q(\omega_1, \omega_2)$ and write f as in the proof of Theorem 5. We then have

$$\begin{aligned}
\|Tf\|_{W\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)} &= \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda, Tf)]^{p/q} \right\}^{1/p} \\
&\le c \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \left[D_{k,\omega_2}\left(\lambda/2, \sum_{l=-\infty}^{k-4} \lambda_l T a_l\right)\right]^{p/q} \right\}^{1/p} \\
&\quad + c \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} \left[D_{k,\omega_2}\left(\lambda/2, \sum_{l=k-3}^{\infty} \lambda_l T a_l\right)\right]^{p/q} \right\}^{1/p} \\
&\equiv I_1 + I_2.
\end{aligned}$$

A desirable estimate for I_2 can be deduced from the boundedness of T from $L^q_{\omega_2}(\boldsymbol{R}^n)$ into $WL^q_{\omega_2}(\boldsymbol{R}^n)$; we omit the details.

For I_1, by (3), for $x \in R_k$, we have

$$\begin{aligned}
\left| \sum_{l=-\infty}^{k-4} \lambda_l T a_l(x) \right| &\le c \sum_{l=-\infty}^{k-4} |\lambda_l| 2^{(l-k)(n+\delta)} [\omega_1(B_l)]^{-\alpha/n} [\omega_2(B_l)]^{-1/q} \\
&\le \frac{c}{[\omega_1(B_k)]^{\alpha/n} [\omega_2(B_k)]^{1/q}} \sum_{l=-\infty}^{k-4} |\lambda_l| \\
&\le \frac{c}{[\omega_1(B_k)]^{\alpha/n} [\omega_2(B_k)]^{1/q}} \|f\|_{H\dot{K}^{\alpha,p}_q(\omega_1,\omega_2)},
\end{aligned}$$

since $p \in (0, 1]$ and $\alpha = n(1 - 1/q) + \delta$.

Now, by a similar computation to that for F_1, we can easily obtain a desirable estimate for I_1.

This finishes the proof of Theorem 6. □

To investigate the boundedness of the operator T in Theorems 5 and 6 on the space $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$, we need T satisfies the following cancellation property; see [14].

DEFINITION 5. Let T be a linear operator. We say $T^*1=0$ if $\int_{\mathbf{R}^n} Ta(x)\,dx = 0$ for all compactly supported bounded measurable functions a such that $\int_{\mathbf{R}^n} a(x)\,dx = 0$.

The following theorem is a strong version of Theorem 5 and generalize Theorem 4 in [7] to both non-convolutional type and the weighted version.

THEOREM 7. *Let $T:\mathscr{S}(\mathbf{R}^n)\to\mathscr{S}'(\mathbf{R}^n)$ be a linear and continuous operator. Assume that the distributional kernel, $k(x,y)$, of T satisfies* (2) *for some $\delta\in(0,1]$. Let $\omega_1\in A_{q_{\omega_1}}(\mathbf{R}^n)$, $\omega_2\in A_{q_{\omega_2}}(\mathbf{R}^n)$, $1\le q_{\omega_1}<\infty$, $1\le q_{\omega_2}<q$ and $n(1-q_{\omega_2}/q)\le \alpha q_{\omega_1}<n(1-q_{\omega_2}/q)+\delta$. If T is bounded on $L^q_{\omega_2}(\mathbf{R}^n)$ and $T^*1=0$, then T is also bounded on $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ for any $p\in(0,\infty]$.*

Proof. Let $f\in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and write f as in the proof of Theorem 5. We then have

$$
\begin{aligned}
&\|Tf\|_{H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}\\
&\quad=\|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Tf)\|_{\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}\\
&\quad=\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\|\chi_k G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Tf)\|^p_{L^q_{\omega_2}(\mathbf{R}^n)}\right\}^{1/p}\\
&\quad\le c\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{l=-\infty}^{k-4}|\lambda_l|\,\|\chi_k G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Ta_l)\|_{L^q_{\omega_2}(\mathbf{R}^n)}\right)^p\right\}^{1/p}\\
&\qquad+c\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left(\sum_{l=k-3}^{\infty}|\lambda_l|\,\|\chi_k G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Ta_l)\|_{L^q_{\omega_2}(\mathbf{R}^n)}\right)^p\right\}^{1/p}\\
&\quad\equiv J_1+J_2.
\end{aligned}
$$

Applying $L^q_{\omega_2}(\mathbf{R}^n)$-boundedness of both $G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}$ and T, we easily deduce a desirable estimate for J_2.

