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THE GROWTH OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS*

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Abstract

In this paper, we investigate the growth of infinite order meromorphic solutions of second order differential equations with transcendental meromorphic coefficients. For most of meromorphic solutions, we obtain some precise estimates of their hyper-order.

1. Introduction and results

Throughout the presentation, we use the standard notations of the Nevanlinna theory (e.g. see [10, 13]). In addition, we use notations $\lambda(f)$ and $\lambda(1/f)$ to denote respectively the exponent of convergence of the zeros and the poles of meromorphic function $f(z)$, $\sigma(f)$ to denote the order of growth of $f(z)$ and $\mu(f)$ to denote the lower order of $f(z)$. In order to express the rate of growth of meromorphic function of infinite order, we recall the following definition (e.g. see [18]).

DEFINITION 1. Let f be a meromorphic function, then we define the hyper-order $\sigma_2(f)$ of $f(z)$

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

For the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

it is well known that let $A(z)$ or $B(z)$ be transcendental, if f_1 and f_2 are two linearly independent meromorphic solutions of (1.1) then by [7, Lemma 3], there is at least one of f_1, f_2 such that it's order is infinite. In [12], S. Hellerstein, J. Miles, and J. Rossi proved that if $A(z)$ and $B(z)$ are entire functions with $\sigma(B) < \sigma(A) \le 1/2$, then any nonconstant solution of (1.1) has infinite order.

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For the case where $\sigma(B) < \sigma(A) < 1/2$, the above assertion was proved in [9] by G. Gundersen. For higher order linear differential equations with entire coefficients of small growth, J. K. Langley investigated the zeros of infinite order solutions in [17]. Then for a great deal of meromorphic solutions with infinite order, more precise estimates for their rate of growth is a very important aspect. Ki-Ho Kwon investigated the problem and obtained the following result in [15]:

THEOREM A. *Let $A(z)$ and $B(z)$ be entire functions such that $\sigma(A) < \sigma(B)$ or $\sigma(B) < \sigma(A) < 1/2$. Then every solution $f(\not\equiv 0)$ of* (1.1) *satisfies $\sigma_2(f) \geq \max\{\sigma(A), \sigma(B)\}$.*

In [16], Ki-Ho Kwon investigated the hyper-order $\sigma_2(f)$ of solutions of (1.1) where $A = h_1(z)e^{P(z)}$, $B = h_2(z)e^{Q(z)}$, $P(z), Q(z)$ are polynomials satisfying $\deg P = \deg Q$, h_1, h_2 are entire functions satisfying $\sigma(h_j) < \deg P$. In [5] Zong-Xuan Chen and Chung-Chun Yang investigated the hyper-order of infinite order entire solutions of higher order linear differential equation with entire coefficients.

In this paper we will consider the hyper-order of meromorphic solutions of the general differential equations with meromorphic coefficients. It is easy to see that most of meromorphic solutions of (1.1) satisfy

$$\overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/f)}{\log r} \leq \max\{\sigma(A), \sigma(B)\}. \tag{1.2}$$

For the solution satisfying (1.2), we obtain the more precise result $\sigma_2(f) = \max\{\sigma(A), \sigma(B)\}$ than $\sigma_2(f) \geq \max\{\sigma(A), \sigma(B)\}$ in Theorem A.

THEOREM 1. *Let $A(z)$ and $B(z)$ be meromorphic functions such that*

$$\max\{\sigma(A), \lambda(1/B)\} < \sigma(B) < +\infty.$$

If the equation (1.1) *has meromorphic solutions, then every meromorphic solution $f(\not\equiv 0)$ satisfies $\sigma_2(f) \geq \sigma(B)$.*

Furthermore, if f satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/f)}{\log r} \leq \sigma(B), \tag{1.3}$$

then $\sigma_2(f) = \sigma(B)$.

THEOREM 2. *Let $A(z)$ and $B(z)(\not\equiv 0)$ be meromorphic functions such that*

$$\max\{\sigma(B), \lambda(1/A)\} < \mu(A) \leq \sigma(A) < 1/2.$$

If the equation (1.1) *has meromorphic solutions, then every meromorphic solution $f(\not\equiv 0)$ satisfies $\sigma_2(f) \geq \sigma(A)$.*

Furthermore, if f satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log\log N(r,1/f)}{\log r} \le \sigma(A), \tag{1.4}$$

then $\sigma_2(f) = \sigma(A)$.

