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PICARD CONSTANTS OF THREE-SHEETED ALGEBROID SURFACES WITH $p(y) = 5$

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Abstract

In 1995 Sawada-Tohge proved that every three-sheeted algebroid Riemann surface with $p(y) = 5$ is of Picard constant 5, unless its discriminant has a form $e^{\delta H}(Ae^{4H} + B)$, where $\delta = 0$ or 1. In this paper we shall prove that the result remains valid with no condition.

1. Introduction

Let $\mathfrak{M}(\boldsymbol{R})$ be the family of non-constant meromorphic functions on a Riemann surface $\boldsymbol{R}$. Let $p(f)$ be the cardinal number of values which are not taken by $f \in \mathfrak{M}(\boldsymbol{R})$. Then we put

$$P(\boldsymbol{R}) = \sup_{f \in \mathfrak{M}(\boldsymbol{R})} p(f),$$

which is called the Picard constant of $\boldsymbol{R}$. We can prove that $P(\boldsymbol{R}) \geqq 2$ if $\boldsymbol{R}$ is open and $P(\boldsymbol{R}) = 0$ if $\boldsymbol{R}$ is compact. Picard constant plays a very important role in the theory of analytic mappings of Riemann surfaces. Indeed Ozawa [5] proved that there exists no non-trivial analytic mapping of $\boldsymbol{R}$ into $\boldsymbol{S}$ if $P(\boldsymbol{R}) < P(\boldsymbol{S})$.

An n-sheeted algebroid surface is the proper existence domain of an n-valued algebroid function, which is defined by the following irreducible equation;

$$S_0(z)y^n - S_1(z)y^{n-1} + \cdots + (-1)^{n-1}S_{n-1}(z)y + (-1)^n S_n(z) = 0,$$

where $S_i(z)$ $(i = 0, 1, \dots, n)$ are entire functions on $\mathbf{C}$ with no common zeros. An algebroid function f is called transcendental if at least one of $S_i(z)/S_0(z)$ $(i = 1, 2, \dots, n)$ is transcendental and f is called entire if all the $S_i(z)/S_0(z)$ $(i = 1, 2, \dots, n)$ are entire. If $\boldsymbol{R}$ is an n-sheeted algebroid surface, then $P(\boldsymbol{R}) \leqq 2n$ by Selberg's theory of algebroid functions [10]. However it is very difficult in general to calculate $P(\boldsymbol{R})$ of a given open Riemann surface $\boldsymbol{R}$, even an algebroid surface.

An n-sheeted algebroid surface is called regularly branched if all its branch points are of order $n - 1$. Then we have

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THEOREM A (Aogai [1], Ozawa [6] and Hiromi-Niino [3]). [1]*Let* $\boldsymbol{R}$ *be an n-sheeted regularly branched algebroid surface. If* $P(\boldsymbol{R}) > 3n/2$, *then* $P(\boldsymbol{R}) = 2n$ *and* $\boldsymbol{R}$ *can be defined by an algebroid function y such that*

$$y^n = (e^{H(z)} - \alpha)(e^{H(z)} - \beta)^{n-1}, \quad H(0) = 0, \quad \alpha\beta(\alpha - \beta) \neq 0,$$

where $H(z)$ *is a non-constant entire function and* α *and* β *are constants.*

We can prove that there exists no three-sheeted regularly branched surface with $P(\boldsymbol{R}) = 5$ by theorem A.

In this paper we shall consider three-sheeted algebroid surfaces defined by three-valued entire algebroid functions. Let $\boldsymbol{R}$ be a three-sheeted algebroid Riemann surface defined by

$$y^3 - S_1(z)y^2 + S_2(z)y - S_3(z) = 0, \tag{1}$$

and $\boldsymbol{X}$ be a three-sheeted algebroid Riemann surface defined by

$$f^3 - U_1(z)f^2 + U_2(z)f - U_3(z) = 0, \tag{2}$$

where $S_i(z)$ $(i = 1, 2, 3)$ and $U_j(z)$ $(j = 1, 2, 3)$ are entire functions. Ozawa and the author proved the following

THEOREM B (Ozawa-Sawada [7]). *Let* $\boldsymbol{X}$ *be a three-sheeted algebroid Riemann surface defined by* (2). *If* $p(f) = 6$, *then we have*

$$\begin{cases} U_1(z) = x_0 e^{L(z)} + x_1, \\ U_2(z) = b_1 x_0 e^{L(z)} + x_2, \\ U_3(z) = x_3, \end{cases} \tag{3}$$

where b_1 $(\neq 0)$, x_0 $(\neq 0)$, x_1, x_2, x_3 $(\neq 0)$ *are constants and* $L(z)$ *is an entire function with* $L(0) = 0$. *And its discriminant* D_X *is*

$$D_X = -b_1^2 x_0^4 e^{4L} + \eta_3 x_0^3 e^{3L} + \eta_2 x_0^2 e^{2L} + \eta_1 x_0 e^L + \eta_0,$$

where

$$\begin{cases} \eta_3 = 4b_1^3 - 2b_1^2 x_1 - 2b_1 x_2 + 4x_3, \\ \eta_2 = 12x_1 x_3 - 18b_1 x_3 - x_2^2 - 4b_1 x_1 x_2 + 12b_1^2 x_2 - b_1^2 x_1^2, \\ \eta_1 = 12x_1^2 x_3 - 18b_1 x_1 x_3 - 18x_2 x_3 - 2x_1 x_2^2 + 12b_1 x_2^2 - 2b_1 x_1^2 x_2, \\ \eta_0 = 4x_1^3 x_3 - x_1^2 x_2^2 + 27x_3^2 - 18x_1 x_2 x_3 + 4x_2^3 \ (\neq 0). \end{cases} \tag{4}$$

And we have

[1] Ozawa [6] and Hiromi-Niino [3] proved above result in the case $n = 2$ and $n = 3$ respectively.

