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BRAID MONODROMY OF COMPLEX LINE ARRANGEMENTS

Dedicated to the memory of Professor N. Sasakura

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Abstract

Let V be the complex vector space C^l, $\mathscr{A}$ an arrangement in V, i.e. a finite family of hyperplanes in V In [11], Moishezon associated to any algebraic plane curve $\mathscr{C}$ of degree n a braid monodromy homomorphism θ $F_s \to B(n)$, where F_s is a free group, $B(n)$ is the Artin braid group. In this paper, we will determine the braid monodromy for the case when $\mathscr{C}$ is an arrangement $\mathscr{A}$ of complex lines in C^2, using the notion of labyrinth of an arrangement. As a corollary we get the braid monodromy presentation for the fundamental group of the complement to the arrangement.

1. Introduction

Let $\mathscr{C} = \{f(x,y) = 0\} \in C^2$ be a plane algebraic curve. From the 1930's, it is well known (see [9], [17]) that the fundamental group of the complement to $\mathscr{C}$, $\pi_1(C^2 \backslash \mathscr{C})$, can be computed using the van Kampen's method. In [11], Moishezon introduced the notion of braid monodromy of $\mathscr{C}$. Suppose that the projection on the x-axis, $pr_1 : C^2 \to C^1$, is generic with respect to the curve $\mathscr{C}$. Let $S(\mathscr{C}) = \{\alpha \in \mathscr{C}; \partial f(\alpha)/\partial y = 0\}$ and $D(\mathscr{C})$ its image under pr_1. Then the braid monodromy of $\mathscr{C}$ is a homeomorphism $\theta : \pi_1(C^1 \backslash D(\mathscr{C})) \to B[pr_1^{-1}(x_0), pr_1^{-1}(x_0) \cap \mathscr{C}]$, where $x_0 \in C^1 \backslash D(\mathscr{C})$ is a base point.

An arrangement $\mathscr{A}$ is a finite family of hyperplanes in C^l. Given an arrangement $\mathscr{A}$, an algorithm to compute the fundamental group of the complement, $\pi_1(C^l \backslash \bigcup_{H \in \mathscr{A}} H)$, was proved in [14] when $\mathscr{A}$ is the complexification of a real arrangement. Similar results were obtained in [5] and [16] by different methods. For an arbitrary complex arrangement a standard argument using the Zariski hyperplane section theorem (see e.g. [7]) reduces the problem to the case when $\mathscr{A}$ is an arrangement of complex lines in C^2. Arvola [1] found an

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algorithm to compute the fundamental group of its complement, using an admissible 2-graph, defined by himself. In [6] we suggest another method to compute this fundamental group. Our method based on a construction called labyrinth, which dues to Rudolph [15]. We found that the labyrinth is still useful to study the braid monodromy of an arrangement $\mathscr{A}$ of complex lines in $\boldsymbol{C}^2$. In this paper we will show how the braid monodromy of the arrangement $\mathscr{A}$ can be obtained from its labyrinth. Note that the braid monodromy gives also a presentation for the fundamental group of the complement of $\mathscr{A}$. Combining this with a result of Libgober [10], we prove in the corollary 4.6 that the labyrinth of an arrangement $\mathscr{A}$ in $\boldsymbol{C}^2$ determines the homotopy type of its complement.

The braid monodromy of a complexified real arrangement was determined by Salvetti [16], Hironaka [8] and Cordovil and Fadacha [4]. Generalizing the notion of admissible graph and the algorithm of Arvola, recently, Cohen and Suciu [3] suggest an algorithm to determine the braid monodromy of an arrangement of complex lines using braided wiring diagram. However, the method we present here is quite different. Moreover, our method gives a concrete algorithm to determine precisely any braid occurring in the conjugation of braid monodromy generators (see the Remark 4.3).

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2. Braid monodromy

In this section we recall briefly the notion of the braid monodromy of a plane curve after B. Moishezon [11].