2. Lemmas that are needed for the proof of Theorem 1

LEMMA 1 ([8, Theorem 4]). *Let f be a meromorphic function, and let $\alpha > 1$ and $\varepsilon > 0$ be given real constants. Then there exists a set $E_1 \subset (0,\infty)$ that has finite linear measure and there exists constant $B > 0$ that depend only on α, such that for all z satisfying $|z| = r \notin E_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B(T(\alpha r, f)r^{\varepsilon}\log T(\alpha r, f))^{j}, \quad (j = 1,2). \tag{2.1}$$

LEMMA 2 ([15]). *Let $g(z)$ be a nonconstant entire function of finite order. Then for any given $\varepsilon > 0$, there exists a set $H_2 \subset (0,\infty)$ with $\overline{dens}\, H_2 = 1$ such that*

$$M(r,g) \ge \exp\{r^{\sigma(g)-\varepsilon}\} \tag{2.2}$$

for all $r \in H_2$.

For a set $H \subset [0,\infty)$, the upper and the lower densities of the H are defined by

$$\overline{dens}\, H = \overline{\lim_{r\to\infty}} \frac{m(H\cap[0,r])}{r}$$

and

$$\underline{dens}\, H = \underline{\lim_{r\to\infty}} \frac{m(H\cap[0,r])}{r}$$

where $m(F)$ is the linear measure of a set F.

LEMMA 3. *Let $w(z)$ be a meromorphic function with $\lambda(1/w) < \sigma(w) < +\infty$. Then for any given $\varepsilon > 0$, there exists a set $H_3 \subset [0,+\infty)$ with $\overline{dens}\, H_3 = 1$ such that*

$$M(r,w) \ge \exp\{r^{\sigma(w)-\varepsilon}\} \tag{2.3}$$

holds for all $r \in H_3$.

Proof. Set $w(z) = z^k g(z)/d(z)$, where k is an integer, $g(z)$ is an entire function, $d(z)$ is the canonical product (or polynomial) formed with the nonzero poles of $w(z)$, hence $\lambda(d) = \sigma(d) = \lambda(1/w) < \sigma(w)$ and $\sigma(g) = \sigma(w)$. By Lemma 2, for any given $\varepsilon(0 < 2\varepsilon < \sigma(w) - \lambda(1/w))$, there exists a set $H_2 \subset [0,+\infty)$ with

$\overline{dens}\, H_2 = 1$ such that

$$M(r, g) \geq \exp\{r^{\sigma(w)-\varepsilon}\}, \quad (r \in H_2). \tag{2.4}$$

Now take a point z_r satisfying $|z_r| = r \in H_2$ and $|g(z_r)| = M(r, g)$. Since there is $R(>0)$ such that for $r > 0$, we have

$$|d(z_r)| \leq \exp\{r^{\lambda(1/w)+\varepsilon}\}. \tag{2.5}$$

Set an $H_3 = H_2 - [0, R]$, then $\overline{dens}\, H_3 = 1$. By (2.4) and (2.5),

$$M(r, w) \geq |z_r^k g(z_r)/d(z_r)| \geq \exp\{r^{\sigma(w)-\varepsilon}\}.$$

holds for $|z_r| = r \in H_3$.

Lemma 4. *Let $F(r)$, $G(r)$ are nondecreasing functions on $[0, \infty)$ such that $F(r) \leq G(r)$ for $r \notin E_4$, where the set $E_4 \subseteq [0, \infty)$ that has finite linear measure. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log F(r)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log G(r)}{\log r},$$

$$\overline{\lim_{r\to\infty}} \frac{\log\log F(r)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log\log G(r)}{\log r}.$$

Proof. Set the linear measure of the E_4, $mE_4 = \delta < +\infty$. For any given sequence $\{r_n\} \subset [0, +\infty)$ $(2\delta < r_1 < r_2 < \cdots, r_n \to \infty)$, there exists a point $r_n' \in [r_n, r_n + 2\delta] - E_4$. From

$$\frac{\log F(r_n)}{\log r_n} \leq \frac{\log F(r_n')}{\log(r_n' - 2\delta)} \leq \frac{\log G(r_n')}{\log r_n' + \log(1 - (2\delta/r_n'))},$$

we have

$$\begin{aligned}\overline{\lim_{n\to\infty}} \frac{\log F(r_n)}{\log r_n} &\leq \overline{\lim_{n\to\infty}} \frac{\log G(r_n')}{\log r_n' + \log(1 - (2\delta/r_n'))} \\ &= \overline{\lim_{n\to\infty}} \frac{\log G(r_n')}{\log r_n'} \leq \overline{\lim_{r\to\infty}} \frac{\log G(r)}{\log r}.\end{aligned}$$

