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A NOTE ON NECESSARY CONDITIONS OF HYPOELLIPTICITY FOR SOME CLASSES OF DIFFERENTIAL OPERATORS WITH DOUBLE CHARACTERISTICS

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Abstract

We construct explicit formulas for fundamental solutions and global non-smooth solutions at degenerate points of some classes of differential operators with double characteristics. A new elementary proof for non-hypoellipticity is given.

§1. Introduction

In this paper we will construct explicit formulas for fundamental solutions and global non-smooth solutions at degenerate points of the following operator

$$G_{k,c}^{a,b} = X_2 X_1 + icx^{k-1}\frac{\partial}{\partial y},$$

where $(x, y) \in \boldsymbol{R}^2$; $a, b, c \in \boldsymbol{C}$, $\mathrm{Re}\, a \cdot \mathrm{Re}\, b \neq 0$; $i = \sqrt{-1}$; k is a positive integer, and $X_1 = (\partial/\partial x) - ibx^k(\partial/\partial y)$, $X_2 = (\partial/\partial x) - iax^k(\partial/\partial y)$. The operator $G_{k,c}^{a,b}$ was studied in [1] when k is odd and in [2] when k is even. For more complete references and generalization we refer to [3], [4], [5], [6], [7] and therein references. We will treat only the case $\mathrm{Re}\, a < 0$. The case $\mathrm{Re}\, a > 0$ can be considered analogously. Recently in [8], [9] we considered a model of the Grushin operator, that is the case when $a = -1$, $b = 1$, and the Kohn-Laplacian on the Heisenberg group. The paper is organized as follows. In §2 we give some definitions of notations used in the paper, and establish some auxiliary lemmas. In §3 we state and prove the main results.

§2. Auxiliary lemmas

We will use the following notation

$$(z, m) = z(z+1)\cdots(z+m-1) = \frac{\Gamma(z+m)}{\Gamma(z)} \quad \text{for } z \in \boldsymbol{C}, m \in \boldsymbol{N}.$$

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We denote by C a general constant which may vary from place to place. For two complex numbers $z_1, z_2 \in \boldsymbol{C}$ we define $z_1^{z_2}$ as $e^{z_2 \ln z_1}$ and if $z_1 = re^{i\varphi}$, $-\pi < \varphi \leq \pi$ then $\ln z_1 = \ln r + i\varphi$. Now let us recall the following lemma from [8].

LEMMA 1. *Assume that* $\omega_1, \omega_3 \in \boldsymbol{C}$, $\operatorname{Re}\omega_3 > -1$. *Then we have*

$$\int_0^\pi (\sin\theta + i\cos\theta)^{\omega_1} \sin^{\omega_3}\theta\, d\theta = \frac{2^{-\omega_3}\pi\Gamma(\omega_3+1)}{\Gamma(1+((\omega_3-\omega_1)/2))\Gamma(1+((\omega_3+\omega_1)/2))}.$$

Proof. We refer to the proof in [8]. □

LEMMA 2. *Assume that* $\omega_1, \omega_2, \omega_3 \in \boldsymbol{C}$, $\operatorname{Re} w_1$, $\operatorname{Re} w_2 > 0$, $\operatorname{Re}\omega_3 > -1$. *Then*

$$(1)\quad \int_0^\pi (w_1\sin\theta + i\cos\theta)^{\omega_1}(w_2\sin\theta - i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\, d\theta$$

$$= \frac{\pi\Gamma(\omega_3+1)F_2(\omega_3+1, -\omega_1, -\omega_2, 1+((\omega_3-\omega_1+\omega_2)/2), 1+((\omega_3+\omega_1-\omega_2)/2), (1-w_1)/2, (1-w_2)/2)}{2^{\omega_3}\Gamma(1+((\omega_3+\omega_1-\omega_2)/2))\Gamma(1+((\omega_3-\omega_1+\omega_2)/2))},$$

$$(2)\quad \int_0^\pi (w_1\sin\theta + i\cos\theta)^{\omega_1}(w_2\sin\theta + i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\, d\theta$$

$$= \frac{\pi\Gamma(\omega_3+1)F_1(\omega_3+1, -\omega_1, -\omega_2, 1+((\omega_3-\omega_1-\omega_2)/2), (1-w_1)/2, (1-w_2)/2)}{2^{\omega_3}\Gamma(1+((\omega_3+\omega_1+\omega_2)/2))\Gamma(1+((\omega_3-\omega_1-\omega_2)/2))},$$

where $F_1(\alpha, \beta, \beta', \gamma, x, y)$, $F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y)$ *are the first and second two-variable hypergeometric functions of Appel* [10] *defined as*

$$F_1(\alpha, \beta, \beta', \gamma, x, y) = \sum \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m+n)(1, m)(1, n)} x^m y^n,$$

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum \frac{(\alpha, m+n)(\beta, m)(\beta', n)}{(\gamma, m)(\gamma', n)(1, m)(1, n)} x^m y^n.$$

Proof. Define the left side of (1), (2) by $F(\omega_1, \omega_2, \omega_3, w_1, w_2)$, $G(\omega_1, \omega_2, \omega_3, w_1, w_2)$. It is clear that F, G are analytic functions of (w_1, w_2) when $\operatorname{Re} w_1$, $\operatorname{Re} w_2 > 0$. First we prove the formula (1). We have

$$\begin{aligned}
&\frac{\partial^{m+n}F(\omega_1, \omega_2, \omega_3, w_1, w_2)}{\partial w_1^m \partial w_2^n} \\
&\quad = (-1)^{m+n} \times \int_0^\pi (-\omega_1, m)(-\omega_2, n)(w_1\sin\theta + i\cos\theta)^{\omega_1-m} \\
&\quad \times (w_2\sin\theta - i\cos\theta)^{\omega_2-n}\sin^{\omega_3+m+n}\theta\, d\theta \\
&\quad = (-1)^{m+n}(-\omega_1, m)(-\omega_2, n)F(\omega_1 - m, \omega_2 - n, \omega_3 + m + n, w_1, w_2).
\end{aligned}$$

By using Lemma 1 we deduce that

$$\begin{aligned}
&(-1)^{m+n}\frac{\partial^{m+n}F(\omega_1,\omega_2,\omega_3,w_1,w_2)}{\partial w_1^m \partial w_2^n}\bigg|_{(w_1,w_2)=(1,1)}\\
&\quad=(-\omega_1,m)(-\omega_2,n)F(\omega_1-m,\omega_2-n,\omega_3+m+n,1,1)\\
&\quad=(-\omega_1,m)(-\omega_2,n)\int_0^\pi(\sin\theta+i\cos\theta)^{\omega_1-m}(\sin\theta-i\cos\theta)^{\omega_2-n}\sin^{\omega_3+m+n}\theta\,d\theta\\
&\quad=\int_0^\pi(-\omega_1,m)(-\omega_2,n)(\sin\theta+i\cos\theta)^{\omega_1-m-\omega_2+n}\sin^{\omega_3+m+n}\theta\,d\theta\\
&\quad=\frac{\pi 2^{-(\omega_3+m+n)}(-\omega_1,m)(-\omega_2,n)\Gamma(\omega_3+1+m+n)}{\Gamma(1+n+((\omega_3+\omega_1-\omega_2)/2))\Gamma(1+m+((\omega_3-\omega_1+\omega_2)/2))}.
\end{aligned}$$

Hence the desired formula follows.