Let $\mathscr{C} = \{f(x, y) = 0\} \in \boldsymbol{C}^2$ be a plane curve. Suppose that the projection $pr_1 : \boldsymbol{C}^2 \to \boldsymbol{C}^1$ onto the x-axis is generic with respect to the curve $\mathscr{C}$. Denote by $S(\mathscr{C})$ the set $\{\alpha \in \mathscr{C};\ \partial f(\alpha)/\partial y = 0\}$ and $D(\mathscr{C})$ the image of $S(\mathscr{C})$ under the projection pr_1. For a point $\tilde{x}$ of the x-plane $\boldsymbol{C}^1$ let $\boldsymbol{C}_{\tilde{x}}$ denote the fiber of the projection pr_1 over the point $\tilde{x}$, $\boldsymbol{C}_{\tilde{x}} = \{(x, y) \in \boldsymbol{C}^2; x = \tilde{x}\}$. For a path $\gamma : \boldsymbol{I} \to \boldsymbol{C}^1 \backslash D(\mathscr{C})$, we see easily that the pull-backs $\gamma^*(pr_1)$ and $\gamma^*(pr_1|_{\mathscr{C}})$ are trivial bundles. We have then the homeomorphisms

$$(pr_1^{-1}(\gamma(0)),\ pr_1^{-1}(\gamma(0)) \cap \mathscr{C}) \to (pr_1^{-1}(\gamma(t)),\ pr_1^{-1}(\gamma(t)) \cap \mathscr{C}),$$

$t \in [0, 1]$, induced naturally by a given trivialization of $(\gamma^*(pr_1), \gamma^*(pr_1|_{\mathscr{C}}))$. We call this homeomorphism the braid homeomorphism defined over the path γ, or simply the braid defined over γ. Let fix a base point x_0 of the x-axis, $x_0 \in \boldsymbol{C}^1 \backslash D(\mathscr{C})$. When γ is a loop beginning and ending at x_0, we obtain a homeomorphism

$$(\boldsymbol{C}_{x_0}, \boldsymbol{C}_{x_0} \cap \mathscr{C}) \to (\boldsymbol{C}_{x_0}, \boldsymbol{C}_{x_0} \cap \mathscr{C}).$$

This defines a homomorphism

$$\theta : \pi_1(\mathbf{C}^1 \backslash D(\mathscr{C}); x_0) \to B[\mathbf{C}_{x_0}, \mathbf{C}_{x_0} \cap \mathscr{C}],$$

which is called the braid monodromy of the curve $\mathscr{C}$. Here by $B[P, K]$ we mean the group of isotopy classes of compact support homeomorphisms of a 2-plane P which preserves a fixed finite subset $K \subset P$.

The determination of the braid monodromy is usually carried out in two steps. First, for a point $x_k \in D(\mathscr{C})$ we denote by $D^\varepsilon_{x_k}$ a small disk of radius ε, centered at x_k. Let fix a point x^ε_k on the boundary $\partial D^\varepsilon_{x_k}$ of this disk and $\mathbf{C}_{x^\varepsilon_k}$ the fiber over this point x^ε_k. By moving this fiber $\mathbf{C}_{x^\varepsilon_k}$ counterclockwise along the boundary of the disk $D^\varepsilon_{x_k}$ we obtain a homeomorphism of $\mathbf{C}_{x^\varepsilon_k}$ into itself, preserving $\mathbf{C}_{x^\varepsilon_k} \cap \mathscr{C}$. I gives rise an element of the braid group $B[\mathbf{C}_{x^\varepsilon_k}, \mathbf{C}_{x^\varepsilon_k} \cap \mathscr{C}]$ and will be called the local braid monodromy of $\mathscr{C}$ at x_k.

Next, suppose that $D(\mathscr{C}) = \{x_1, \dots, x_N\}$. Let $\Gamma_1, \dots, \Gamma_N$ be a system of simple paths in $\mathbf{C}^1 \backslash D(\mathscr{C})$ satisfying

1) $\Gamma_i \cap \Gamma_j = x_0$, $1 \le i < j \le N$.

2) Each Γ_i connects x_0 with x^ε_i and $\Gamma_i \cap D(\mathscr{C}) = \emptyset$.

Denote by γ_i the element of $\pi_1(\mathbf{C}^1 \backslash D(\mathscr{C}))$, represented by $\Gamma_i \,.\, \partial D^\varepsilon_{x_i} \,.\, \Gamma_i^{-1}$. The set of all those γ_i's is called a good ordered system of generators of $\pi_1(\mathbf{C}^1 \backslash D(\mathscr{C}))$. To find the braid monodromy θ it suffices to find all $\theta(\gamma_i)$, $1 \le i \le N$. Let $\theta(\Gamma_i)$ be the braid homeomorphism defined over the path Γ_i. Then it is clear that $\theta(\gamma_i)$ is completely determined by the local braid monodromy at x_i and the braid $\theta(\Gamma_i)$.