Since the sequence $\{r_n\} \subset [0, +\infty)$ is arbitrary, we have

$$\overline{\lim_{r\to\infty}} \frac{\log F(r)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log G(r)}{\log r}.$$

Similarly, we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log F(r)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log\log G(r)}{\log r}.$$

Lemma 5 (see [6]). *Let $f(z)$ is a meromorphic function, $f(0) \neq \infty$. Then for $\tau > 1$ and $r > 0$, we have*

$$T(r, f) < C_\tau T(\tau r, f') + \log^+(\tau r) + 4 + \log^+|f(0)|,$$

where $C_\tau(>0)$ is a constant depending only on τ.

Lemma 6 ([14, Theorem 12.4]). *Let f be an entire function with $\sigma(f) = \infty$. Then f can be represented in the form $f(z) = g(z)e^{h(z)}$ where $g(z)$ and $h(z)$ are entire functions such that*

$$\sigma_2(f) = \max\{\sigma_2(g), \sigma_2(e^h)\}$$

$$\sigma_2(g) = \overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/g)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/f)}{\log r}.$$

Lemma 7. *Let $A(z)$, $B(z)$ be meromorphic functions of finite order. If $f(z)$ is a meromorphic solution of (1.1) and satisfies*

$$\overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/f)}{\log r} \le \max\{\sigma(A), \sigma(B)\} = \sigma, \tag{2.6}$$

then $\sigma_2(f) \le \sigma$.

Proof. Set $f(z) = w(z)e^{h(z)} = z^k(g(z)/d(z))e^{h(z)}$, where $w(z) = z^k(g(z)/d(z))$, k is an integer, $h(z)$ is an entire function, $g(z)$ and $d(z)$ are canonical products (or polynomial) formed respectively with the nonzero zeros and nonzero poles of $f(z)$. Since the poles of $f(z)$ can only occur at the poles of $A(z)$ and $B(z)$, hence $\sigma(d) = \lambda(1/f) \le \sigma$.

If $\lambda(f) = \sigma(N(r, 1/f)) < +\infty$, then $\sigma(w) < +\infty$ and

$$m\left(r, \frac{w^{(j)}}{w}\right) = O(\log r) \quad (j = 1, 2). \tag{2.7}$$

If $\lambda(f) = +\infty$, then $\sigma(w) = \lambda(f) = \sigma(g) = +\infty$. By (2.6) and Lemma 6 for any given $\varepsilon > 0$, we have

$$\log T(r, g) \le r^{\sigma+\varepsilon}, \quad \log T(r, d) \le (\sigma + \varepsilon)\log r$$

for sufficiently large r, hence

$$\log T(r, w) \le M\{r^{\sigma+\varepsilon}\}$$

and

$$m(r, w^{(j)}/w) = O(\log rT(r, w)) \le M\{r^{\sigma+\varepsilon}\} \quad (j = 1, 2) \tag{2.8}$$

for $|z| = r$ outside a set $E_7 \subset [0, +\infty)$ with finite linear measure, where $M(>0)$ is some constant.

Substituting $f = we^h$ into (1.1), we obtain

$$-(h')^2 = \left(\frac{w''}{w} + 2\frac{w'}{w}h' + h''\right) + A\left(\frac{w'}{w} + h'\right) + B. \tag{2.9}$$

Using the similar reasoning as in the proof of Tumura-Clunie's Theorems [10, pp. 67–69], we can get

$$m(r,h') \leq M_1\left\{m\left(r,\frac{w'}{w}\right) + m\left(r,\frac{w''}{w}\right) + m(r,A) + m(r,B) + S(r,h')\right\} \tag{2.10}$$

for $|z| = r$ outside a set $E_7' \subset [0,+\infty)$ with finite linear measure, where M_1 is some positive constant.