Now we proceed to prove the formula (2). We can repeat the above proof with F replaced by G. The only difference is

$$\begin{aligned}
&(-\omega_1,m)(-\omega_2,n)G(\omega_1-m,\omega_2-n,\omega_3+m+n,1,1)\\
&\quad=(-\omega_1,m)(-\omega_2,n)\int_0^\pi(\sin\theta+i\cos\theta)^{\omega_1-m}(\sin\theta+i\cos\theta)^{\omega_2-n}\sin^{\omega_3+m+n}\theta\,d\theta\\
&\quad=\int_0^\pi(-\omega_1,m)(-\omega_2,n)(\sin\theta+i\cos\theta)^{\omega_1-m+\omega_2-n}\sin^{\omega_3+m+n}\theta\,d\theta\\
&\quad=\frac{\pi 2^{-(\omega_3+m+n)}(-\omega_1,m)(-\omega_2,n)\Gamma(\omega_3+1+m+n)}{\Gamma(1+((\omega_3+\omega_1+\omega_2)/2))\Gamma(1+m+n+((\omega_3-\omega_1-\omega_2)/2))}.
\end{aligned}$$

Hence the desired formula follows. $\square$

COROLLARY 1. *Under the assumptions of Lemma 2, if moreover $\omega_1+\omega_2+\omega_3=-2$, then*

$$\begin{aligned}
&\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\,d\theta\\
&\quad=\frac{2\pi\Gamma(\omega_3+1)}{(w_1+w_2)^{\omega_3+1}\Gamma(1+((\omega_3+\omega_1-\omega_2)/2))\Gamma(1+((\omega_3-\omega_1+\omega_2)/2))}.
\end{aligned}$$

Proof. By Lemma 2 we have

$$\begin{aligned}
&\int_0^\pi(w_1\sin\theta+i\cos\theta)^{\omega_1}(w_2\sin\theta-i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\,d\theta\\
&\quad=\frac{\pi\Gamma(\omega_3+1)F_2(1+\omega_3,-\omega_1,-\omega_2,-\omega_1,-\omega_2,(1-w_1)/2,(1-w_2)/2)}{2^{\omega_3}\Gamma(1+((\omega_3+\omega_1-\omega_2)/2))\Gamma(1+((\omega_3-\omega_1+\omega_2)/2))}.
\end{aligned}$$

Now using the following relation (see [10, p. 15])

$$F_2(\alpha,\beta,\beta',\beta,\beta',x,y) = (1-x-y)^{-\alpha}$$

gives the desired result. □

COROLLARY 2. *Under the assumptions of Lemma* 2, *if moreover* $\omega_1+\omega_2+\omega_3$ *or* $-\omega_1-\omega_2+\omega_3$ *is some even negative integer then*

$$\int_0^\pi (w_1\sin\theta + i\cos\theta)^{\omega_1}(w_2\sin\theta+i\cos\theta)^{\omega_2}\sin^{\omega_3}\theta\, d\theta = 0.$$

Proof. Indeed, in that case $1+((\omega_3+\omega_1+\omega_2)/2)$ or $1+((\omega_3-\omega_1-\omega_2)/2)$ will be a pole of the function $\Gamma(.)$. □

§3. Main results

In the case $\mathrm{Re}\,a<0$, and $\mathrm{Re}\,b>0$ we will consider a function of the following form

$$F^{\alpha,\beta,\gamma}_{k,a,b}(x,y) = (bx^{k+1}-i(k+1)y)^\alpha(-ax^{k+1}+i(k+1)y)^\beta x^\gamma.$$

If $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b<0$ and $a\neq b$ then we will consider the following function

$$E^{\alpha,\beta,\gamma}_{k,a,b}(x,y) = (-bx^{k+1}+i(k+1)y)^\alpha(-ax^{k+1}+i(k+1)y)^\beta x^\gamma.$$

In the resonance case $a=b$, $\mathrm{Re}\,a<0$ we consider the following function

$$R^{\kappa,\eta,\gamma}_{k,a}(x,y) = x^\gamma(-ax^{k+1}+i(k+1)y)^\eta e^{\kappa x^{k+1}/(-ax^{k+1}+i(k+1)y)}.$$

It is clear that the differentiating of $F^{\alpha,\beta,\gamma}_{k,a,b}(x,y)$ and $E^{\alpha,\beta,\gamma}_{k,a,b}(x,y)$ formally differ from each other by a factor -1. Note that

$$X_1(-ax^{k+1}+i(k+1)y) = (k+1)(b-a)x^k, X_1(bx^{k+1}-i(k+1)y) = 0,$$
$$X_2(-ax^{k+1}+i(k+1)y) = 0, X_2(bx^{k+1}-i(k+1)y) = (k+1)(b-a)x^k.$$

Therefore we have

$$\begin{aligned} G^{a,b}_{k,c}F^{\alpha,\beta,\gamma}_{k,a,b}(x,y) &= (bx^{k+1}-i(k+1)y)^{\alpha-1}(-ax^{k+1}+i(k+1)y)^{\beta-1}x^{\gamma-2} \\ &\times\{[(k+1)^2(b-a)^2\alpha\beta + (k+1)(k+\gamma)(b-a)b\beta \\ &-(k+1)a(b-a)\alpha\gamma - \gamma(\gamma-1)ab \\ &-(k+1)c(a\alpha+b\beta)]x^{2k+2} + (k+1)^2\gamma(\gamma-1)y^2 \\ &+ i[-(k+1)^2(k+\gamma)(b-a)\beta \\ &+(k+1)^2c(\alpha+\beta)+(k+1)^2(b-a)\alpha\gamma \\ &+(k+1)\gamma(\gamma-1)(a+b)]x^{k+1}y\}. \end{aligned}$$

Hence in the non-resonance case $a \neq b$ we formally have $G_{k,c}^{a,b} F_{k,a,b}^{\alpha,\beta,\gamma}(x,y) = -G_{k,c}^{a,b} E_{k,a,b}^{\alpha,\beta,\gamma}(x,y) = 0$ if

$\gamma = 0$, $\alpha = 0$, $\beta = 0$, the solution is a constant.
$\gamma = 0$, $c = k(b-a)$, $\alpha = 0$, β arbitrary $\neq 0$.
$\gamma = 0$, $c = 0$, α arbitrary $\neq 0$, $\beta = 0$.
$\gamma = 0$, $\alpha = (c - k(b-a))/((k+1)(b-a)) =: \alpha_1$, $\beta = -c/((k+1)(b-a)) =: \beta_1$.
$\gamma = 1$, $\alpha = 0$, $\beta = 0$, the solution is the linear function x.
$\gamma = 1$, $c = (b-a)(k+1)$, $\alpha = 0$, β arbitrary $\neq 0$.
$\gamma = 1$, $c = -(b-a)$, α arbitrary $\neq 0$, $\beta = 0$.
$\gamma = 1$, $\alpha = (c - (k+1)(b-a))/((k+1)(b-a)) =: \alpha_2$, $\beta = -(c + (b-a))/((k+1)(b-a)) =: \beta_2$.