In this paper we will deal with the case when $\mathscr{C}$ is an arrangement $\mathscr{A}$ of n complex lines in $\mathbf{C}^2$ defined by a polynomial of the form $\prod_{i=1}^n (y - \alpha_i(x))$.

3. Labyrinth

Let $\mathscr{A}$ be an arrangement of n complex lines in $\mathbf{C}^2$. Suppose that each line $H_i \in \mathscr{A}$ is defined by an equation $y = \alpha_i(x)$, where α_i is a linear function $\alpha_i : \mathbf{C} \to \mathbf{C}$. Let $R_i(x) = \mathrm{Re}(\alpha_i(x))$ and $I_i(x) = \mathrm{Im}(\alpha_i(x))$. For any $1 \le i < j \le n$, the subset $L_{i,j}$ of the x-axis $\mathbf{C}^1$, defined by

$$L_{i,j} = \{x \in \mathbf{C}^1;\ R_i(x) = R_j(x)\},$$

is a (*real*) line in $\mathbf{C}^1$.

Definition 3.1. We call the set

$$\mathscr{L}(\mathscr{A}) = \{L_{i,j};\ 1 \le i < j \le n\}$$

the labyrinth of the arrangement $\mathscr{A}$.

Remark 3.2. (i) The notion of labyrinth was introduced by L. Rudolph [15] for any plane curve. Here we consider it in the arrangement context and call it by the name *labyrinth*. This notion was also used by Y. Orevkov in [12], where he called it the *Rudolph's graph*.

(ii) For each line $L \in \mathscr{L}(\mathscr{A})$, there might be $i_1, \ldots, i_k$ with $1 \le i_1 < \cdots < i_k \le n$ such that

$$L = \{x \in \boldsymbol{C}^1;\ R_{i_s}(x) = R_{i_t}(x),\ 1 \le s < t \le k\}.$$

The number k will be called the multiplicity of L. It is easy to see that after a suitable change of coordinates we can always assume that the multiplicity of any line L in $\mathscr{L}(\mathscr{A})$ equals to 2.

Clearly, each line $L_{i,j}$ divides the x-plane $\boldsymbol{C}^1$ into two parts $L_{i,j}^+ = \{x \in \boldsymbol{C}^1; R_i(x) < R_j(x)\}$ and $L_{i,j}^- = \{x \in \boldsymbol{C}^1; R_i(x) > R_j(x)\}$. Each component of $\boldsymbol{C}^1 \backslash \mathscr{L}(\mathscr{A})$ can be then defined by $R_{s(1)} < \cdots < R_{s(n)}$ for a certain permutation s of the set $\{1, 2, \ldots, n\}$.

Similarly, for $1 \le i < j \le n$ we have the (*real*) line $L'_{i,j}$ in $\boldsymbol{C}^1$, defined by

$$L'_{i,j} = \{x \in \boldsymbol{C}^1;\ I_i(x) = I_j(x)\}.$$

This line $L'_{i,j}$ also divides $\boldsymbol{C}^1$ into two parts $L'^+_{i,j} = \{x \in \boldsymbol{C}^1; I_i(x) < I_j(x)\}$ and $L'^-_{i,j} = \{x \in \boldsymbol{C}^1; I_i(x) > I_j(x)\}$. These lines $L'_{i,j}$ will help to determine the braid $\theta(\Gamma_i)$ mentioned in the above section.

In the arrangement context, the points of $S(\mathscr{C})$ are usually called the multiple points of the arrangement $\mathscr{A}$. By definition, a multiple point P of the arrangement $\mathscr{A}$ is the nonempty intersection of two or more hyperplanes of $\mathscr{A}$. The assumption on the genericity of the projection pr_1 implies that the multiple points of the arrangement $\mathscr{A}$ are distinct by their x- coordinates. In other words, the images of multiple points of $\mathscr{A}$ on the x-plane $\boldsymbol{C}^1$ are pairwise distinct.