Theorem C (Ozawa-Sawada [7]). *Let $\boldsymbol{R}$ be a three-sheeted algebroid surface defined by* (1). *If $p(y) = 5$, then we have*

$$\begin{cases} S_1(z) = y_1, \\ S_2(z) = y_0 e^{H(z)} + y_2, \\ S_3(z) = y_3, \end{cases} \tag{5}$$

where y_0 $(\neq 0)$, y_1, y_2, y_3 $(\neq 0)$ are constants and $H(z)$ is a non-constant entire function with $H(0) = 0$. We denote this surface by $\boldsymbol{R}_A$. Furthermore its discriminant $D_{\boldsymbol{R}_A}$ is

$$D_{\boldsymbol{R}_A} = 4y_0^3 e^{3H} + \zeta_2 y_0^2 e^{2H} + \zeta_1 y_0 e^H + \zeta_0,$$

where

$$\begin{cases} \zeta_0 = 4y_1^3 y_3 - y_1^2 y_2^2 - 18 y_1 y_2 y_3 + 4y_2^3 + 27 y_3^2 \ (\neq 0), \\ \zeta_1 = 12 y_2^2 - 18 y_1 y_3 - 2 y_1^2 y_2, \\ \zeta_2 = 12 y_2 - y_1^2. \end{cases} \tag{6}$$

Remark. Ozawa-Sawada [7] proved that there exist the following three surfaces $\boldsymbol{R}_A$, $\boldsymbol{R}_B$ and $\boldsymbol{R}_G$ with $p(y) = 5$:

$$\boldsymbol{R}_A : y^3 - y_1 y^2 + (y_0 e^{H(z)} + y_2) y - y_3 = 0,$$

$$\boldsymbol{R}_B : y^3 - (z_0 e^{H(z)} + z_1) y^2 + z_2 y - z_3 = 0,$$

and

$$\boldsymbol{R}_G : y^3 - (w_0 e^{-H(z)} + a) y^2 + w_1 w_0 e^{-H(z)} y - w_2 w_0 e^{-H(z)} = 0,$$

where $H(z)$ is a non-constant entire function and y_0 $(\neq 0)$, y_1, y_2, y_3 $(\neq 0)$, z_0 $(\neq 0)$, z_1, z_2, z_3 $(\neq 0)$, a $(\neq 0)$, w_0 $(\neq 0)$, w_1 and w_2 $(\neq 0)$ are constants. Furthermore

$$x^3 - y_1 x^2 + y_2 x - y_3 = 0$$

has 3 distinct solutions.

However we may consider '**only one**' surface $\boldsymbol{R}_A$. In fact we can investigate that $\boldsymbol{R}_A$, $\boldsymbol{R}_B$ and $\boldsymbol{R}_G$ are conformally equivalent. Putting $y = 1/Y$, then we can deduce $\boldsymbol{R}_B$ from $\boldsymbol{R}_A$. And putting $y = A(1 - a/Y)$, where A is a solution of $A^3 - y_1 A^2 + y_2 A - y_3 = 0$, then we can deduce $\boldsymbol{R}_G$ from $\boldsymbol{R}_A$.

Furthermore we have

Theorem D (Ozawa-Sawada [7], Sawada-Tohge [9]). [2]*Let $\boldsymbol{R}$ be the surface defined by* (1) *with $p(y) = 5$. If $(\zeta_1, \zeta_2) \neq (0, 0)$, then $P(\boldsymbol{R}) = 5$.*

[2] Ozawa-Sawada [7] proved the above result under the condition that $\boldsymbol{R}$ is of finite order and Sawada-Tohge [9] proved that the result remains valid without the order condition.