By (2.7) (or (2.8)) and (2.10), we know that for $r \notin E_7 \cup E_7'$ $(r \to +\infty)$,

$$m(r,h') \leq M_2(2\{r^{\sigma+\varepsilon}\} + 2r^{\sigma+\varepsilon} + S(r,h')), \tag{2.11}$$

where M_2 is some positive constant. By Lemma 5, we have

$$T(r,h) = m(r,h) \leq M_3(m(2r,h') + \log 2r). \tag{2.12}$$

Since Lemma 4 and $S(r,h') = o\{m(r,h')\}$ as $r \to +\infty$ possibly outside a set of r of finite linear measure, we get $\sigma_2(f) \leq \sigma$ from (2.11) and (2.12).

LEMMA 8 ([4]). *Suppose that $w(z)$ is a meromorphic function with $\sigma(w) = \beta < \infty$. Then for any given $\varepsilon > 0$, there is a set $E_8 \subset (1,+\infty)$ that has finite linear measure and finite logarithmic measure, such that*

$$|w(z)| \leq \exp\{r^{\beta+\varepsilon}\}$$

holds for $|z| = r \notin [0,1] \cup E_8, r \to \infty$.

3. Proof of Theorem 1

Suppose that α and β are real numbers satisfying

$$\max\{\sigma(A), \lambda(1/B)\} < \alpha < \beta < \sigma(B). \tag{3.1}$$

If $f(z)$ is a meromorphic solution of (1.1), then by (1.1) it is easy to see that $\sigma(f) = \infty$. By (1.1), we have

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + |A(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.2}$$

By Lemma 1, there is a set $E_1 \subset [0,+\infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq r[T(2r,f)]^3, \quad (j = 1,2). \tag{3.3}$$

By Lemma 8, there is a set $E_8 \subset (1, +\infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_8$, we have

$$|A(z)| \leq \exp\{r^\alpha\}. \tag{3.4}$$

On the other hand by Lemma 3, we know that there is a set $H_3 \subset [0, +\infty)$ with $\overline{dens}\, H_3 = 1$ such that

$$M(r, B) \geq \exp\{r^\beta\} \tag{3.5}$$

holds for all $r \in H_3$. Now set an $H = H_3 - ([0,1] \cup E_1 \cup E_5)$, then $\overline{dens}\, H = 1$. For all $r \in H$, we have by (3.1)–(3.5)

$$\begin{aligned} &\exp\{r^\beta\} \leq 2r \exp\{r^\alpha\}[T(2r, f)]^3, \\ &\exp\{(1 + o(1))r^\beta\} \leq [T(2r, f)]^3 \end{aligned} \tag{3.6}$$

as $r \to +\infty$. Therefore by (3.6), $\sigma_2(f) \geq \beta$ holds. Since β is arbitrary, we get

$$\sigma_2(f) \geq \sigma(B). \tag{3.7}$$

Furthermore, if $f(z)$ satisfies (1.3), then by Lemma 7 and (3.7), we get $\sigma_2(f) = \sigma(B)$.

4. Lemmas that are needed for the proof of Theorem 2

LEMMA 9 ([8]). *Let $w(z)$ be a transcendental meromorphic function with $\sigma(w) = \sigma < +\infty$. Let $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geq 0$ for $i = 1, \ldots, m$, and let $\varepsilon > 0$ be a given constant. Then there exists a subset $E_9 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_9$, and $(k, j) \in \Gamma$, we have*

$$|w^{(k)}(z)/w^{(j)}(z)| \leq r^{(k-j)(\sigma-1+\varepsilon)}.$$

Also there exists a subset $E_9' \subset [0, +\infty)$ with finite linear measure such that for all z satisfying $|z| = r \notin E_9'$, and $(k, j) \in \Gamma$, we have

$$|w^{(k)}(z)/w^{(j)}(z)| \leq r^{(k-j)(\sigma+\varepsilon)}.$$

LEMMA 10. *Let $f(z)$ be a meromorphic function with $\lambda(1/f) < \sigma(f) = \infty$. Then there exists a set $H_{10} \subset [0, \infty)$ with $\overline{dens}\, H_{10} = 1$ such that for $r \in H_{10}$, there is a point z_r satisfying $|z_r| = r$ and*

$$|f(z_r)/f'(z_r)| \leq r. \tag{4.1}$$

Proof. Set $\lambda(1/f) = \alpha < \infty$. We first suppose that $\lambda(f) = \infty$, then

$$\overline{\lim_{r\to\infty}} \frac{\log n(r, 1/f)}{\log r} = \infty. \tag{4.2}$$