Let $x_k \in \boldsymbol{C}^1$ be the image of a multiple point $P_k = (x_k, y_k)$ of $\mathscr{A}$ under the projection pr_1. Suppose that $P_k = \bigcap_{j=1}^r H_{i_j}$. Then it is clear that x_k belongs to the lines L_{i_s,i_t}, $1 \le s < t \le r$ of the labyrinth $\mathscr{L}(\mathscr{A})$. However, there might be another line $L \in \mathscr{L}(\mathscr{A})$, which does not belong to $\{L_{i_s,i_t};\ 1 \le s < t \le r\}$, going through this point x_k.

Definition 3.3. (i) The labyrinth $\mathscr{L}(\mathscr{A})$ is said to be good with respect to the multiple point $P_k = \bigcap_{j=1}^r H_{i_j}$ if there is not any line of $\mathscr{L}(\mathscr{A})$ except L_{i_s,i_t}; $1 \le s < t \le r$, going through x_k.

(ii) The labyrinth $\mathscr{L}(\mathscr{A})$ of an arrangement $\mathscr{A}$ is said to be proper if any line of $\mathscr{L}(\mathscr{A})$ has multiplicity 2 and it is good with respect to all multiple points of $\mathscr{A}$.

Remark 3.4. After a suitable change of coordinates we can assume that the labyrinth $\mathscr{L}(\mathscr{A})$ is good with respect to all multiple points of $\mathscr{A}$. So, from now on we always assume that the labyrinth $\mathscr{L}(\mathscr{A})$ of an arrangement $\mathscr{A}$ is proper.

4. The braid monodromy of complex arrangement

In this section we will determine the braid monodromy of an arrangement $\mathscr{A} = \{H_i;\ i = 1, \ldots, n\}$ of complex lines in $\boldsymbol{C}^2$.

Suppose that $f(x, y) = \prod_{i=1}^{n}(y - \alpha_i(x))$ is the defining polynomial for $\mathscr{A}$. Let $\mathscr{P} = \{P_1, \dots, P_N\}$ denote the set of multiple points of $\mathscr{A}$. These multiple points $P_i = (x_i, y_i)$ are the only singularities of $\mathscr{A}$. Remind that we always assume that the projection $pr_1 : \boldsymbol{C}^2 \to \boldsymbol{C}^1$ is generic with respect to the arrangement $\mathscr{A}$ and the labyrinth $\mathscr{L}(\mathscr{A})$ is proper. As indicated in the section §1, in order to determine the braid monodromy of $\mathscr{A}$ we have to determine all $\theta(\gamma_i)$ for a given good ordered system of generators $\{\gamma_1, \dots, \gamma_N\}$ of $\pi_1(\boldsymbol{C}^1 \backslash D(\mathscr{C}))$. And these $\theta(\gamma_i)$'s are determined by local braid monodromies at x_i and the braid homeomorphisms defined by moving the fiber of pr_1 along a chosen system of paths Γ_i. In [1] and [3], these datas was recorded by using the admissible 2-graph or its generalization, the braided wiring diagram, respectively. Here we will read these datas from the labyrinth $\mathscr{L}(\mathscr{A})$ of the arrangement $\mathscr{A}$.

The intersection of $\boldsymbol{C}_{x_0} = \{(x, y) \in \boldsymbol{C}^2;\ x = x_0\}$, the fiber of pr_1 over the base point x_0, with lines of $\mathscr{A}$, $\boldsymbol{C}_{x_0} \cap (\bigcup_{i=1}^{n} H_i)$, consists of n distinct points. When we move the fiber along a path in x-axis $\boldsymbol{C}^1$, these points form a braid on n strings. We will call the string corresponding to the hyperplane H_i the i^{th} string. In general these points have distinct real parts. A braiding will occur when the path intersects a line of the labyrinth $\mathscr{L}(\mathscr{A})$. In order to express $\theta(\gamma_i)$ in terms of the braid generator we need to recall of braids and braid groups.

A braid on n strings can be viewed as the graph of the motion of n points on a complex line from time $t = 0$ to time $t = 1$, satisfying

(i) These points remain distinct throughout the motion.