In this paper we shall prove that the above result remains valid without the condition $(\zeta_1, \zeta_2) \neq (0,0)$. In fact we shall prove the following

THEOREM. *The surface* $\boldsymbol{R}_A$ *is of Picard constant* 5 *with no condition.*

2. Preparations

In this paper we shall consider the surfaces, defined by theorem C, satisfying the additional condition $\zeta_1 = \zeta_2 = 0$ and the surfaces, defined by theorem B, satisfying the additional condition $\eta_1 = \eta_2 = \eta_3 = 0$. First of all we list up all the surfaces $\boldsymbol{X}$ defined by (2) and (3) with the condition $\eta_1 = \eta_2 = \eta_3 = 0$. By (4) we have

$$\begin{cases} \eta_3 = 4b_1^3 - 2b_1^2 x_1 - 2b_1 x_2 + 4x_3 = 0, \\ \eta_2 = 12x_1x_3 - 18b_1x_3 - x_2^2 - 4b_1x_1x_2 + 12b_1^2x_2 - b_1^2x_1^2 = 0, \\ \eta_1 = 12x_1^2x_3 - 18b_1x_1x_3 - 18x_2x_3 - 2x_1x_2^2 + 12b_1x_2^2 - 2b_1x_1^2x_2 = 0. \end{cases}$$

To eliminate x_3 from $\eta_1 = 0$ and $\eta_3 = 0$, let us calculate the resultant of $\eta_1 = 0$ and $\eta_3 = 0$, then we have

$$(3b_1 - 2x_1)(6b_1^2 - 3b_1x_1 + x_2)(b_1x_1 + x_2) = 0.$$

Similarly eliminating x_3 from $\eta_2 = 0$ and $\eta_3 = 0$, we have

$$18b_1^4 - 21b_1^3x_1 + 5b_1^2x_1^2 + 3b_1^2x_2 + 2b_1x_1x_2 - x_2^2 = 0. \tag{7}$$

First of all we assume that $3b_1 - 2x_1 = 0$. Let us put $B = b_1$, then $x_1 = 3B/2$. And from $x_1 = 3B/2$ and (7), we have $x_2 = dB^2$, where d is a constant such that $4d^2 - 24d + 9 = 0$. Furthermore we have $x_3 = (2d-1)B^3/4$ from $\eta_3 = 0$.

Next we assume that $6b_1^2 - 3b_1x_1 + x_2 = 0$. Eliminating x_2 from $6b_1^2 - 3b_1x_1 + x_2 = 0$ and (7), we have

$$b_1^2(18b_1^2 - 6b_1x_1 - x_1^2) = 0.$$

Similarly we put $B = b_1$, then $x_1 = dB$, where d is a constant such that $d^2 + 6d - 18 = 0$. Furthermore we have $x_2 = 3(d-2)B^2$ and $x_3 = 2(d-2)B^3$ from $6b_1^2 - 3b_1x_1 + x_2 = 0$ and $\eta_3 = 0$, respectively.

Last we assume that $b_1x_1 + x_2 = 0$. Eliminating x_2 from $b_1x_1 + x_2 = 0$ and (7), we have

$$b_1^2(9b_1^2 - 12b_1x_1 + x_1^2) = 0.$$

Therefore, putting $B = b_1$, we have $x_1 = dB$ and $x_2 = -dB^2$, where d is a constant such that $d^2 - 12d + 9 = 0$. Furthermore we have $x_3 = -B^3$ from $\eta_3 = 0$. Therefore there exist only three surfaces $\boldsymbol{X}$ satisfying the condition $\eta_1 = \eta_2 = \eta_3 = 0$:

$$
\boldsymbol{X}\text{-(i)}\begin{cases} U_1(z) = x_0 e^{L(z)} + \dfrac{3}{2}B, \\ U_2(z) = Bx_0 e^{L(z)} + dB^2, \\ U_3(z) = \dfrac{2d-1}{4}B^3, \end{cases}
$$

where B $(\neq 0)$ is a constant and d is a solution of $4d^2 - 24d + 9 = 0$, and its discriminant is

$$
D_{\boldsymbol{X}\text{-(i)}} = -B^2 x_0^4 e^{4L} + \frac{1}{16}(4d-3)^3 B^6,
$$

$$
\boldsymbol{X}\text{-(ii)}\begin{cases} U_1(z) = x_0 e^{L(z)} + dB, \\ U_2(z) = Bx_0 e^{L(z)} + 3(d-2)B^2, \\ U_3(z) = 2(d-2)B^3, \end{cases}
$$

where B $(\neq 0)$ is a constant and d is a solution of $d^2 + 6d - 18 = 0$, and its discriminant is

$$
D_{\boldsymbol{X}\text{-(ii)}} = -B^2 x_0^4 e^{4L} - (d-2)(d-6)^3 B^6,
$$

and

$$
\boldsymbol{X}\text{-(iii)}\begin{cases} U_1(z) = x_0 e^{L(z)} + dB, \\ U_2(z) = Bx_0 e^{L(z)} - dB^2, \\ U_3(z) = -B^3, \end{cases}
$$

where B $(\neq 0)$ is a constant and d is a solution of $d^2 - 12d + 9 = 0$, and its discriminant is

$$
D_{\boldsymbol{X}\text{-(iii)}} = -B^2 x_0^4 e^{4L} - (d-1)(d+3)^3 B^6.
$$

Next we list up all the surfaces $\boldsymbol{R}_A$ defined by (1) and (5) with the condition $\zeta_1 = \zeta_2 = 0$. By (6) we have

$$
\begin{cases} \zeta_1 = 12y_2^2 - 18y_1y_3 - 2y_1^2y_2 = 0, \\ \zeta_2 = 12y_2 - y_1^2 = 0. \end{cases}
$$