In the resonance case $a = b$ we can consider $R_{k,a}^{\kappa,\eta,\gamma}(x,y)$ as the limit of $E_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)$, $E_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)$ when $b \to a$. Thus we will have

$$\gamma = 0, \eta = -\frac{k}{k+1} =: \eta_1, \quad \kappa = -\frac{c}{k+1} =: \kappa_1$$

$$\text{or} \quad \gamma = 1, \eta = -\frac{k+2}{k+1} =: \eta_2, \quad \kappa = -\frac{c}{k+1} =: \kappa_2,$$

and the formal equations $G_{k,c}^{a,a} R_{k,a}^{\kappa_1,\eta_1,0}(x,y) = 0$, $G_{k,c}^{a,a} R_{k,a}^{\kappa_2,\eta_2,1}(x,y) = 0$.

THEOREM 1. *Assume that k is odd. If* $\operatorname{Re} a < 0$, *and* $\operatorname{Re} b > 0$ *then*

I) $G_{k,c}^{a,b} F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)$
$$= -\frac{4(b-a)^{1/(k+1)} \pi \Gamma(k/(k+1))}{\Gamma((k(b-a)-c)/((k+1)(b-a))) \Gamma(c/((k+1)(b-a)))} \delta(x,y) =: A_{k,c}^{a,b} \delta(x,y).$$

II) $G_{k,k(b-a)}^{a,b} F_{k,a,b}^{0,\beta,0}(x,y) = 0$ *if* $\operatorname{Re} \beta > -k/(k+1)$.

III) $G_{k,0}^{a,b} F_{k,a,b}^{\alpha,0,0}(x,y) = 0$ *if* $\operatorname{Re} \alpha > -k/(k+1)$.

If $\operatorname{Re} a < 0$, $\operatorname{Re} b < 0$ *and* $a \neq b$ *then*

IV) $G_{k,c}^{a,b} E_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = 0$.

If $\operatorname{Re} a < 0$ *then*

V) $G_{k,c}^{a,a} R_{k,a}^{\kappa_1,\eta_1,0}(x,y) = 0$.

Proof. I) We begin by noting that if k is odd and $\operatorname{Re} a < 0$, and $\operatorname{Re} b > 0$ then $(bx^{k+1} - i(k+1)y)^{\alpha}$ and $(-ax^{k+1} + i(k+1)y)^{\beta} \in C^{\infty}(\mathbf{R}^2 \backslash (0,0))$ for every α and β. Let us introduce the following "polar coordinate"

$$x = \rho (\sin\theta)_{\pm}^{1/(k+1)}, \quad y = \frac{\rho^{k+1}}{k+1}\cos\theta, \quad dxdy = \frac{\rho^{k+1}}{k+1} |\sin\theta|^{-k/(k+1)} \, d\rho d\theta.$$

Here we use the following notation $(\sin\theta)_{\pm}^{r} = \operatorname{sign}(\sin\theta)|\sin\theta|^{r}$ for every $r \in \mathbf{R}$. Note that the map $(x,y) \to (\rho,\theta)$ is not a differmorphism along the line $x = 0$.

But it is good enough for us because in the future we will use it only for integration, and if necessary we can take integrals as a limit. It is easy to verify that $\rho^{2k+2} = x^{2k+2} + (k+1)^2 y^2$. Let us write $F_k^1(x,y) = F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)$. First we prove that $F_k^1(x,y) \in L_{loc}^{((k+2)/k)-\tau}(\boldsymbol{R}^2)$ for any small positive τ. Indeed, since $F_k^1(x,y) \in C^\infty(\boldsymbol{R}^2 \backslash (0,0))$ it suffices to prove that $F_k^1(x,y) \in L^{((k+2)/k)-\tau}(B_\varepsilon)$, where $B_\varepsilon = \{(x,y) | \rho(x,y) < \varepsilon\}$. We have

$$\int_{B_\varepsilon} |F_k^1(x,y)|^{((k+2)/k)-\tau} \, dxdy \le C \int_{-\pi}^{\pi} |\sin\theta|^{-k/(k+1)} \, d\theta \int_0^\varepsilon \rho^{k+1} (\rho^{-k})^{((k+2)/k)-\tau} \, d\rho$$
$$\le C \int_0^\varepsilon \rho^{-1+\tau k} \, d\rho < \infty.$$

Note that $F_k^1(x,y) \notin L_{loc}^{(k+2)/k}(\boldsymbol{R}^2)$. Let $\boldsymbol{R}_\varepsilon^2 = \{(x,y) \in \boldsymbol{R}^2 | \rho(x,y) \ge \varepsilon\}$. By applying Green's formula we have

$$\begin{aligned}
&\int_{\mathrm{R}_\varepsilon^2} f(x,y) G_{k,-c}^{b,a} v(x,y) \, dxdy \\
&\quad = \int_{\mathrm{R}_\varepsilon^2} v(x,y) G_{k,c}^{a,b} f(x,y) \, dxdy \\
&\quad - \int_{\rho=\varepsilon} v(x,y) \{(\nu_1 - iax^k \nu_2) X_1 f(x,y) + icx^{k-1} \nu_2 .\ f(x,y)\} \, ds \\
&\quad + \int_{\rho=\varepsilon} f(x,y)(\nu_1 - ibx^k \nu_2) X_2 v \, ds =: \int_{\mathrm{R}_\varepsilon^2} V(f,v,k,a,b,c) \, dxdy \\
&\quad - \int_{\rho=\varepsilon} v(x,y) B_1(f,k,a,b,c) \, ds + \int_{\rho=\varepsilon} f(x,y) B_2(v,k,a,b) \, ds
\end{aligned} \tag{3}$$

for every $v(x,y) \in C_0^\infty(\boldsymbol{R}^2)$, $f(x,y) \in C^\infty(\boldsymbol{R}^2 \backslash (0,0))$, where $\nu = (\nu_1, \nu_2)$ is the unit outward normal to $\partial \boldsymbol{R}_\varepsilon^2$. Replace $f(x,y)$ in (3) by $F_k^1(x,y)$ we obtain that

$$\begin{aligned}
&\int_{\mathrm{R}_\varepsilon^2} F_k^1(x,y) G_{k,-c}^{b,a} v(x,y) \, dxdy \\
&\quad = \int_{\mathrm{R}_\varepsilon^2} V(F_k^1, v, k, a, b, c) \, dxdy \\
&\quad - \int_{\rho=\varepsilon} v(x,y) B_1(F_k^1, k, a, b, c) \, ds + \int_{\rho=\varepsilon} F_k^1(x,y) B_2(v,k,a,b) \, ds.
\end{aligned} \tag{4}$$