For J_1, we first estimate $G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Ta_l)(x)$ for $x\in R_k$ and $l\le k-4$. In this case, choosing any $\phi\in\mathscr{A}_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(\mathbf{R}^n)$ with $\|\phi\|_{\mathscr{A}_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(\mathbf{R}^n)}\le 1$, for $x\in R_k$, $|x-y|<t$ and $l\le k-4$, by $T^*1=0$, we have

$$
\begin{aligned}
|(Ta*\phi_t)(y)|&=\left|\int_{\mathbf{R}^n}Ta_l(z)\frac{1}{t^n}\phi\left(\frac{y-z}{t}\right)dz\right|\\
&=\left|\int_{\mathbf{R}^n}Ta_l(z)\frac{1}{t^n}\left(\phi\left(\frac{y-z}{t}\right)-\phi\left(\frac{y}{t}\right)\right)dz\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{t^n}\int_{|z|<2^{l+1}} |Ta_l(z)| \left|\phi\left(\frac{y-z}{t}\right) - \phi\left(\frac{y}{t}\right)\right| dz \\
&\quad + \frac{1}{t^n}\int_{2^{l+1}\leq|z|<|x|/2} \cdots + \frac{1}{t^n}\int_{|z|\geq|x|/2} \cdots \\
&\equiv L_1 + L_2 + L_3.
\end{aligned}
$$

By the mean value theorem and Hölder's inequality, we have

$$
\begin{aligned}
L_1 &\leq \frac{1}{t^{n+1}} \|Ta_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \left(\int_{|z|<2^{l+1}} [\omega_2(z)]^{-q'/q} \left|\nabla\phi\left(\frac{y-\theta z}{t}\right)\right|^{q'} |z|^{q'}\,dz\right)^{1/q'} \\
&\leq c2^l \|a_l\|_{L^q_{\omega_2}(\boldsymbol{R}^n)} \left(\int_{|z|<2^{l+1}} [\omega_2(z)]^{-q'/q} (t+|y-\theta z|)^{-(n+1)q'}\,dz\right)^{1/q'} \\
&\leq \frac{c2^l}{|x|^{n+1}[\omega_1(B_l)]^{\alpha/n}} \left(\int_{|z|<2^{l+1}} [\omega_2(z)]^{-q'/q}\,dz\right)^{1/q'} \\
&\leq \frac{c2^{l(n+1)}}{|x|^{n+1}[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}},
\end{aligned}
$$

where $\nabla = (\partial/\partial x_1, \dots, \partial/\partial x_n)$, c is independent of ϕ, x, t and y, and $\theta \in (0,1)$. Here we used the inequalities

$$
\left(1 + \frac{|y-\theta z|}{t}\right)^{n+1} \left|\nabla\phi\left(\frac{y-\theta z}{t}\right)\right| \leq c\|\phi\|_{\mathscr{A}^{q\omega_1, q\omega_2}_{\alpha,q}(\boldsymbol{R}^n)} \leq c
$$

and $t + |y-\theta z| \geq |x-y| + |y-\theta z| \geq |x-\theta z| \geq |x|/2$. Using the same estimates, we have

$$
\begin{aligned}
L_2 &= \frac{1}{t^{n+1}} \int_{2^{l+1}\leq|z|<|x|/2} \left|\int_{|u|\leq 2^l} a(u)(k(z,u)-k(z,0))\,du\right| \\
&\quad \times \left|\nabla\phi\left(\frac{y-\theta z}{t}\right)\right| |z|\,dz \\
&\leq \frac{c}{t^{n+1}} \int_{2^{l+1}\leq|z|<|x|/2} \left(\int_{|u|<2^l} |a(u)| \frac{|u|^\delta}{|z|^{n+\delta}}\,du\right) \\
&\quad \times \left|\nabla\phi\left(\frac{y-\theta z}{t}\right)\right| |z|\,dz \\
&\leq c2^{l\delta} \|a\|_{L^1(\boldsymbol{R}^n)} \frac{1}{t^{n+1}} \int_{2^{l+1}\leq|z|<|x|/2} \left|\nabla\phi\left(\frac{y-\theta z}{t}\right)\right| \frac{1}{|z|^{n+\delta-1}}\,dz
\end{aligned}
$$