If $y_1 = 0$, we have $y_2 = 0$ and $y_3 = A$, where A is a non-zero constant. If $y_1 \neq 0$, putting $y_1 = 6A$ $(\neq 0)$, we have $y_2 = 3A^2$ from $\zeta_2 = 0$ and $y_3 = -A^3$ from $\zeta_1 = 0$. Therefore there exist only two surfaces $\boldsymbol{R}_A$ satisfying the condition $\zeta_1 = \zeta_2 = 0$:

$$\boldsymbol{R}_A\text{-(i)} \begin{cases} S_1(z) = 0, \\ S_2(z) = y_0 e^{H(z)}, \\ S_3(z) = A, \end{cases}$$

where A ($\neq 0$) is a constant. Its discriminant is

$$D_{\boldsymbol{R}_A\text{-(i)}} = 4y_0^3 e^{3H} + 27A^2,$$

and

$$\boldsymbol{R}_A\text{-(ii)} \begin{cases} S_1(z) = 6A, \\ S_2(z) = y_0 e^{H(z)} + 3A^2, \\ S_3(z) = -A^3, \end{cases}$$

where A ($\neq 0$) is a constant. Its discriminant is

$$D_{\boldsymbol{R}_A\text{-(ii)}} = 4y_0^3 e^{3H} - 729A^6.$$

Now we suppose that $\boldsymbol{R}$, defined by theorem C, is of Picard constant 6. There exists a meromorphic function f on $\boldsymbol{R}$ such that $p(f) = 6$. Without loss of generality we may assume that the function f is entire, which does not take 5 finite values. The function f can be represented by

$$f = f_0 + f_1 y + f_2 y^2, \tag{8}$$

where f_0, f_1 and f_2 are "**single-valued**" meromorphic functions, which have poles at most on $\{z|H'(z) = 0\}$ (see Ozawa-Sawada [7]). Eliminating y from (1) and (8), we have

$$f^3 - U_1 f^2 + U_2 f - U_3 = 0,$$

where

$$U_1 = 3f_0 + f_1 S_1 + f_2(S_1^2 - 2S_2), \tag{9}$$

$$\begin{aligned} U_2 = {} & 3f_0^2 + 2f_0\{f_1 S_1 + f_2(S_1^2 - 2S_2)\} + f_1^2 S_2 + f_1 f_2(S_1 S_2 - 3S_3) \\ & + f_2^2(S_2^2 - 2S_1 S_3), \end{aligned} \tag{10}$$

$$\begin{aligned} U_3 = {} & f_0^3 + f_0^2\{f_1 S_1 + f_2(S_1^2 - 2S_2)\} \\ & + f_0\{f_1^2 S_2 + f_1 f_2(S_1 S_2 - 3S_3) + f_2^2(S_2^2 - 2S_1 S_3)\} \\ & + f_1^3 S_3 + f_1^2 f_2 S_1 S_3 + f_1 f_2^2 S_2 S_3 + f_2^3 S_3^2. \end{aligned} \tag{11}$$

Because of $p(f) = 6$, the function f defines the surface $\boldsymbol{X}$ described by theorem B. And we have the following relation between the discriminants of $\boldsymbol{R}$ and $\boldsymbol{X}$ (see Ozawa-Sawada [7]):

$$D_{\boldsymbol{X}} = D_{\boldsymbol{R}} \cdot G^2, \tag{12}$$

where

$$G = f_1^3 + 2f_1^2 f_2 S_1 + (S_1^2 + S_2) f_1 f_2^2 + (S_1 S_2 - S_3) f_2^3. \tag{13}$$

Now we may assume that the surface $\boldsymbol{R}$ satisfies the condition $\zeta_1 = \zeta_2 = 0$, then we have that the surface $\boldsymbol{X}$ satisfies the condition $\eta_1 = \eta_2 = \eta_3 = 0$ and $G = Ke^M$, where K is a non-zero constant and M is an entire function with $M(0) = 0$ (see Sawada-Tohge [9]).

Eliminating f_0 from (9) and (10), we have

$$\begin{aligned} &-3f_1^2(S_1^2 - 3S_2) - 3f_1 f_2(2S_1^3 - 7S_1S_2 + 9S_3) \\ &\quad - 3f_2^2(S_1^4 - 4S_1^2S_2 + S_2^2 + 6S_1S_3) + 3U_1^2 - 9U_2 = 0. \end{aligned} \tag{14}$$

Similarly eliminating f_0 from (9) and (11) we have

$$\begin{aligned} &f_1^3(2S_1^3 - 9S_1S_2 + 27S_3) \\ &\quad + 3f_1^2\{2f_2(S_1^4 - 5S_1^2S_2 + 3S_2^2 + 9S_1S_3) - U_1(S_1^2 - 3S_2)\} \\ &\quad + 3f_1f_2\{f_2(2S_1^5 - 11S_1^3S_2 + 15S_1^2S_3 + 11S_1S_2^2 - 9S_2S_3) \\ &\qquad - U_1(2S_1^3 - 7S_1S_2 + 9S_3)\} \\ &\quad + f_2^3(2S_1^6 - 12S_1^4S_2 + 18S_1^3S_3 + 15S_1^2S_2^2 - 36S_1S_2S_3 + 2S_2^3 + 27S_3^2) \\ &\quad - 3f_2^2U_1(S_1^4 - 4S_1^2S_2 + S_2^2 + S_1S_3) + U_1^3 - 27U_3 = 0. \end{aligned} \tag{15}$$

