M. SHIROSAKI
KODAI MATH. J.
23 (2000), 376–379

A HYPERBOLIC HYPERSURFACE OF DEGREE 10

MANABU SHIROSAKI

§1. Introduction

In [K], Kobayashi posed a problem whether all 'generic' hypersurfaces in $\boldsymbol{P}^n(\boldsymbol{C})$ of degree enough large with respect to n are hyperbolic. For $n=2$ this conjecture is true. In fact, a non-singular curve of degree not less than 4 is hyperbolic. However, for $n \geq 3$ it is open. On the other hand, Masuda and Noguchi [MN] defined the number $d(n)$ by the minimum number such that there exists a hyperbolic hypersurface of $\boldsymbol{P}^n(\boldsymbol{C})$ of each integer not less than it. By Demailly [D], $d(3) \leq 11$.

In this paper, we give a hyperbolic hypersurface of degree 10 in $\boldsymbol{P}^3(\boldsymbol{C})$, and hence, $d(3) \leq 10$.

§2. Lemmas

We use the terminology in [S]. Let $f_0, \dots, f_n$ be entire functions on $\boldsymbol{C}$ such that $f_j \not\equiv 0$ for at least one j $(0 \leq j \leq n)$. Then $\tilde{f} := (f_0, \dots, f_n)$ becomes a representation of a holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^n(\boldsymbol{C})$. If $f(z) = (c_0 : \cdots : c_n)$ for all $z \in \boldsymbol{C} - \tilde{f}^{-1}(\mathbf{o})$, where $c_0, \dots, c_n$ are constants at least one of which are not 0, then we say that f or $(f_0 : \cdots : f_n)$ is constant.

We will need the following:

LEMMA 1 ([S, p. 291]). *Let f be a nonconstant meromorphic function on $\boldsymbol{C}$ and a_j $(1 \leq j \leq q)$ distinct points in $\bar{\boldsymbol{C}} := \boldsymbol{C} \cup \{\infty\}$. If all the zeros of $f - a_j$ have the multiplicities at least m_j for each j, where m_j are arbitrarily fixed positive integers $(1 \leq j \leq q)$ and $f - \infty$ means $1/f$, then*

$$\sum_{j=1}^{q}\left(1 - \frac{1}{m_j}\right) \leq 2.$$

Remark. If $f - a_j$ has no zero, then we may consider $1 - 1/m_j$ as 1.

LEMMA 2. *Let a, b, c be nonzero constants and $d \geq 3$ an integer. Then, $P(z) = az^d + bz^{d-1} + c$ has at least $d-2$ simple zeros.*

Received August 23, 1999.

Proof. Let z_0 be a multiple zero of $P(z)$. Then, $P'(z_0) = daz_0^{d-1} + (d-1)bz_0^{d-2} = 0$. Trivially $z_0 \neq 0$ because of $P(0) = c \neq 0$. Hence we have $z_0 = -(d-1)b/da$ and $P''(z_0) = d(d-1)az_0^{d-2} + (d-1)(d-2)bz_0^{d-3} = -(d-1)bz_0^{d-3} \neq 0$. Therefore, $P(z)$ has at most one multiple zero, and its multiplicity is 2. This implies Lemma. Q.E.D.

§3. A hyperbolic hypersurface of degree 10

Now, we prove the following theorem:

THEOREM 3. *Let a_1, a_2, a_3 be nonzero constants and $d \geq 5$ an integer. Define the hypersurface X in $\boldsymbol{P}^3(\boldsymbol{C})$ by*

$$w_0^{2d} + w_1^{2d} - (a_1 w_1^{d-1} w_2 + a_2 w_2^d + a_3 w_3^d)^2 = 0.$$

Then there exists no nonconstant holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^3(\boldsymbol{C})$ such that $f(\boldsymbol{C}) \subset X$, i.e., X is hyperbolic.