$$\leq \frac{c2^{l\delta}}{|x|^{n+1}}\|a\|_{L^q_{\omega_2}(\boldsymbol{R}^n)}\left(\int_{B_l}[\omega_2(x)]^{-1/(q-1)}dx\right)^{1/q'}$$

$$\times \int_{2^{l+1}\leq|z|<|x|/2}\frac{1}{|z|^{n+\delta-1}}dz$$

$$\leq \begin{cases} \dfrac{c2^{l(n+1)}}{|x|^{n+1}[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}\ln\left(\dfrac{|x|}{2^{l+2}}\right), & \textit{if } \delta=1 \\ \dfrac{c2^{l(n+\delta)}}{|x|^{n+\delta}[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}, & \textit{if } \delta\in(0,1), \end{cases}$$

by $\omega_2 \in A_q(\boldsymbol{R}^n)$. For L_3, we have

$$L_3 \leq \frac{1}{t^n}\int_{|z|\geq|x|/2}\left|\int_{|u|\leq 2^l}a(u)(k(z,u)-k(z,0))\,du\right|$$

$$\times\left(\left|\phi\left(\frac{y-z}{t}\right)\right|+\left|\phi\left(\frac{y}{t}\right)\right|\right)dz$$

$$\leq \frac{c2^{l(n+\delta)}}{t^n[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}\int_{|z|\geq|x|/2}\frac{1}{|z|^{n+\delta}}\left(\left|\phi\left(\frac{y-z}{t}\right)\right|+\left|\phi\left(\frac{y}{t}\right)\right|\right)dz$$

$$\leq \frac{c2^{l(n+\delta)}}{|x|^{n+\delta}[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}.$$

Thus, for $x \in R_k$ and $l \leq k-4$, we have

$$G^{q_{\omega_1},q_{\omega_2}}_{\alpha,q}(Ta_l)(x) \leq c_\varepsilon\frac{2^{l(n+\varepsilon)}}{|x|^{n+\varepsilon}[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}} \tag{4}$$

for any $\varepsilon \in (0,1)$ where $\delta = 1$ and $\varepsilon = \delta$ when $\delta \in (0,1)$, where c_ε is independent of x and l. From this, it follows that

$$J_1 \leq c_\varepsilon\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-4}|\lambda_l|2^{(l-k)(n+\varepsilon)}\left[\frac{\omega_1(B_k)}{\omega_1(B_l)}\right]^{\alpha/n}\times\left[\frac{\omega_2(B_k)}{\omega_2(B_l)}\right]^{1/q}\right)^p\right\}^{1/p}$$

$$\leq c_\varepsilon\left\{\sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{k-4}|\lambda_l|2^{(l-k)(n+\varepsilon-\alpha q_{\omega_1}-q_{\omega_2}n/q)}\right)^p\right\}^{1/p}$$

$$\leq c_\varepsilon\left\{\sum_{k=-\infty}^{\infty}|\lambda_l|^p\right\}^{1/p}$$

$$\leq c_\varepsilon\|f\|_{HK^{\alpha,p}_q(\omega_1,\omega_2)},$$

where we choose $\varepsilon \in (0,1)$ such that $n+\varepsilon > \alpha q_{\omega_1} + q_{\omega_2} n/q$ when $\delta = 1$ and $\varepsilon = \delta$ when $\delta \in (0,1)$.

This finishes the proof of Theorem 7. □

To discuss the extreme case, $\alpha q_{\omega_1} = n(1 - q_{\omega_2}/q) + \delta$, of Theorem 7, we introduce the weighted weak Herz-type Hardy space $WH\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$.

DEFINITION 6. Let $\alpha \in \boldsymbol{R}$, $0 < p$, $q \le \infty$, $\omega_1 \in A_{q_{\omega_1}}(\boldsymbol{R}^n)$, $\omega_2 \in A_{q_{\omega_2}}(\boldsymbol{R}^n)$ and $1 \le q_{\omega_1}$, $q_{\omega_2} < \infty$. The homogeneous weighted weak Herz-type Hardy space $WH\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ on $\boldsymbol{R}^n$ is defined by

$$WH\dot{K}_q^{\alpha,p}(\omega_1,\omega_2) = \{f \in \mathscr{S}'(\boldsymbol{R}^n) : G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f \in W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)\}$$

and

$$\|f\|_{WH\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)} = \|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f\|_{W\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)}.$$