We can construct the following liner equation with respect to f_1 from (13) and (14):

$$\begin{aligned} &\frac{1}{(S_1^2 - 3S_2)^2}[f_1\{-3f_2^2(4S_1^3S_3 - S_1^2S_2^2 - 18S_1S_2S_3 + 4S_2^3 + 27S_3^2) \\ &\qquad - (S_1^2 - 3S_2)(U_1^2 - 3U_2)\} \\ &\qquad - 2f_2^3S_1(4S_1^3S_3 - S_1^2S_2^2 - 18S_1S_2S_3 + 4S_2^3 + 27S_3^2) \\ &\qquad - f_2(S_1S_2 - 9S_3)(U_1^2 - 3U_2) + G(S_1^2 - 3S_2)^2] = 0. \end{aligned} \tag{16}$$

Similarly we can construct the following linear equation with respect to f_1 from (13) and (15):

$$\begin{aligned} &\frac{1}{(S_1^2 - 3S_2)^2\{2f_2(S_1^2 - 3S_2) - 3U_1\}} \\ &\quad \times [f_1\{4f_2^3(S_1^2 - 3S_2)(4S_1^3S_3 - S_1^2S_2^2 - 18S_1S_2S_3 + 4S_2^3 + 27S_3^2) \\ &\qquad - 9f_2^2U_1(4S_1^3S_3 - S_1^2S_2^2 - 18S_1S_2S_3 + 4S_2^3 + 27S_3^2) \\ &\qquad - (S_1^2 - 3S_2)(U_1^3 - 27U_3) - G(S_1^2 - 3S_2)(2S_1^3 - 9S_1S_2 + 27S_3)\} \\ &\quad + 2f_2^4(2S_1^3 - 7S_1S_2 + 9S_3)(4S_1^3S_3 - S_1^2S_2^2 - 18S_1S_2S_3 + 4S_2^3 + 27S_3^2) \end{aligned} \tag{17}$$

$$
\begin{aligned}
&- 6f_2^3 S_1 U_1 (4S_1^3 S_3 - S_1^2 S_2^2 - 18 S_1 S_2 S_3 + 4S_2^3 + 27 S_3^2) \\
&+ f_2 \{ G(-2S_1^6 + 16 S_1^4 S_2 + 18 S_1^3 S_3 - 45 S_1^2 S_2^2 - 108 S_1 S_2 S_3 \\
&\qquad + 54 S_2^3 + 243 S_3^2) - (S_1 S_2 - 9 S_3)(U_1^3 - 27 U_3) \} \\
&+ 3 G U_1 (S_1^2 - 3S_2)^2] = 0.
\end{aligned}
$$

Therefore eliminating f_1 from (16) and (17) we have the following equation, which plays an important role:

$$
\frac{E_1 \cdot E_2}{(S_1^2 - 3S_2)^4 \{ 2f_2 (S_1^2 - 3S_2) - 3U_1 \}} = 0, \tag{18}
$$

where

$$
\begin{aligned}
E_1 = {} & f_2^3 (4S_1^3 S_3 - S_1^2 S_2^2 - 18 S_1 S_2 S_3 + 4S_2^3 + 27 S_3^2) \\
& \times (2S_1^3 - 9 S_1 S_2 + 27 S_3) - G(S_1^2 - 3S_2)^3
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
E_2 = {} & 2(4S_1^3 S_3 - S_1^2 S_2^2 - 18 S_1 S_2 S_3 + 4S_2^3 + 27 S_3^2) f_2^3 \\
& + 2(U_1^2 - 3U_2)(S_1^2 - 3S_2) f_2 \\
& + (2S_1^3 - 9 S_1 S_2 + 27 S_3) G - (2U_1^3 - 9 U_1 U_2 + 27 U_3).
\end{aligned}
$$

It is easy to verify that there exists no single-valued meromorphic function f_2 satisfying $E_1 = 0$. In fact, in the case $\boldsymbol{R} = \boldsymbol{R}_A$-(i), we have

$$
27 A f_2^3 (4 y_0^3 e^{3H} + 27 A^2) + 27 G y_0^3 e^{3H} = 0,
$$

from (19). In this case the function f_2 mast have an algebraic branch point of order 2 at every zero of $4y_0^3 e^{3H} + 27A^2$, because that the function $G = Ke^M$ has no zero. This is a contradiction. And, in the case $\boldsymbol{R} = \boldsymbol{R}_A$-(ii), we have

$$
27 A f_2^3 (2 y_0 e^H - 9A^2)(4 y_0^3 e^{3H} - 729 A^6) - 27 G (y_0 e^H - 9A^2)^3 = 0,
$$

from (19). In this case the function f_2 mast have an algebraic branch point of order 2 at every zero of $4y_0^3 e^{3H} - 729 A^6$ and $2y_0 e^H - 9A^2$. This is also a contradiction.

In the following section we shall consider the equation $E_2 = 0$. And we shall prove that there exists no single-valued meromorphic function f_2 satisfying the equation $E_2 = 0$.