There exists $\{r_n\}(r_n \to \infty)$ satisfying $r_n \geq n$ and

$$n(r_n, 1/f) \geq r_n^{2(\alpha+1)} \geq (nr_n)^{\alpha+1}. \tag{4.3}$$

Set a set $H'_{10} = \bigcup_{n=1}^{\infty}[r_n, nr_n]$. Now we take $\{R_n\}$ such that $nr_n/2 \leq R_n \leq nr_n$, then

$$\begin{aligned} \overline{dens}\, H'_{10} &= \overline{\lim_{r\to\infty}} \frac{m(H'_{10} \cap [0, r])}{r} \\ &\geq \overline{\lim_{n\to\infty}} \frac{m\{[r_n, nr_n] \cap [0, R_n]\}}{R_n} = \overline{\lim_{n\to\infty}}\left(1 - \frac{2}{n}\right) = 1. \end{aligned} \tag{4.4}$$

For $r \in H'_{10}$, there is an n such that $r \in [r_n, nr_n]$. By (4.3) we get

$$n\left(r, \frac{1}{f}\right) \geq n\left(r_n, \frac{1}{f}\right) \geq r^{\alpha+1}. \tag{4.5}$$

On the other hand, for any $\varepsilon(0 < 3\varepsilon < 1)$, there is $R(>0)$ such that for $r > R$,

$$r^{\varepsilon-1} < \frac{1}{2}, n(r, f) \leq r^{\alpha+\varepsilon} \tag{4.6}$$

hold. Set a set $H_{10} = H'_{10} \cap [R, +\infty)$, then $\overline{dens}\, H_{10} = 1$. For $r \in H_{10}$ by the residue theorem and (4.5), (4.6), we have

$$\frac{1}{2\pi}\int_{|z|=r} \frac{f'(z)}{f(z)}\, dz = n\left(r, \frac{1}{f}\right) - n(r, f) \geq r^{\alpha+1} - r^{\alpha+\varepsilon} > \frac{1}{2}r^{\alpha+1}. \tag{4.7}$$

So, from (4.7) it is easy to see that there is a point z_r with $|z_r| = r$ such that $|f'(z_r)/f(z_r)| > 1/r$. Hence (4.1) holds.

Now we suppose that $\lambda(f) < +\infty$. Then by $\lambda(1/f) < +\infty$, f can be expressed in the form $f = we^h$ such that $\sigma(w) = \beta < \infty$ and h is a transcendental entire function. By Lemma 9, we know that there is a set $E'_9 \subset [0, +\infty)$ with linear measure $mE'_9 < \infty$ such that for $|z| = r \notin E'_9$, we have

$$\left|\frac{w'(z)}{w(z)}\right| < r^{\beta+1}. \tag{4.8}$$

Since h' is a transcendental entire function, there is $R(>0)$ such that for $r > R$, we can take a point z_r satisfying $|z_r| = r \notin [0, R] \cup E'_9$ and

$$|h'(z_r)| = M(r, h') > r^{\beta+2}. \tag{4.9}$$

By (4.8), (4.9) and

$$\left|\frac{f'(z)}{f(z)}\right| \geq |h'(z)| - \left|\frac{w'(z)}{w(z)}\right|,$$

we get

$$\left|\frac{f'(z_r)}{f(z_r)}\right| \geq \frac{1}{r}$$

for $r \in [R, +\infty) - E_9'$. Set a set $H_{10} = [R, +\infty) - E_9'$, then it is easy to see that $\overline{dens}\, H_{10} = 1$. Hence (4.1) holds for $r \in H_{10}$.