The non-homogeneous weighted weak Herz-type Hardy space $WHK_q^{\alpha,p}(\omega_1,\omega_2)$ on $\boldsymbol{R}^n$ is defined by

$$WHK_q^{\alpha,p}(\omega_1,\omega_2) = \{f \in \mathscr{S}'(\boldsymbol{R}^n) : G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f \in WK_q^{\alpha,p}(\omega_1,\omega_2)\}$$

and

$$\|f\|_{WHK_q^{\alpha,p}(\omega_1,\omega_2)} = \|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}} f\|_{WK_q^{\alpha,p}(\omega_1,\omega_2)}.$$

If $\omega_1(x) \equiv \omega_2(x) \equiv 1$, we will denote $WH\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and $WHK_q^{\alpha,p}(\omega_1,\omega_2)$ simply by $WH\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n)$ and $WHK_q^{\alpha,p}(\boldsymbol{R}^n)$ which are introduced by Hu, Lu and Yang in [7]. Obviously, $WH\dot{K}_p^{0,p} = WHK_p^{0,p} = WH_{\omega_2}^p(\boldsymbol{R}^n)$ for any $p \in (0,\infty)$ which are studied by Quek and Yang in [14]; see also [3], [9], [1] and [16].

The following theorem is the end case of Theorem 7 and generalize Theorem 5 in [7] to both the non-convolutional case and the weighted case.

THEOREM 8. *Let T, k, ω_1, ω_2 and q be the same as in Theorem 7 with $\delta \in (0,1)$ and $q_{\omega_1} = q_{\omega_2} = 1$. If $\alpha = n(1 - 1/q) + \delta$ and $p \in (0,1]$, then T is bounded from $H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ into $WH\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$.*

Proof. Let $f \in H\dot{K}_q^{\alpha,p}(\omega_1,\omega_2)$ and write f as in the proof of Theorem 5. We then have

$$\begin{aligned}
&\|Tf\|_{WHK_q^{\alpha,p}(\omega_1,\omega_2)} \\
&\quad = \|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Tf)\|_{WK_q^{\alpha,p}(\omega_1,\omega_2)} \\
&\quad = \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} [\omega_1(B_k)]^{\alpha p/n} [D_{k,\omega_2}(\lambda, G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Tf))]^{p/q} \right\}^{1/p}
\end{aligned}$$

$$\leq c\sup_{\lambda>0}\lambda\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left[D_{k,\omega_2}\left(\lambda/2,\sum_{l=-\infty}^{k-4}|\lambda_l|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Ta_l)\right)\right]^{p/q}\right\}^{1/p}$$

$$+c\sup_{\lambda>0}\lambda\left\{\sum_{k=-\infty}^{\infty}[\omega_1(B_k)]^{\alpha p/n}\left[D_{k,\omega_2}\left(\lambda/2,\sum_{l=k-3}^{\infty}|\lambda_l|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Ta_l)\right)\right]^{p/q}\right\}^{1/p}$$

$$\equiv M_1+M_2.$$

By the $L^q_{\omega_2}(\boldsymbol{R}^n)$-boundedness of $G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}$ and T, we easily obtain a desirable estimate for M_2.

For M_1, when $x\in R_k$, by (4) , we have

$$\sum_{l=-\infty}^{k-4}|\lambda_l|G_{\alpha,q}^{q_{\omega_1},q_{\omega_2}}(Ta_l)(x)$$

$$\leq c_\delta\sum_{l=-\infty}^{k-4}|\lambda_l|2^{(l-k)(n+\delta)}\frac{1}{[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}$$

$$\leq\frac{c_\delta}{[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}\sum_{l=-\infty}^{k-4}|\lambda_l|2^{(l-k)(n+\delta-\alpha q_{\omega_1}-nq_{\omega_2}/q)}$$

$$\leq\frac{c_\delta}{[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}\left(\sum_{l=-\infty}^{k-4}|\lambda_l|^p\right)^{1/p}$$

$$\leq\frac{c_\delta}{[\omega_1(B_l)]^{\alpha/n}[\omega_2(B_l)]^{1/q}}\|f\|_{HK_q^{\alpha,p}(\omega_1,\omega_2)},$$

since $p\leq 1$ and $\alpha=n(1-1/q)+\delta$.

Now, similar to the above, we can obtain a desirable estimate for M_1. This finishes the proof of Theorem 8. □

Finally, we remark that if we assume more regularity on the kernel, we could extend the range of α in Theorems 5 and 6; and moreover, if T satisfies higher vanishing moment conditions, the range of α in Theorems 7 and 8 can also be extended. We omit the details.

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