3. Proof of theorem

In this section we shall consider the following equation

$$
\begin{aligned}
E_2 = {} & 2(4S_1^3 S_3 - S_1^2 S_2^2 - 18 S_1 S_2 S_3 + 4S_2^3 + 27 S_3^2) f_2^3 \\
& + 2(U_1^2 - 3U_2)(S_1^2 - 3S_2) f_2 \\
& + (2S_1^3 - 9 S_1 S_2 + 27 S_3) G - (2U_1^3 - 9 U_1 U_2 + 27 U_3) = 0,
\end{aligned} \tag{20}
$$

where S_i $(i = 1, 2, 3)$ are entire functions, the pair of which defines either $\boldsymbol{R}_A$-(i) or $\boldsymbol{R}_A$-(ii), U_j $(j = 1, 2, 3)$ are entire functions, the pair of which defines one of the surfaces $\boldsymbol{X}$-(i), $\boldsymbol{X}$-(ii) and $\boldsymbol{X}$-(iii) and $G = Ke^M$. We shall prove that there exists no single-valued meromorphic function f_2 satisfying the equation (20).

Let us consider the case $\boldsymbol{R} = \boldsymbol{R}_A$-(i) and $\boldsymbol{X} = \boldsymbol{X}$-(i). Then we have

$$-B^2x_0^4e^{4L} + \frac{1}{16}(4d-3)^3B^6 = (4y_0^3e^{3H} + 27A^2)\cdot K^2e^{2M},$$

from (12) and their discriminants of $\boldsymbol{R}_A$-(i) and $\boldsymbol{X}$-(i). In this case there exist only two possibilities:

$$\text{(I)}\begin{cases} M \equiv 0, \\ 4L \equiv 3H, \\ -B^2x_0^4 = 4y_0^3K^2, \\ \dfrac{1}{16}(4d-3)^3B^6 = 27A^2K^2, \end{cases} \qquad \text{(II)}\begin{cases} 2M \equiv 4L \equiv -3H, \\ -B^2x_0^4 = 27A^2K^2, \\ \dfrac{1}{16}(4d-3)^3B^6 = 4y_0^3K^2. \end{cases}$$

First of all we consider the case (I). Let us put $J = L/3 = H/4$, $X = e^J$ and $w = f_2$. Then we have the following algebraic equation:

$$\begin{aligned} &2(4y_0^3X^{12} + 27A^2)w^3 - \frac{3}{2}y_0/\left(4x_0^2X^6 - 3(4d-3)B^2\right)X^4w \\ &\quad - 2x_0^3X^9 + 9dB^2x_0X^3 + 27AK = 0, \end{aligned} \tag{21}$$

from (20). Next we consider the case (II) and let us put $J = -L/3 = H/4 = -M/6$, $X = e^J$ and $w = f_2$. Then, from (20), we have

$$\begin{aligned} &2(4y_0^3X^{12} + 27A^2)w^3 + \frac{3}{2}y_0\frac{3(4d-3)B^2X^6 - 4x_0^2}{X^2}w \\ &\quad + \frac{9dB^2x_0X^6 + 27AKX^3 - 2x_0^3}{X^9} = 0, \end{aligned}$$

and

$$\begin{aligned} &2(4y_0^3X^{12} + 27A^2)X^9w^3 + \frac{3}{2}y_0\left(3(4d-3)B^2X^6 - 4x_0^2\right)X^7w \\ &\quad + 9dB^2x_0X^6 + 27AKX^3 - 2x_0^3 = 0. \end{aligned} \tag{22}$$

In the case $\boldsymbol{R} = \boldsymbol{R}_A$-(i) and $\boldsymbol{X} = \boldsymbol{X}$-(ii), by the similar way of above, we have the following two algebraic equations:

$$\begin{aligned} &2(4y_0^3X^{12} + 27A^2)w^3 \\ &\quad - 6y_0\left(x_0^2X^6 + (2d-3)Bx_0X^3 + (d-3)(d-6)B^2\right)X^4w \\ &\quad - 2x_0^3X^9 - 3(2d-3)Bx_0^2X^6 - 6(d-3)^2B^2x_0X^3 \\ &\quad - (d-6)^2(2d-3)B^3 + 27AK = 0, \end{aligned} \tag{23}$$

where $M \equiv 0$, $J = L/3 = H/4$, $X = e^J$ and $w = f_2$, and

$$\begin{aligned}
&2(4y_0^3X^{12} + 27A^2)X^9w^3 \\
&\quad - 6y_0\Big((d-3)(d-6)B^2X^6 + (2d-3)Bx_0X^3 + x_0^2\Big)X^7w \\
&\quad - (d-6)^2(2d-3)B^3X^9 - 6(d-3)^2B^2x_0X^6 \\
&\quad - 3\Big((2d-3)Bx_0^2 - 9AK\Big)X^3 - 2x_0^3 = 0,
\end{aligned} \tag{24}$$

where $J = -L/3 = H/4 = -M/6$, $X = e^J$ and $w = f_2$, from (20).