LEMMA 11 ([3]). *Let $g(z)$ be an entire function of order σ where $0 < \sigma < 1/2$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H_{11} \subset [0, +\infty)$ with $\overline{dens}\, H_{11} \geq 1 - 2\sigma$ such that for all z satisfying $|z| = r \in H_{11}$, we have*

$$|g(z)| \geq \exp\{r^{\sigma-\varepsilon}\}. \tag{4.10}$$

LEMMA 12. *Let $f(z)$ be a meromorphic function with $\lambda(1/f) < \sigma(f) = \sigma < 1/2$. Then for $\varepsilon > 0$, there exists a set $H_{12} \subset [0, \infty)$ with $\overline{dens}\, H_{12} > 0$ such that for all z satisfying $|z| = r \in H_{12}$, we have*

$$|f(z)| \geq \exp\{(1 + o(1))r^{\sigma-\varepsilon}\}. \tag{4.11}$$

Proof. Set $f(z) = z^k g(z)/d(z)$, where k is an integer, $d(z)$ is a canonical product (or polynomial) formed with the nonzero poles of $f(z), \sigma(d) = \lambda(1/f) < \sigma(f), g(z)$ is an entire function with $\sigma(g) = \sigma < 1/2$. By Lemma 11, for a given $\varepsilon(0 < 3\varepsilon < \sigma - \lambda(1/f))$, there exists a set $H_{11} \subset [0, +\infty)$ with $\overline{dens}\, H_{11} > 0$, such that for all z satisfying $|z| = r \in H_{11}$, (4.10) holds. And there is $R(>0)$ such that

$$|d(z)| \leq \exp\{r^{\lambda(1/f)+\varepsilon}\} \tag{4.12}$$

holds for $r > R$ Set an $H_{12} = H_{11} \cap [R, +\infty)$, then $\overline{dens}\, H_{12} > 0$. For all z satisfying $|z| = r \in H_{12}$, we get by (4.10) and (4.12)

$$|f(z)| \geq \exp\{r^{\sigma-\varepsilon} - r^{\lambda(1/f)+\varepsilon}\} = \exp\{(1 + o(1))r^{\sigma-\varepsilon}\}.$$

LEMMA 13 ([1]). *Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < 1/2$ and denote $A(r) = \inf_{|z|=r} \log|f(z)|$, $B(r) = \sup_{|z|=r} \log|f(z)|$. If $\sigma < \alpha < 1$, then*

$$\underline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r)\} \geq 1 - \frac{\sigma}{\alpha}$$

where

$$\underline{\log dens}(H) = \underline{\lim}_{r\to\infty} \left(\int_1^r (\chi_H(t)/t)\, dt\right) \Big/ \log r$$

and

$$\overline{\log dens}(H) = \overline{\lim}_{r\to\infty} \left(\int_1^r (\chi_H(t)/t)\, dt\right) \Big/ \log r$$

where $\chi_H(t)$ is the characteristic function of a set H.

LEMMA 14 ([2]). *Let $f(z)$ be entire with $\mu(f) = \mu < 1/2$ and $\mu < \sigma = \sigma(f)$. If $\mu \le \delta < \min(\sigma, 1/2)$ and $\delta < \alpha < 1/2$, then*

$$\overline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r) > r^{\delta}\} > C(\sigma, \delta, \alpha),$$

where $C(\sigma, \delta, \alpha)$ is a positive constant depending only on σ, δ and α.

LEMMA 15. *Let $A(z), B(z)$ be meromorphic functions such that $B \not\equiv 0$ and*

$$\max\{\sigma(B), \lambda(1/A)\} < \mu(A) \le \sigma(A) < \frac{1}{2}.$$

If $f(\not\equiv 0)$ is a meromorphic solution of the equation (1.1), then $\sigma(f) = \infty$.

Proof. Assume that $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.1), then f is transcendental by $B \not\equiv 0$. Now suppose that $\sigma(f) < \infty$. We set $f(z) = g(z)/\ d(z)$, where $g(z)$ is an entire function, $d(z)$ is the canonical product (or polynomial) formed with the nonzero poles of $f(z)$. By the fact that the poles of $f(z)$ can only occur at the poles of A, B, it follows that $\sigma(d) \le \max\{\sigma(B), \lambda(1/A)\} = \beta < \mu(A)$. Since

$$T(r, f') \le 2T(r, f) + O(\log r),$$

$$T(r, f'') \le 2T(r, f') + m\left(r, \frac{f''}{f'}\right) \le 4T(r, f) + O(\log r)$$