The first integral in the right side of (4) vanishes. We now compute the third integral in the right side of (4). It is easy to check that

$$ds|_{\partial B_\varepsilon} = \frac{1}{k+1} (\varepsilon^2 |\sin\theta|^{-2k/(k+1)} \cos^2\theta + \varepsilon^{2k+2} \sin^2\theta)^{1/2} \, d\theta \quad \text{and}$$

$$v|_{\partial B_\varepsilon} = (v_1, v_2)|_{\partial B_\varepsilon} = -\left(\frac{x^{2k+1}}{(x^{4k+2}+(k+1)^2y^2)^{1/2}}, \frac{(k+1)y}{(x^{4k+2}+(k+1)^2y^2)^{1/2}}\right)\Bigg|_{\partial B_\varepsilon}.$$

Hence

$$(x^{4k+2}+(k+1)^2y^2)^{-1/2}\,ds|_{\partial B_\varepsilon} = \frac{1}{k+1}\varepsilon^{-k}|\sin\theta|^{-k/(k+1)}\,d\theta.$$

It follows that

$$\begin{aligned}\left|\int_{\rho=\varepsilon} F_k^1(x,y)B_2(v,k,a,b)\,ds\right| &\leq C\int_{\rho=\varepsilon}|F_k^1(x,y)|.\ (|v_1|+|v_2.\ x^k|)\,ds \\ &\leq C\varepsilon\int_{-\pi}^{\pi}(|\sin\theta|+|\cos\theta|)\,d\theta \to 0 \quad \text{as } \varepsilon\to 0.\end{aligned} \tag{5}$$

Next we evaluate $B_1(F_k^1,k,a,b,c)$. We have

$$\begin{aligned}B_1(F_k^1,k,a,b,c) = {} & cx^{k-1}(bx^{k+1}-i(k+1)y)^{\alpha_1}(-ax^{k+1}+i(k+1)y)^{\beta_1-1} \\ & \times(x^{2k+2}+(k+1)^2y^2)(x^{4k+2}+(k+1)^2y^2)^{-1/2}.\end{aligned}$$

Therefore applying Corollary 1 with $\omega_1=\beta_1-1=-(c+(k+1)(b-a))/((k+1)(b-a))$, $\omega_2=\alpha_1=(c-k(b-a))/((k+1)(b-a))$, $\omega_3=-(1/(k+1))$, $w_1=-a$, $w_2=b$ we obtain

$$\begin{aligned}&-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^1,k,a,b,c)\,ds \\ &= -\frac{c}{k+1}\int_{-\pi}^{\pi}|\sin\theta|^{-1/(k+1)}(-a|\sin\theta|+i\cos\theta)^{\beta_1-1} \\ &\qquad\times(b|\sin\theta|-i\cos\theta)^{\alpha_1}v(\varepsilon,\theta)\,d\theta \\ &= -\frac{c}{k+1}\int_{-\pi}^{\pi}(v(0,0)+\bar{o}(1))|\sin\theta|^{-1/(k+1)} \\ &\qquad\times(-a|\sin\theta|+i\cos\theta)^{\beta_1-1}(b|\sin\theta|-i\cos\theta)^{\alpha_1}\,d\theta \\ &= -\frac{2v(0,0)c}{k+1}\int_0^{\pi}\sin^{-1/(k+1)}\theta(-a\sin\theta+i\cos\theta)^{\beta_1-1} \\ &\qquad\times(b\sin\theta-i\cos\theta)^{\alpha_1}\,d\theta+\bar{o}(1) \\ &= -\frac{4(b-a)^{1/(k+1)}\pi\Gamma(k/(k+1))}{\Gamma((k(b-a)-c)/((k+1)(b-a)))\Gamma(c/((k+1)(b-a)))}v(0,0)+\bar{o}(1).\end{aligned} \tag{6}$$

Now from (4), (5), (6) we deduce that

$$\begin{aligned}(G_{k,c}^{a,b}F_k^1(x,y),v(x,y)) &= (F_k^1(x,y),G_{k,-c}^{a,b}v(x,y))\\ &= \lim_{\varepsilon\to 0}\int_{\rho\geq\varepsilon}F_k^1(x,y)G_{k,-c}^{a,b}v(x,y)\,dxdy\\ &= -\frac{4(b-a)^{1/(k+1)}\pi\Gamma(k/(k+1))}{\Gamma((k(b-a)-c)/((k+1)(b-a)))\Gamma(c/((k+1)(b-a)))}v(0,0).\end{aligned}$$

Hence $G_{k,c}^{a,b}F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)=A_{k,c}^{a,b}\delta(x,y)$.

II) Assume that $\operatorname{Re}\beta>-k/(k+1)$. We then have $F_{k,a,b}^{0,\beta,0}(x,y)=(-ax^{k+1}+i(k+1)y)^{\beta}\in L_{loc}^{p}(\boldsymbol{R}^2)$ for every $1\leq p<-(k+2)/((k+1)\operatorname{Re}\beta)$ if $\operatorname{Re}\beta<0$ and $F_{k,a,b}^{0,\beta,0}(x,y)\in L_{loc}^{\infty}(\boldsymbol{R}^2)$ if $\operatorname{Re}\beta\geq 0$. It is easy to compute that

$$\begin{aligned}&B_1(F_{k,a,b}^{0,\beta,0},k,a,b,k(b-a))|_{\rho=\varepsilon}\\ &\quad=(k+1)(b-a)x^{k-1}(-ax^{k+1}+i(k+1)y)^{\beta-1}\\ &\qquad\times\{-\beta x^{2k+2}+ia(k+(k+1)\beta)x^{k+1}y+k(k+1)y^2\}\\ &\qquad\times(x^{4k+2}+(k+1)^2y^2)^{-1/2}|_{\rho=\varepsilon}\\ &\quad=\varepsilon^{2k+(k+1)\beta}(b-a)(-\beta(k+1)\sin^2\theta+ia(k+k\beta+\beta)\cos\theta|\sin\theta|+k\cos^2\theta)\\ &\qquad\times|\sin\theta|^{(k-1)/(k+1)}(-a|\sin\theta|+i\cos\theta)^{\beta-1}\\ &\qquad\times(\varepsilon^{4k+2}|\sin\theta|^{(4k+2)/(k+1)}+\varepsilon^{2k+2}\cos^2\theta)^{-1/2}\end{aligned}$$

and

$$|(|\nu_1|+|\nu_2.\ x^k|)F_{k,a,b}^{0,\beta,0}|\,ds|_{\rho=\varepsilon}\leq C\varepsilon^{(k+1)(1+\operatorname{Re}\beta)}|(-a|\sin\theta|+i\cos\theta)^{\beta}|\,d\theta.$$

Using the assumption that $\operatorname{Re}\beta>-k/(k+1)$ we deduce that

$$-\int_{\rho=\varepsilon}v(x,y)B_1(F_{k,a,b}^{0,\beta,0},k,a,b,k(b-a))\,ds+\int_{\rho=\varepsilon}F_{k,a,b}^{0,\beta,0}(x,y)B_2(v,k,a,b)\,ds\to 0$$

as $\varepsilon\to 0$. Hence $G_{k,k(b-a)}^{a,b}F_{k,a,b}^{0,\beta,0}(x,y)=0$.