Similarly, in the case $\boldsymbol{R} = \boldsymbol{R}_A$-(i) and $\boldsymbol{X} = \boldsymbol{X}$-(iii), we have

$$\begin{aligned}
&2(4y_0^3X^{12} + 27A^2)w^3 \\
&\quad - 6y_0\Big(x_0^2X^6 + (2d-3)Bx_0X^3 + d(d+3)B^2\Big)X^4w \\
&\quad - 2x_0^3X^9 - 3(2d-3)Bx_0^2X^6 - 6d^2B^2x_0X^3 \\
&\quad - (d+3)^2(2d-3)B^3 + 27AK = 0,
\end{aligned} \tag{25}$$

where $M \equiv 0$, $J = L/3 = H/4$, $X = e^J$ and $w = f_2$, and

$$\begin{aligned}
&2(4y_0^3X^{12} + 27A^2)X^9w^3 \\
&\quad - 6y_0\Big(d(d+3)B^2X^6 + (2d-3)Bx_0X^3 + x_0^2\Big)X^7w \\
&\quad - (d+3)^2(2d-3)B^3X^9 - 6d^2B^2x_0X^6 \\
&\quad - 3\Big((2d-3)Bx_0^2 - 9AK\Big)X^3 - 2x_0^3 = 0,
\end{aligned} \tag{26}$$

where $J = -L/3 = H/4 = -M/6$, $X = e^J$ and $w = f_2$.

Similarly, in the case $\boldsymbol{R} = \boldsymbol{R}_A$-(ii) and $\boldsymbol{X} = \boldsymbol{X}$-(i), we have

$$\begin{aligned}
&2(4y_0^3X^{12} - 729A^6)w^3 - \frac{3}{2}(y_0X^4 - 9A^2)\Big(4x_0^2X^6 - 3(4d-3)B^2\Big)w \\
&\quad - 2x_0^3X^9 - 54AKy_0X^4 + 9dB^2x_0X^3 + 243A^3K = 0,
\end{aligned} \tag{27}$$

where $M \equiv 0$, $J = L/3 = H/4$, $X = e^J$ and $w = f_2$, and

$$\begin{aligned}
&2(4y_0^3X^{12} - 729A^6)X^9w^3 \\
&\quad + \frac{3}{2}(y_0X^4 - 9A^2)\Big(3(4d-3)B^2X^6 - 4x_0^2\Big)X^3w \\
&\quad - 54AKy_0X^7 + 9dB^2x_0X^6 + 243A^3KX^3 - 2x_0^3 = 0,
\end{aligned} \tag{28}$$

where $J = -L/3 = H/4 = -M/6$, $X = e^J$ and $w = f_2$.

Similarly, in the case $\boldsymbol{R} = \boldsymbol{R}_A$-(ii) and $\boldsymbol{X} = \boldsymbol{X}$-(ii), we have

$$
(29)\quad \begin{aligned}
&2(4y_0^3X^{12} - 729A^6)w^3 \\
&\quad - 6(y_0X^4 - 9A^2)\Big(x_0^2X^6 + (2d-3)Bx_0X^3 + (d-3)(d-6)B^2\Big)w \\
&\quad - 2x_0^3X^9 - 3(2d-3)Bx_0^2X^6 - 54AKy_0X^4 - 6(d-3)^2B^2x_0X^3 \\
&\quad - (d-6)^2(2d-3)B^3 + 243A^3K = 0,
\end{aligned}
$$

where $M \equiv 0$, $J = L/3 = H/4$, $X = e^J$ and $w = f_2$, and

$$
(30)\quad \begin{aligned}
&2(4y_0^3X^{12} - 729A^6)X^9w^3 \\
&\quad - 6(y_0X^4 - 9A^2)\Big((d-3)(d-6)B^2X^6 + (2d-3)Bx_0X^3 + x_0^2\Big)X^3w \\
&\quad - (d-6)^2(2d-3)B^3X^9 - 54AKy_0X^7 - 6(d-3)^2B^2x_0X^6 \\
&\quad - 3\Big((2d-3)Bx_0^2 - 81A^3K\Big)X^3 - 2x_0^3 = 0,
\end{aligned}
$$

where $J = -L/3 = H/4 = -M/6$, $X = e^J$ and $w = f_2$.

Similarly, in the case $\boldsymbol{R} = \boldsymbol{R}_A$-(ii) and $\boldsymbol{X} = \boldsymbol{X}$-(iii), we have

$$
(31)\quad \begin{aligned}
&2(4y_0^3X^{12} - 729A^6)w^3 \\
&\quad - 6(y_0X^4 - 9A^2)\Big(x_0^2X^6 + (2d-3)Bx_0X^3 + d(d+3)B^2\Big)w \\
&\quad - 2x_0^3X^9 - 3(2d-3)Bx_0^2X^6 - 54AKy_0X^4 - 6d^2B^2x_0X^3 \\
&\quad - (d+3)^2(2d-3)B^3 + 243A^3K = 0,
\end{aligned}
$$

where $M \equiv 0$, $J = L/3 = H/4$, $X = e^J$ and $w = f_2$, and

$$
(32)\quad \begin{aligned}
&2(4y_0^3X^{12} - 729A^6)X^9w^3 \\
&\quad - 6(y_0X^4 - 9A^2)\Big(d(d+3)B^2X^6 + (2d-3)Bx_0X^3 + x_0^2\Big)X^3w \\
&\quad - (d+3)^2(2d-3)B^3X^9 - 54AKy_0X^7 - 6d^2B^2x_0X^6 \\
&\quad - 3\Big((2d-3)Bx_0^2 - 81A^3K\Big)X^3 - 2x_0^3 = 0,
\end{aligned}
$$

where $J = -L/3 = H/4 = -M/6$, $X = e^J$ and $w = f_2$.