and

$$-A = \frac{f''}{f'} + B\frac{f}{f'},$$

we can get

$$T(r, A) \le cT(r, f) + T(r, B) + O(\log r), \tag{4.13}$$

where c is a constant. By (4.13) and $\sigma(B) \le \beta < \mu(A)$, we conclude $\mu(f) \ge \mu(A)$. By $f = g/d$ we have $T(r, f) \le T(r, g) + T(r, d) + O(1)$. For any given $\varepsilon(0 < 2\varepsilon < \mu(f) - \sigma(d))$, $T(r, f) \ge r^{\mu(f)-\varepsilon}$ and $T(r, d) \le r^{\sigma(d)+\varepsilon}$ hold for sufficiently large r, hence $(1 - o(1))r^{\mu(f)-\varepsilon} \le T(r, g) + O(1)$, so $\mu(f) \le \mu(g)$. If $\mu(f) < \mu(g)$, then by $g = fd$, we have

$$r^{\mu(g)-\varepsilon_1} \le T(r, f) + T(r, d) \le T(r, f) + r^{\sigma(d)+\varepsilon_1},$$

$$(0 < 2\varepsilon_1 < \mu(g) - \mu(f) < \mu(g) - \sigma(d))$$

and

$$(1 - o(1))r^{\mu(g)-\varepsilon_1} \le T(r, f)$$

for sufficiently large r, hence $\mu(f) \ge \mu(g)$. This is a contradiction. Therefore, $\mu(f) = \mu(g) > \beta$. Similarly, we can get $\sigma(f) = \sigma(g)$.

From the Wiman-Valiron theory (see [11, 13]), there exists a subset $E \subset (1, \infty)$ with finite logarithmic measure $lm(E) < \infty$, such that for a point z satisfying $|z| = r \notin E$ and $|g(z)| = M(r, g)$, we have

$$\frac{g'(z)}{g(z)} = \left(\frac{\nu_g(r)}{z}\right)(1 + o(1)), \tag{4.14}$$

where $\nu_g(r)$ denotes the central-index of $g(z)$. By Lemma 9 we know that for any $\varepsilon_2 (0 < 2\varepsilon_2 < \mu(A) - \beta)$, there exists a subset $E_9 \subset (1, +\infty)$ with $lm E_9 < \infty$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_9$, we have

$$\left|\frac{d'(z)}{d(z)}\right| \leq r^{\beta - 1 + \varepsilon_2}. \tag{4.15}$$

By $\mu(g) = \mu(f) \geq \mu(A)$, there is $R (> 1)$ such that

$$\nu_g(r) \geq r^{\mu(A) - \varepsilon_2} \tag{4.16}$$

holds for $r > R$. Hence for a point z on which $|g(z)| = M(r, g)$, and $|z| = r \notin [0, R] \cup E \cup E_9$, we have by (4.14)

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} - \frac{d'(z)}{d(z)} = \frac{\nu_g(r)}{z}\left[(1 + o(1)) - \left(\frac{\nu_g(r)}{z}\right)^{-1} \frac{d'(z)}{d(z)}\right]. \tag{4.17}$$

By (4.15)–(4.17), we get

$$\frac{f'(z)}{f(z)} = \frac{\nu_g(r)}{z}(1 + o(1)). \tag{4.18}$$

By (1.1) and (4.18), we get

$$-A\frac{\nu_g(r)}{z}(1 + o(1)) = \frac{f''(z)}{f(z)} + B(z). \tag{4.19}$$

Set $A(z) = a(z)/h(z)$, where $a(z)$ is transcendental and $h(z)$ is the canonical product (or polynomial) formed with nonzero poles of $A(z)$. Then $\sigma(h) = \lambda(h) = \lambda(1/A) \leq \beta$, $\mu(a) = \mu(A) \leq \sigma(A) = \sigma(a) < 1/2$. We can choose α, δ such that

$$\beta < \alpha < \delta < \mu(A). \tag{4.20}$$

By Lemma 13 (if $\mu(a) = \sigma(a)$) or Lemma 14 (if $\mu(a) < \sigma(a)$), there exists a subset $H \subset (1, \infty)$ with logarithmic measure $lm H = \infty$ such that for all z satisfying $|z| = r \in H$, we have

$$|a(z)| \geq \exp\{r^{\delta}\}. \tag{4.21}$$

By Lemma 8, there exists a set $E_8 \subset (1, \infty)$ with logarithmic measure $lm E_8 < \infty$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, $r \to \infty$, the following estimations hold:

$$|B(z)| \leq \exp\{r^{\alpha}\}, \quad |h(z)| \leq \exp\{r^{\alpha}\}. \tag{4.22}$$