III) The proof of this part is the same as the proof of part II) with $-a,\beta$ replaced by b,α.

IV) We note that if k is odd and $\operatorname{Re}a<0$, and $\operatorname{Re}b<0$ then $(-bx^{k+1}+i(k+1)y)^{\alpha}$ and $(-ax^{k+1}+i(k+1)y)^{\beta}\in C^{\infty}(\boldsymbol{R}^2\backslash(0,0))$ for every α and β. Therefore we can repeat all the arguments in part I). The only difference is

$$\begin{aligned}&-\int_{\rho=\varepsilon}v(x,y)B_1(E_{k,a,b}^{\alpha_1,\beta_1,0},k,a,b,c)\,ds+\int_{\rho=\varepsilon}E_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)B_2(v,k,a,b)\,ds\\ &\quad=-\frac{2v(0,0)c}{k+1}\int_0^{\pi}\sin^{-1/(k+1)}\theta(-a\sin\theta+i\cos\theta)^{\beta_1-1}(-b\sin\theta-i\cos\theta)^{\alpha_1}\,d\theta+\bar{o}(1).\end{aligned}$$

Applying Corollary 2 with $\omega_1 = \beta_1 - 1 = -(c + (k+1)(b-a))/((k+1)(b-a))$, $\omega_2 = \alpha_1 = (c - k(b-a))/((k+1)(b-a))$, $\omega_3 = -1/(k+1)$, $w_1 = -a$, $w_2 = -b$ we obtain

$$(7)\quad -\frac{2v(0,0)c}{k+1}\int_0^{\pi} \sin^{-1/(k+1)}\theta(-a\sin\theta + i\cos\theta)^{\beta_1-1}(-b\sin\theta + i\cos\theta)^{\alpha_1}\,d\theta = 0.$$

Now clearly we have $G_{k,c}^{a,b}E_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = 0$.

V) Argue as in part I) we arrive at

$$-\int_{\rho=\varepsilon} v(x,y)B_1(R_{k,a}^{\kappa_1,\eta_1,0}, k, a, a, c)\,ds + \int_{\rho=\varepsilon} R_{k,a}^{\kappa_1,\eta_1,0}(x,y)B_2(v,k,a,a)\,ds$$
$$= -\frac{2v(0,0)c}{k+1}\int_0^{\pi}\sin^{-1/(k+1)}\theta(-a\sin\theta + i\cos\theta)^{\eta_1-1}$$
$$\times e^{\kappa_1\sin\theta/(-a\sin\theta+i\cos\theta)}\,d\theta + \bar{o}(1) = \bar{o}(1).$$

The last equality is a consequence of (7) when we let $b \to a$. Therefore we conclude that $G_{k,c}^{a,a}R_{k,a}^{\kappa_1,\eta_1,0}(x,y) = 0$. This concludes the proof of Theorem 1. □

Corollary 3. *If either* $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b > 0$, $c = -(k+1)(b-a)N$, *or* $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b > 0$, $c = ((k+1)N + k)(b-a)$, *where N is a non-negative integer, or* $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b < 0$, *then $G_{k,c}^{a,b}$ is not hypoelliptic (nor analytic hypoelliptic).*

Proof. Indeed, if $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b > 0$, $c = -(k+1)(b-a)N$, or $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b > 0$, $c = ((k+1)N + k)(b-a)$ then $\Gamma((k(b-a) - c)/((k+1)(b-a))) = \infty$ or $\Gamma(c/((k+1)(b-a))) = \infty \Rightarrow A_{k,c}^{a,b} = 0 \Rightarrow G_{k,c}^{a,b}F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = 0$. If $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b < 0$, then $G_{k,c}^{a,b}E_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = 0$ in the non-resonance case or $G_{k,c}^{a,a}R_{k,a}^{\kappa_1,\eta_1,0}(x,y) = 0$ in the resonance case. □

Theorem 2. *Assume that k is odd. If* $\mathrm{Re}\,a < 0$, *and* $\mathrm{Re}\,b > 0$ *then*

I) $G_{k,c}^{a,b}F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)$
$$= \frac{4\pi\Gamma((k+2)/(k+1))}{(b-a)^{1/(k+1)}\Gamma(((k+1)(b-a)-c)/((k+1)(b-a)))\Gamma((c+b-a)/((k+1)(b-a)))}$$
$$\cdot\frac{\partial\delta(x,y)}{\partial x} =: B_{k,c}^{a,b}\frac{\partial\delta(x,y)}{\partial x}.$$

II) $G_{k,(k+1)(b-a)}^{a,b}F_{k,a,b}^{0,\beta,1}(x,y) = 0$ *if* $\mathrm{Re}\,\beta > -(k+2)/(k+1)$.

III) $G_{k,-(b-a)}^{a,b}F_{k,a,b}^{\alpha,0,1}(x,y) = 0$ *if* $\mathrm{Re}\,\alpha > -(k+2)/(k+1)$.

If $\mathrm{Re}\,a < 0$, $\mathrm{Re}\,b < 0$ *and* $a \neq b$ *then*

IV) $G_{k,c}^{a,b}E_{k,a,b}^{\alpha_2,\beta_2,1}(x,y) = 0$.

If $\mathrm{Re}\,a < 0$ *then*

V) $G_{k,c}^{a,a}R_{k,a}^{\kappa_2,\eta_2,1}(x,y) = 0$.