(ii) The sets of points at $t = 0$ and $t = 1$ are equal. There is a natural way to compose braids and to take the inverse of a braid. The isotopy classes of braids on n strings form a group, $B(n)$, called the braid group. It has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and defining relations

$$\sigma_i . \sigma_j = \sigma_j . \sigma_i \quad \text{if } |i - j| \geq 2$$

$$\sigma_i . \sigma_{i+1} . \sigma_i = \sigma_{i+1} . \sigma_i . \sigma_{i+1},$$

where the generator σ_i is illustrated in Figure 1.

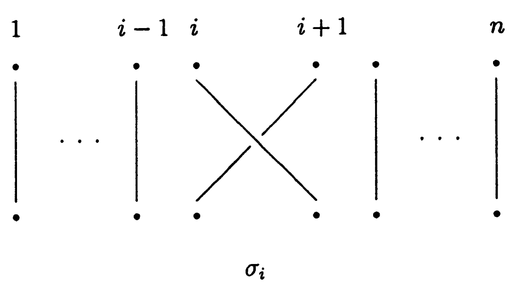
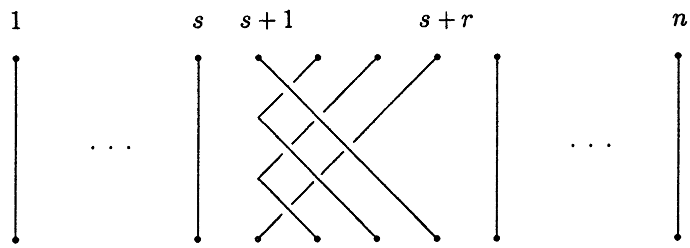


Figure 1

For a subset $I = \{s+1, \dots, s+r\}$ of $\{1, \dots, n\}$, we call the "half-twist" on I (see [2]) the following braid

$$\Delta_I = (\sigma_{s+1}\sigma_{s+2} \cdots \sigma_{s+r-1}) \cdot (\sigma_{s+1}\sigma_{s+2}\sigma_{s+r-2}) \cdots (\sigma_{s+1}\sigma_{s+2}) \cdot \sigma_{s+1}.$$

Geometrically, the half-twist Δ_I can be accomplished by holding the top of j^{th} strings fixed, $s+1 \leq j \leq s+r$, and attaching the bottom of these strings to a rot and then turn it over once, while keeping fixed all other strings. See Figure 2.



The half twist Δ_I

Figure 2

The pure braid group, $P(n)$, is the kernel of the natural surjection $B(n) \to S(n)$, where $S(n)$ denotes the symmetric group on n letters. It is well known that $P(n)$ has a presentation with generators

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1},$$

$1 \leq i < j \leq n$, and some certain defining relations.

Suppose that the base point $x_0 \in \boldsymbol{C}^1 \backslash D(\mathscr{C})$ is chosen in the component of $\boldsymbol{C}^1 \backslash \mathscr{L}(\mathscr{A})$, defined by $R_1 < \cdots < R_n$. For a multiple point $P_k = (x_k, y_k)$, let $\mathscr{I}_k = \{i_1, \dots, i_r\}$ be the set of all indices of those lines of $\mathscr{A}$ passing through P_k. It gives rise a partition of $\{1, \dots, n\}$ as $\mathscr{L}_k \cup \mathscr{I}_k \cup \mathscr{U}_k$, where $\mathscr{L}_k = \{l_1, \dots, l_s\}$, $\mathscr{I}_k = \{i_1, \dots, i_r\}$, $\mathscr{U}_k = \{u_1, \dots, u_t\}$, as follows. On the x-axis $\boldsymbol{C}^1$, all components of $\boldsymbol{C}^1 \backslash \mathscr{L}(\mathscr{A})$ incident with x_k must be of the following form

$$R_{l_1} < \cdots < R_{l_s} < R_{i_{\sigma(1)}} < \cdots < R_{i_{\sigma(r)}} < R_{u_1} < \cdots < R_{u_t},$$

where σ is a certain permutation of $\{1, \dots, r\}$. We call then the set $I_k = \{s+1, \dots, s+r\}$ the local index of the multiple point P_k.