By Lemma 9, there exists a subset $E_9 \subset (1, \infty)$ with $lm\, E_9 < \infty$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_9$, we have

$$\left| \frac{f''(z)}{f(z)} \right| \le r^M, \tag{4.23}$$

where M is a constant. It follows from (4.19)–(4.23) that as $r \in H - ([0, R] \cup E \cup E_8 \cup E_9)$ $(lm([0, R] \cup E \cup E_8 \cup E_9)) = \infty)$, $r \to \infty$,

$$v_g(r) \to 0 \tag{4.24}$$

holds. But (2.24) contradicts (4.16). Therefore $\sigma(g) = \infty$, i.e. $\sigma(f) = \infty$.

5. Proof of Theorem 2

Assume that $f(z)(\not\equiv 0)$ is a meromorphic solution of (1.1), then $\sigma(f) = \infty$ by Lemma 15. We can rewrite (1.1) as

$$-A(z) = f''(z)/f'(z) + B(z)f(z)/f'(z). \tag{5.1}$$

By Lemma 1, there is a set $E_1 \subset [0, \infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_1$, we have

$$|f''(z)/f'(z)| \le r[T(2r, f')]^2. \tag{5.2}$$

We can choose α, ρ such that

$$\max\{\sigma(B), \lambda(1/A)\} = \beta < \alpha < \rho < \sigma(A). \tag{5.3}$$

By Lemma 8, there is a set $E_8 \subset [0, +\infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_8$,

$$|B(z)| \le \exp\{r^\alpha\} \tag{5.4}$$

holds. By Lemma 12, there is a set $H_{12} \subset [0, +\infty)$ with $\overline{dens}\, H_{12} > 0$ such that for all z satisfying $|z| = r \in H_{12}$, we have

$$|A(z)| \ge \exp\{(1 + o(1))r^\rho\}. \tag{5.5}$$

Hence by (5.1)–(5.5), for all z satisfying $|z| = r \in H_{12} - ([0, 1] \cup E_1 \cup E_8)$ $(\overline{dens}\{H_{12} - ([0, 1] \cup E_1 \cup E_8)\} > 0)$, we have

$$\exp\{(1 + o(1))r^\rho\} \le r[T(2r, f')]^2 + \exp\{r^\alpha\} \left| \frac{f(z)}{f'(z)} \right|. \tag{5.6}$$

By Lemma 10, there exists a set $H_{10} \subset [0, +\infty)$ with $\overline{dens}\, H_{10} = 1$ such that for $r \in H_{10}$, there is a point z_r satisfying $|z_r| = r$ and

$$|f(z_r)/f'(z_r)| \le r. \tag{5.7}$$

By (5.6) and (5.7), we know that for $r \in H_{10} \cap (H_{12} - ([0, 1] \cup E_1 \cup E_8))$

$(\overline{dens}(H_{10} \cap (H_{12} - ([0,1] \cup E_1 \cup E_8))) > 0)$, we have

$$\exp\{(1+o(1))r^{\rho}\} \leq r[T(2r,f')]^2 + \exp\{r^{\alpha}\}r \leq 2r\exp\{r^{\alpha}\}[T(2r,f')]^2. \tag{5.8}$$

Hence, $\sigma_2(f') \geq \rho$. Since ρ is arbitrary and

$$T(r,f') \leq 3T(r,f) + O(\log rT(r,f))$$

holds possibly outside a set of r of finite linear measure, combining Lemma 4 we get $\sigma_2(f) \geq \sigma(A)$.

Furthermore, if f satisfies (1.4), then by Lemma 7, we have $\sigma_2(f) \leq \sigma(A)$. Hence $\sigma_2(f) = \sigma(A)$.

6. Examples

EXAMPLE 1. The equation

$$f'' + \left(2\cot z - 2z - \frac{1}{z}\right)f' - \left(4z^2e^{2z^2} + \left(2z + \frac{1}{z}\right)\cot z + 1\right)f = 0$$

have a solution $f = (1/\sin z)e^{e^{z^2}}$, where $\sigma(B) = 2$, $\sigma(A) = 1$, $\lambda(1/A) = \lambda(1/B) = 1$, and $\sigma_2(f) = 2$.

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