Proof. I) Let us write $F_k^2(x,y) = F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)$. As in the proof of Theorem 1 it is easy to check that $F_k^2(x,y) \in L_{loc}^{((k+2)/(k+1))-\tau}(\boldsymbol{R}^2)$ for any small positive τ. We see that

$$\left|\int_{\rho=\varepsilon} ax^k(v_1 - iv_2bx^k.\ F_k^2(x,y).\frac{\partial v(x,y)}{\partial y}\, ds\right| \tag{8}$$

$$\leq C\varepsilon\int_{-\pi}^{\pi}\left|\frac{\partial v(x,y)}{\partial y}\right| d\theta \to 0 \quad \text{as } \varepsilon \to 0.$$

Next we have

$$\int_{\rho=\varepsilon} F_k^2(x,y).(v_1 - ibx^kv_2).\ \frac{\partial v(x,y)}{\partial x}\, ds$$

$$= \frac{1}{k+1}\int_{-\pi}^{\pi}\frac{\partial v(\varepsilon,\theta)}{\partial x}\sin^{1/(k+1)}\theta(-a|\sin\theta| + i\cos\theta)^{\beta_2-1}(b|\sin\theta| - i\cos\theta)^{\alpha_2}$$

$$\times (a\sin^2\theta - i(ab+1)|\sin\theta|\cos\theta - b\cos^2\theta)\, d\theta \tag{9}$$

$$= \frac{2(\partial v(0,0)/\partial x)}{k+1}\int_0^{\pi}\sin^{1/(k+1)}\theta(-a|\sin\theta| + i\cos\theta)^{\beta_2-1}(b|\sin\theta| - i\cos\theta)^{\alpha_2}$$

$$\times (a\sin^2\theta - i(ab+1)|\sin\theta|\cos\theta - b\cos^2\theta)\, d\theta + \bar{o}(1).$$

Now let us compute $B_1(F_k^2, k, a, b, c)$. We have

$$B_1(F_k^2, k, a, b, c)|_{\rho=\varepsilon}$$

$$= x^k(-ax^{k+1} + i(k+1)y)^{\beta_2-1}(bx^{k+1} - i(k+1)y)^{\alpha_2}$$

$$\times ((c+b)x^{2k+2} - i(k+1)(ab+1)yx^{k+1} + (k+1)^2(c-a)y^2)$$

$$\times (x^{4k+2} + (k+1)^2y^2)^{-1/2}|_{\rho=\varepsilon} = \varepsilon^{k-1}(\sin\theta)_{\pm}^{k/(k+1)}(-a|\sin\theta| + i\cos\theta)^{\beta_2-1}$$

$$\times (b|\sin\theta| - i\cos\theta)^{\alpha_2}(\varepsilon^{4k+2}|\sin\theta|^{(4k+2)/(k+1)} + \varepsilon^{2k+2}\cos^2\theta)^{-1/2}$$

$$\times ((c+b)\sin^2\theta - i(ab+1)|\sin\theta|\cos\theta + (c-a)\cos^2\theta).$$

Note that $v(\varepsilon,\theta) = v(0,0) + \varepsilon(\sin\theta)_{\pm}^{1/(k+1)}(\partial v(0,0)/\partial x) + \bar{o}(\varepsilon)$. It follows that

$$-\int_{\rho=\varepsilon} v(x,y)B_1(F_k^2, k, a, b, c)\, ds$$

$$= -\frac{v(0,0)}{(k+1)\varepsilon}\int_{-\pi}^{\pi}\operatorname{sign}(\sin\theta)(-a|\sin\theta| + i\cos\theta)^{\beta_2-1}(b|\sin\theta| - i\cos\theta)^{\alpha_2}$$

$$\times ((c+b)\sin^2\theta - i(ab+1)|\sin\theta|\cos\theta + (c-a)\cos^2\theta)\, d\theta \tag{10}$$

$$-\frac{2(\partial v(0,0)/\partial x)}{k+1}\int_0^\pi \sin^{1/(k+1)}\theta$$
$$\times(-a|\sin\theta|+i\cos\theta)^{\beta_2-1}(b|\sin\theta|-i\cos\theta)^{\alpha_2}$$
$$\times((c+b)\sin^2\theta-i(ab+1)|\sin\theta|\cos\theta+(c-a)\cos^2\theta)\,d\theta+\bar{o}(1).$$

We see that the first integral in the right side of (10) vanishes since the integrand is an odd function of θ. Therefore summing (8), (9), and (10) and applying Corollary 1 with $\omega_1=\beta_2-1=-(c+(k+2)(b-a))/((k+1)(b-a))$, $\omega_2=\alpha_2=(c-(k+1)(b-a))/((k+1)(b-a))$, $\omega_3=1/(k+1)$, $w_1=-a$, $w_2=b$ yields

$$(11)\quad -\int_{\rho=\varepsilon} v(x,y)B_1(F_k^2,k,a,b,c)\,ds+\int_{\rho=\varepsilon} F_2^k(x,y)B_2(v,k,a,b)\,ds$$
$$=-\frac{2(c+b-a)(\partial v(0,0)/\partial x)}{k+1}$$
$$\times\int_0^\pi \sin^{1/(k+1)}\theta(-a|\sin\theta|+i\cos\theta)^{\beta_2-1}(b|\sin\theta|-i\cos\theta)^{\alpha_2}\,d\theta+\bar{o}(1)$$
$$=-\frac{4\pi\Gamma((k+2)/(k+1))}{(b-a)^{1/(k+1)}\Gamma(((k+1)(b-a)-c)/((k+1)(b-a)))\Gamma((c+b-a)/((k+1)(b-a)))}$$
$$\times\frac{\partial v(0,0)}{\partial x}+\bar{o}(1).$$

By (4) with $F_k^1(x,y)$ replaced by $F_k^2(x,y)$ and (11) we deduce that

$$(G_{k,c}^{a,b}F_k^2(x,y),v(x,y))$$
$$=(F_k^2(x,y),G_{k,-c}^{a,b}v(x,y))$$
$$=\lim_{\varepsilon\to 0}\int_{\rho\geq\varepsilon} F_k^2(x,y)G_{k,-c}^{a,b}v(x,y)\,dxdy$$
$$=-\frac{4\pi\Gamma((k+2)/(k+1))(\partial v(0,0)/\partial x)}{(b-a)^{1/(k+1)}\Gamma(((k+1)(b-a)-c)/((k+1)(b-a)))\Gamma((c+b-a)/((k+1)(b-a)))}.$$

It follows that $G_{k,c}^{a,b}F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)=B_{k,c}^{a,b}(\partial\delta(x,y)/\partial x)$.

II) Assume that $\mathrm{Re}\,\beta>-(k+2)/(k+1)$. We then have $F_{k,a,b}^{0,\beta,1}(x,y)=x(-ax^{k+1}+i(k+1)y)^\beta\in L_{loc}^p(\boldsymbol{R}^2)$ for every $1\leq p<-(k+2)/(1+(k+1)\mathrm{Re}\,\beta)$ if $\mathrm{Re}\,\beta<-1/(k+1)$ and $F_{k,a,b}^{0,\beta,1}(x,y)\in L_{loc}^\infty(\boldsymbol{R}^2)$ if $\mathrm{Re}\,\beta\geq-1/(k+1)$. It is easy to compute that