Now we need the following

LEMMA 1 (Picard [8]). *If the curve $\varphi(X, w) = 0$ is of genus $g > 1$, then there exists no pair of meromorphic functions $X(z)$ and $w(z)$ such that $\varphi(X(z), w(z)) \equiv 0$.*

Proof of Theorem. Let $\boldsymbol{R}$ be the surface $\boldsymbol{R}_A$-(i). And let us assume that $\boldsymbol{R}$ is of Picard constant 6. Then there exists an entire function $f = f_0 + f_1y + f_2y^2$ on $\boldsymbol{R}$, which does not take 5 finite values. Furthermore we assume that the function f defines the surface $\boldsymbol{X} = \boldsymbol{X}$-(i). In this case the single-valued meromorphic function $w = f_2$ satisfies either (21) or (22).

First of all we assume that (21) is not irreducible. Then there exists a single-valued meromorphic function $w = w_1(X)$ satisfying (21). It is easy to verify that there exists no common zero of $4y_0^3X^{12} + 27A^2$ and $4x_0^2X^6 - 3(4d-3)B^2$. We assume that there is a finite pole of $w = w_1(X)$, say X_0, which is of order p, then X_0 is a zero of $4y_0^3X^{12} + 27A^2$. By (21) we have $p = 1/2$, which is absurd. Hence $w = w_1(X)$ has no pole on $\boldsymbol{C}$. Next let us put $X = 1/t$, then we have

$$2(4y_0^3 + 27A^2t^{12})w^3 - \frac{3}{2}y_0\left(4x_0^2 - 3(4d-3)B^2t^6\right)t^2w$$
$$+ (-2x_0^3 + 9dB^2x_0t^6 + 27AKt^9)t^3 = 0,$$

from (21). Therefore $w = w_1(X)$ has a simple zero over $X = \infty$. Therefore we have $w = w_1(X) \equiv 0$ by Liouville's theorem. This is a contradiction. Hence the equation (21) is irreducible. So we can consider the 3-valued algebraic function defined by (21). The function $w = w(X)$ has 12 poles on $\{X \mid 4y_0^3X^{12} + 27A^2 = 0\}$, all of which are algebraic branch points of order 1. Therefore the compact Riemann surface, defined by $w = w(X)$, is of genus $g \geqq 4$. By lemma 1, there exist no pair of meromorphic functions $X = e^J$ and $w = f_2$ satisfying the equation (21). This is absurd.

Next let us consider (22). And let us assume that (22) is not irreducible. Then there exists a single-valued meromorphic function $w = w_2(X)$ satisfying (22). It is easy to verify that there exists no common zero of $4y_0^3X^{12} + 27A^2$ and $3(4d-3)B^2X^6 - 4x_0^2$. We assume that there is a finite non-zero pole of $w = w_2(X)$, say X_0, which is of order p, then X_0 is a zero of $4y_0^3X^{12} + 27A^2$. And by (22) we have $p = 1/2$, which is absurd. Hence $w = w_2(X)$ has only one pole at $X = 0$, which is of order 3. Putting $X = 1/t$, we have

$$2(4y_0^3 + 27A^2t^{12})w^3 + \frac{3}{2}y_0\left(3(4d-3)B^2 - 4x_0^2t^6\right)t^8w$$
$$+ (9dB^2x_0 + 27AKt^3 - 2x_0^3t^6)t^{15} = 0,$$

from (22). Therefore $w = w_2(X)$ has a zero of order at least 4 at $X = \infty$. This is a contradiction. Hence the equation (22) is irreducible. So we can consider the 3-valued algebraic function $w = w(X)$ defined by (22). The function $w = w(X)$ has 12 branch points of order 1 on $\{X \mid 4y_0^3X^{12} + 27A^2 = 0\}$, therefore the compact Riemann surface, defined by $w = w(X)$, is of genus $g \geqq 4$. By lemma 1, there exist no pair of meromorphic functions $X = e^J$ and $w = f_2$ satisfying the equation (22). This is absurd. Therefore there exists no entire function f on $\boldsymbol{R}_A$-(i), which defines the surface $\boldsymbol{X}$-(i).

By the similar way of above, we can verify that there exists no single-valued meromorphic function $w = f_2$ satisfying each of the equations (23), (24), (25) and (26). Therefore there exists no entire function f on $\boldsymbol{R}_A$-(i), which does not take 5 finite values. Hence $\boldsymbol{R}_A$-(i) is of Picard constant 5.

The similar way of above remains valid in the case of $\boldsymbol{R}_A$-(ii). Q.E.D

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