Proof. Assume that a holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^3(\boldsymbol{C})$ with redeuced representation (f_0, f_1, f_2, f_3) satisfies $f(\boldsymbol{C}) \subset X$, *i.e.*,

$$f_0^{2d} + f_1^{2d} - (a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d)^2 = 0. \tag{1}$$

(I) The case of $f_0 = 0$. From (1), we have $\varepsilon f_1^d + a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d = 0$, where $\varepsilon = \pm 1$. (i) If $f_1 = 0$, then $a_2 f_2^d + a_3 f_3^d = 0$. Trivially $(f_2 : f_3)$ is constant, and hence, f is constant. (ii) If $f_2 = 0$, then $\varepsilon f_1^d + a_3 f_3^d = 0$. Trivially $(f_1 : f_3)$ is constant, and hence, f is constant. (iii) If $f_1 \neq 0$, $f_2 \neq 0$, then

$$\varepsilon g^d + a_1 g^{d-1} + a_2 = -a_3 (f_3/f_2)^d, \tag{2}$$

where $g = f_1/f_2$. By Lemma 2, $\varepsilon z^d + a_1 z^{d-1} + a_2 = 0$ has at least $d-2$ simple roots $\omega_j (j = 1, \ldots, d-2)$. For each $j = 1, \ldots, d-2$, multiplicities of zeros of $g - \omega_j$ are multiples of d by (2). The inequality $(1 - 1/d)(d-2) = d - 3 + 2/d > 2$ and Lemma 1 imply that g is constant. Hence f is constant.

(II) The case of $f_0 \neq 0$, $f_1 = 0$. From (1), we have $\varepsilon f_0^d + a_2 f_2^d + a_3 f_3^d = 0$, where $\varepsilon = \pm 1$. In this case, it is obvious that f is constant.

(III) The case of $f_0 \neq 0$, $f_1 \neq 0$. From (1), we have $f_0^{2d} + f_1^{2d} = (a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3)^2$. As in (I)(iii), we can conclude by $(1 - 1/2) \cdot (2d) = d > 2$ that $(f_0 : f_1)$ is constant. Hence it is possible to write $f_0 = c f_1$ by a nonzero constant c. By substituting this to (1), we get

$$b f_1^d + a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d = 0,$$

where b is a constant such that $b^2 = c^{2d} + 1$. (i) If $f_2 = 0$, then $b f_1^d + a_3 f_3^d = 0$. In this case, if $b = 0$, then $f_3 = 0$ and f is constant. If $b \neq 0$, then $(f_1 : f_3)$ is constant and so is f. (ii) Assume that $f_2 \neq 0$. If $b \neq 0$, then we can conclude as in (I)(iii) that f is constant. If $b = 0$, then $f_2(a_1 f_1^{d-1} + a_2 f_2^{d-1}) = -a_3 f_3^d$.

From the inequality $(1-1/d)d = d-1 > 2$ and Lemma 1 it follows that $(f_1 : f_2)$ is constant, and so is f. Q.E.D.

For each $d \geq 11$, Demailly [D] gave a hyperbolic hypersurface of $\boldsymbol{P}^3(\boldsymbol{C})$ of degree d. Therefore, $d(3) \leq 10$ is obtained.

§4. Complements in $\boldsymbol{P}^2(\boldsymbol{C})$

In this section we give (reducible) hypersurfaces with hyperbolic complements.

THEOREM 4. *Let a_0, a_1, a_2 be nonzero constants and $d \geq 4$ an integer. Define a hypersurface X in $\boldsymbol{P}^2(\boldsymbol{C})$ by*

$$w_0^{2d} - (a_0 w_0^{d-1} w_1 + a_1 w_1^d + a_2 w_2^d)^2 = 0.$$

Then there exists no nonconstant holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^2(\boldsymbol{C})$ such that $f(\boldsymbol{C}) \subset \boldsymbol{P}^2(\boldsymbol{C}) \backslash X$.

Proof. Assume that a holomorphic mapping f of $\boldsymbol{C}$ into $\boldsymbol{P}^2(\boldsymbol{C})$ with redeuced representation (f_0, f_1, f_2) satisfies $f(\boldsymbol{C}) \subset \boldsymbol{P}^2(\boldsymbol{C}) \backslash X$, *i.e.*,

$$\alpha^{2d} + f_0^{2d} - (a_0 f_0^{d-1} f_1 + a_1 f_1^d + a_2 f_2^d)^2 = 0, \tag{3}$$

where α is an entire function without zeros.

In the case of $d \geq 5$, it follows from Theorem 3 that f is constant. Hence, it is enough to consider the case of $d = 4$, and from now on take $d = 4$.

(I) The case of $f_0 = 0$. From (3), we have $\varepsilon\alpha^4 = a_1 f_1^4 + a_2 f_2^4$, where $\varepsilon = \pm 1$. By the Little Picard Theorem, $(f_1 : f_2)$ is constant, and hence, so is f.