$$
\begin{aligned}
&B_1(F^{0,\beta,1}_{k,a,b}, k, a, b, (k+1)(b-a))|_{\rho=\varepsilon} \\
&\quad = x^k(-ax^{k+1} + i(k+1)y)^{\beta-1} \times \{-((k+1)(b-a)\beta - a)x^{2k+2} \\
&\qquad + i(k+1)((k+1)a(b-a)(\beta+1) - a^2 - 1)x^{k+1}y + (k+1)^2 \\
&\qquad \times ((k+1)(b-a) - a)y^2\}(x^{4k+2} + (k+1)^2y^2)^{-1/2}|_{\rho=\varepsilon} = \varepsilon^{2k+1+(k+1)\beta} \\
&\qquad \times (-((k+1)(b-a)\beta - a)\sin^2\theta + i((k+1)a(b-a)(\beta+1) - a^2 - 1) \\
&\qquad \times \cos\theta|\sin\theta| + ((k+1)(b-a) - a)\cos^2\theta)(\sin\theta)_{\pm}^{k/(k+1)} \\
&\qquad \times (-a|\sin\theta| + i\cos\theta)^{\beta-1}(\varepsilon^{4k+2}|\sin\theta|^{(4k+2)/(k+1)} + \varepsilon^{2k+2}\cos^2\theta)^{-1/2}
\end{aligned}
$$

and

$$
\begin{aligned}
&F^{0,\beta,1}_{k,a,b} B_2(v, k, a, b)\, ds|_{\rho=\varepsilon} \\
&\quad = -\frac{\varepsilon^{k+2+(k+1)\beta}}{k+1}|\sin\theta|^{1/(k+1)}(-a|\sin\theta| + i\cos\theta)^{\beta}(|\sin\theta| - ib\cos\theta)X_2 v(x,y)\, d\theta.
\end{aligned}
$$

Therefore we deduce that

$$
\begin{aligned}
&-\int_{\rho=\varepsilon} v(x,y)B_1(F^{0,\beta,1}_{k,a,b}, k, a, b, (k+1)(b-a))\, ds \\
&\qquad + \int_{\rho=\varepsilon} F^{0,\beta,1}_{k,a,b}(x,y)B_2(v,k,a,b)\, ds \\
&\quad = \frac{\varepsilon^{(k+1)(\beta+1)}v(0,0)}{k+1}\int_{-\pi}^{\pi}\big(-((k+1)(b-a)\beta - a)\sin^2\theta \\
&\qquad + i((k+1)a(b-a)(\beta+1) - a^2 - 1) \\
&\qquad \times \cos\theta|\sin\theta| + ((k+1)(b-a) - a)\cos^2\theta\big)\,\mathrm{sign}(\sin\theta) \\
&\qquad \times (-a|\sin\theta| + i\cos\theta)^{\beta-1}\, d\theta + O(\varepsilon^{k+2+(k+1)\operatorname{Re}\beta}).
\end{aligned}
\tag{12}
$$

The integral in the right side of (12) vanishes for every ε since its integrand is an odd function of θ. Therefore using the assumption that $\operatorname{Re}\beta > -(k+2)/(k+1)$ we see that the expression in the left side of (12) tends to 0 as ε tends to 0. Hence $G^{a,b}_{k,(k+1)(b-a)}F^{0,\beta,1}_{k,a,b}(x,y) = 0$.

III) The proof of this part is the same as the proof of part II) with $-a, \beta$ replaced by b, α.

IV) We note that if k is odd and $\operatorname{Re} a < 0$, and $\operatorname{Re} b < 0$ then $(-bx^{k+1} + i(k+1)y)^{\alpha}$ and $(-ax^{k+1} + i(k+1)y)^{\beta} \in C^{\infty}(\boldsymbol{R}^2 \backslash (0,0))$ for every α and β. Therefore we can repeat all the arguments in part I). The only difference is

$$-\int_{\rho=\varepsilon} v(x,y)B_1(E_{k,a,b}^{\alpha_2,\beta_2,1},k,a,b,c)\,ds+\int_{\rho=\varepsilon} E_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)B_2(v,k,a,b)\,ds$$
$$\to -\frac{2(c+b-a)(\partial v(0,0)/\partial x)}{k+1}$$
$$\times\int_0^\pi \sin^{1/(k+1)}\theta(-a|\sin\theta|+i\cos\theta)^{\beta_2-1}(-b|\sin\theta|+i\cos\theta)^{\alpha_2}\,d\theta.$$

Applying Corollary 2 with $\omega_1=\beta_2-1=-(c+(k+2)(b-a))/((k+1)(b-a))$, $\omega_2=\alpha_2=(c-(k+1)(b-a))/((k+1)(b-a))$, $\omega_3=-1/(k+1)$, $w_1=-a$, $w_2=-b$ yields

$$-\frac{2(c+b-a)(\partial v(0,0)/\partial x)}{k+1}\times\int_0^\pi \sin^{1/(k+1)}\theta(-a|\sin\theta|+i\cos\theta)^{\beta_2-1}(-b|\sin\theta|+i\cos\theta)^{\alpha_2}\,d\theta=0. \tag{13}$$

Now clearly we have $G_{k,c}^{a,b}E_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)=0$.

V) Argue as in part I) we arrive at

$$-\int_{\rho=\varepsilon} v(x,y)B_1(R_{k,a}^{\kappa_2,\eta_2,1},k,a,a,c)\,ds+\int_{\rho=\varepsilon} R_{k,a}^{\kappa_2,\eta_2,1}(x,y)B_2(v,k,a,a)\,ds$$
$$\to -\frac{2c(\partial v(0,0)/\partial x)}{k+1}\int_0^\pi \sin^{1/(k+1)}\theta$$
$$\times(-a\sin\theta+i\cos\theta)^{\eta_2-1}e^{\kappa_2\sin\theta/(-a\sin\theta+i\cos\theta)}\,d\theta=0.$$

The last equality is a consequence of (13) when we let $b\to a$. Therefore we conclude that $G_{k,c}^{a,a}R_{k,a}^{\kappa_2,\eta_2,1}(x,y)=0$. This concludes the proof of Theorem 2. □

Corollary 4. *If either* $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b>0$, $c=-((k+1)N+1)(b-a)$ *or* $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b>0$, $c=(k+1)(b-a)(N+1)$ *where N is a non-negative integer, or* $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b<0$, *then* $G_{k,c}^{a,b}$ *is not hypoelliptic (nor analytic hypoelliptic).*

Proof. Indeed, if either $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b>0$, $c=-((k+1)N+1)(b-a)$ or $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b>0$, $c=(k+1)(b-a)(N+1)$ then $\Gamma(((k+1)(b-a)-c)/((k+1)(b-a)))=\infty$ or $\Gamma((c+b-a)/((k+1)(b-a)))=\infty\Rightarrow B_{k,c}^{a,b}=0\Rightarrow G_{k,c}^{a,b}F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)=0$. If $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b<0$, then $G_{k,c}^{a,b}E_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)=0$ in the non-resonance case or $G_{k,c}^{a,a}R_{k,a}^{\kappa_2,\eta_2,1}(x,y)=0$ in the resonance case. □

Theorem 3. *Assume that k is even and* $\mathrm{Re}\,a<0$, $\mathrm{Re}\,b>0$. *If* $c=(k+1)(b-a)N+((2k+1)(b-a)/2)$, *where N is an integer, then*

$$G_{k,c}^{a,b}F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)=0,\quad G_{k,c}^{a,b}F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)=0.$$