(II) The case of $f_0 \neq 0$. From (3), we have $\alpha^8 + f_0^8 = (a_1 f_0^3 f_1 + a_1 f_1^4 + a_2 f_2^4)^2$. By the inequality $(1 - 1/2) \cdot 8 = 4 > 2$ and Lemma 1, we have $(f_0 : \alpha)$ is constant, and we can write $\alpha = c f_0$ by a nonzero constant c. By substituting this into (3),

$$b f_0^4 + a_0 f_0^3 f_1 + a_1 f_1^4 + a_2 f_2^4 = 0 \tag{4}$$

is obtained, where b is a constant such that $b^2 = c^8 + 1$. (i) If $f_1 = 0$, then $b f_0^4 + a_2 f_2^4 = 0$. In this case, if $b = 0$, then $f_2 = 0$; otherwise, $(f_0 : f_2)$ is constant. In any case, f is constant. (ii) The case of $f_1 \neq 0$. If $b = 0$, then from (4) $f_1(a_0 f_0^3 + a_1 f_1^3) = -a_2 f_2^4$. By the inequality $(1 - 1/4) \cdot 4 = 3 > 2$ and Lemma 1, it is obtained that $(f_0 : f_1)$ is constant. Hence f is constant. Consider the case of $b \neq 0$. We rewrite (4) as

$$b f_0^4 + a_0 f_0^3 f_1 + a_1 f_1^4 = -a_2 f_2^4.$$

If $b z^4 + a_0 z^3 + a_1 = 0$ has no multiple roots, then we conclude that $(f_0 : f_1)$ is constant by Lemma 1 and the inequality $(1 - 1/4) \cdot 4 = 3 > 2$. Hence f is constant. Otherwise, we can factorize

$$bz^4 + a_0 z^3 + a_1 = b(z - \omega_1)(z - \omega_2)(z - \omega_3)^2$$

by Lemma 2, where $\omega_1, \omega_2, \omega_3$ are distinct nonzero constants. Put $g = f_0/f_1$. Then

$$b(g - \omega_1)(g - \omega_2)(g - \omega_3)^2 = -a_2(f_2/f_1)^4.$$

Therefore, multiplicities of zeros of $g - \omega_1$, $g - \omega_2$ and $g - \omega_3$ are multiples of 4, 4 and 2, respectively. Moreover, g has no zeros. The inequality $1 + (1 - 1/4) + (1 - 1/4) + (1 - 1/2) = 3 > 2$ and Lemma 1 imply that g is constant, and hence f is constant. Q.E.D.

COROLLARY 5. *Let X be as in Theorem 4. (i) If $d \geq 5$, then $\boldsymbol{P}^2(\boldsymbol{C})\backslash X$ is completely hyperbolic and hyperbolically imbeded in $\boldsymbol{P}^2(\boldsymbol{C})$. (ii) If $d = 4$ and $\pm 3^3 a_0^4 + 4^4 a_1 \neq 0$, then $\boldsymbol{P}^2(\boldsymbol{C})\backslash X$ is completely hyperbolic and hyperbolically imbeded in $\boldsymbol{P}^2(\boldsymbol{C})$.*

Proof. (i) In this case the result is obvious by the above two theorems and Brody-Green's theorem.

(ii) The hypersurface X has two irreducible components of degree 4. By the condition $\pm 3^3 a_0^4 + 4^4 a_1 \neq 0$, they are non-singular, and hence, Riemann surfaces of genus 3. Therefore, all holomorphic mappings of $\boldsymbol{C}$ into X are constant. From Theorem 4 and Brody-Green's theorem, the conclusion follows. Q.E.D.

REFERENCES

[D] J.-P DEMAILLY, Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, Algebraic Geometry held at Santa Cruz, July 1995 (J. Kollár et al eds.), Proc. Sympos. Pure Math., **62-2**, Amer. Math. Soc., Providence, 1997, 285–360.

[K] S. KOBAYASHI, Hyperbolic Manifolds and Holomorphic Mappings, Dekker, 1987

[MN] K. MASUDA AND J. NOGUCHI, A construction of hyperbolic hypersurface of $\boldsymbol{P}^n(\boldsymbol{C})$, Math. Ann., **304** (1996), 339–362.

[S] M. SHIROSAKI, On polynomials which determine holomorphic mappings, J. Math. Soc. Japan, **49** (1997), 289–298.

DEPARTMENT OF MATHMATICAL SCIENCES
COLLEGE OF ENGINEERING
OSAKA PREFECTURE UNIVERSITY
SAKAI 599-8531, JAPAN