If $c = (k+1)(b-a)N + (k(b-a)/2)$, *where* N *is an integer, then*

$$G_{k,c}^{a,b} F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = A_{k,c}^{a,b}\delta(x,y), \quad G_{k,c}^{a,b} F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y) = B_{k,c}^{a,b}(\partial\delta(x,y)/\partial x).$$

Proof. If $c = (k+1)(b-a)N + ((2k+1)(b-a)/2)$, or $c = (k+1)(b-a)N + (k(b-a)/2)$, then $(k/(k+1)) + 2\beta_1$ and $((k+2)/(k+1)) + 2\beta_2$ are integers. Therefore $F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y)$, $F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y) \in C^\infty(\boldsymbol{R}^2\backslash(0,0))$. Again we have $F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) \in L_{loc}^{((k+2)/k)-\tau}(\boldsymbol{R}^2)$ and $F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y) \in L_{loc}^{((k+2)/(k+1))-\tau}(\boldsymbol{R}^2)$ for any small positive τ. First we prove the theorem for $F_{k,a,b}^{\alpha_1,\beta_1,0}$. As in the proof of Theorem 1 we can show that

$$\int_{\rho=\varepsilon} F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) B_2(v,k,a,b)\, ds \to 0 \quad \text{as } \varepsilon \to 0.$$

Next we have

$$\begin{aligned}
&-\int_{\rho=\varepsilon} v(x,y) B_1(F_{k,a,b}^{\alpha_1,\beta_1,0}, k, a, b, c)\, ds \\
&\quad = -\frac{c}{k+1}\int_{-\pi}^{\pi} (\sin\theta)_{\pm}^{-1/(k+1)}(-a\sin\theta + i\cos\theta)^{\beta_1-1} \\
&\qquad\qquad \times (b\sin\theta - i\cos\theta)^{\alpha_1} v(\varepsilon,\theta)\, d\theta \\
&\quad = -\frac{c}{k+1}\int_{-\pi}^{\pi} (v(0,0) + \bar{o}(1))(\sin\theta)_{\pm}^{-1/(k+1)} \qquad (14)\\
&\qquad\qquad \times (-a\sin\theta + i\cos\theta)^{\beta_1-1}(b\sin\theta - i\cos\theta)^{\alpha_1}\, d\theta \\
&\quad = -\frac{cv(0,0)}{k+1}\int_{-\pi}^{\pi} (\sin\theta)_{\pm}^{-1/(k+1)}(-a\sin\theta + i\cos\theta)^{\beta_1-1} \\
&\qquad\qquad \times (b\sin\theta - i\cos\theta)^{\alpha_1}\, d\theta + \bar{o}(1).
\end{aligned}$$

If $c = (k+1)(b-a)N + ((2k+1)(b-a)/2)$ then the integrand in the right side of (14) changes sign when we replace θ by $\theta - \pi$. Therefore the integral vanishes. Hence $G_{k,c}^{a,b} F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = 0$. If $c = (k+1)(b-a)N + (k(b-a)/2)$ then it follows that

$$\begin{aligned}
&-\frac{cv(0,0)}{k+1}\int_{-\pi}^{\pi} (\sin\theta)_{\pm}^{-1/(k+1)}(-a\sin\theta + i\cos\theta)^{\beta_1-1}(b\sin\theta - i\cos\theta)^{\alpha_1}\, d\theta \\
&\quad = -\frac{2cv(0,0)}{k+1}\int_0^{\pi} (\sin\theta)^{-1/(k+1)}(-a\sin\theta + i\cos\theta)^{\beta_1-1}(b\sin\theta - i\cos\theta)^{\alpha_1}\, d\theta \\
&\quad = A_{k,c}^{a,b} v(0,0).
\end{aligned}$$

Therefore $G_{k,c}^{a,b} F_{k,a,b}^{\alpha_1,\beta_1,0}(x,y) = A_{k,c}^{a,b}\delta(x,y)$.

Next we prove the theorem for $F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)$. As in Theorem 2 we have

$$-\int_{\rho=\varepsilon} v(x,y)B_1(F_{k,a,b}^{\alpha_2,\beta_2,1},k,a,b,c)\,ds+\int_{\rho=\varepsilon} F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)B_2(v,k,a,b)\,ds\to$$

$$(15)\qquad =-\frac{(c+b-a)(\partial v(0,0)/\partial x)}{k+1}\times\int_{-\pi}^{\pi}(\sin\theta)_{\pm}^{1/(k+1)}(-a\sin\theta+i\cos\theta)^{\beta_2-1}(b\sin\theta-i\cos\theta)^{\alpha_2}\,d\theta.$$

If $c=(k+1)(b-a)N+((2k+1)(b-a)/2)$ then the integrand in the right side of (15) changes sign when we replace θ by $\theta-\pi$, therefore the integral vanishes. Hence $G_{k,c}^{a,b}F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)=0$. If $c=(k+1)(b-a)N+(k(b-a)/2)$ then we deduce that

$$\begin{aligned}&-\frac{(c+b-a)(\partial v(0,0)/\partial x)}{k+1}\\&\quad\times\int_{-\pi}^{\pi}(\sin\theta)_{\pm}^{1/(k+1)}(-a\sin\theta+i\cos\theta)^{\beta_2-1}(b\sin\theta-i\cos\theta)^{\alpha_2}\,d\theta\\&=-\frac{2(c+b-a)(\partial v(0,0)/\partial x)}{k+1}\\&\quad\times\int_{0}^{\pi}(\sin\theta)_{\pm}^{1/(k+1)}(-a\sin\theta+i\cos\theta)^{\beta_2-1}(b\sin\theta-i\cos\theta)^{\alpha_2}\,d\theta\\&=-B_{k,c}^{a,b}\frac{\partial v(0,0)}{\partial x}.\end{aligned}$$

If follows that $G_{k,c}^{a,b}F_{k,a,b}^{\alpha_2,\beta_2,1}(x,y)=B_{k,c}^{a,b}(\partial\delta(x,y)/\partial x)$. □

Corollary 5. *Assume that* k *is even and* $\operatorname{Re}a<0$, $\operatorname{Re}b>0$. *If* $c=(k+1)(b-a)N+(((2k+1)(b-a))/2)$, *where* N *is an integer, then* $G_{k,c}^{a,b}$ *is not hypoelliptic* (*nor analytic hypoelliptic*).

Remark 1. Altogether Corollary 3, Corollary 4 and Corollary 5 give all the values k,a,b,c, where $G_{k,c}^{a,b}$ is not hypoelliptic, as stated in [1], [2].

Remark 2. Since $G_{k,c}^{a,b}$ is invariant under the translation $(x,y)\to(x,y+y_0)$ it is easy to have the fundamental solutions or singular solutions at points $(0,y_0)$ in all cases considered above.

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