LEMMA 4.1. *The local braid monodromy of $\mathscr{A}$ at x_k is the full twist A_{I_k} on the set I_k, where $A_{I_k} = \Delta_{I_k}^2$.*

Proof. We consider the local situation at the multiple point P_k. Let $\mathscr{I}_k = \{i_1, \dots, i_r\}$ as above. Let choose the point x_k^ε in the component of $\boldsymbol{C}^1 \backslash \mathscr{L}(\mathscr{A})$ defined by $R_{l_1} < \cdots < R_{l_s} < R_{i_1} < \cdots < R_{i_r} < R_{u_1} < \cdots < R_{u_t}$. For the sake of simplicity, the component $R_{l_1} < \cdots < R_{l_s} < R_{i_{\sigma(1)}} < \cdots < R_{i_{\sigma(r)}} < R_{u_1} < \cdots < R_{u_t}$ will be denoted simply by $R_{i_{\sigma(1)}} < \cdots < R_{i_{\sigma(r)}}$.

First we consider the case $r = 2$. It is illustrated in the Figure 3. We can choose $x_k^\varepsilon \in L'^{-}_{i_1,i_2}$. Then we move the fiber of pr_1 from x_k^ε to a point in the component $R_{i_2} < R_{i_1}$ by a path lying in $L'^{-}_{i_1,i_2}$. When the path crosses the line L_{i_1,i_2} of the labyrinth a braid of i_1^{th} and i_2^{th} strings occurs. Because the cross point lies in $L'^{-}_{i_1,i_2}$, the i_1^{th} string overcrosses the i_2^{th} string at this cross point. Then if the local index of P_k is $(i, i+1)$ we will get the braid σ_i. Now we continue to move the fiber of the projection pr_1 around x_k and back to x_k^ε. Because the path, along which we move the fiber of pr_1, must go around x_k, when it crosses the line L_{i_1,i_2} again the cross point must lie in $L'^{+}_{i_1,i_2}$. But this time we move from the component $R_{i_2} < R_{i_1}$ to the component $R_{i_1} < R_{i_2}$. So, once again we get the braid σ_i. It implies that the local braid monodromy at x_k is $\sigma_i^2 = A_{\{i,i+1\}}$.

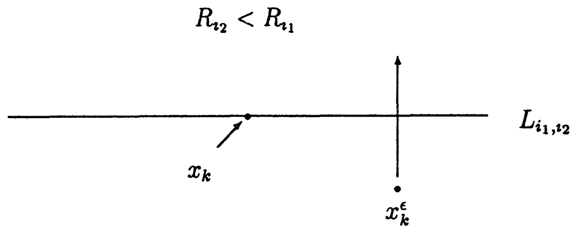


Figure 3

Next we consider the case $r > 2$. As above let $I_k = \{s+1, \dots, s+r\}$ be the local index of P_k. After a suitable isotopy, we may assume that locally at x_k, the point x_k^ε is chosen in $L'^{-}_{i_1,j}$, $j \in \{i_2, \dots, i_r\}$. It implies that when we move the fiber of pr_1 from x_k^ε to a point in the opposite component of $\mathbf{C}^1 \backslash \mathscr{L}(\mathscr{A})$ at $x_k, R_{i_r} < \cdots < R_{i_1}$, the i_1^{th} string will overcross all other i_j^{th} strings, $2 \leq j \leq r$. An inductive argument shows that we obtain then the half twist Δ_{I_k} on the subset I_k. We continue to move the fiber of pr_1 around the point x_k and back to x_k^ε. Similarly to the case $r = 2$, here we obtain again the half twist Δ_{I_k}. It proves that the local braid monodromy at x_k is the full twist $A_{I_k} = \Delta_{I_k}^2$.

THEOREM 4.2. *The braid monodromy of $\mathscr{A}$ is determined by*

$$\theta(\gamma_k) = \beta_k \,.\, A_{I_k} \,.\, \beta_k^{-1},$$

$1 \leq k \leq N$, *where A_{I_k} is determined as in* Lemma 4.1, *β_k is a braid which can be read off from the labyrinth $\mathscr{L}(\mathscr{A})$.*

Proof. We do the global step in the usual way to construct the braid monodromy (cf. §1). First we need to choose the paths Γ_k. For the multiple point $P_k = (x_k, y_k)$ let choose the point x_k^ε, near to x_k, as in Lemma 4.1, i.e. in the component of $\mathbf{C}^1 \backslash \mathscr{L}(\mathscr{A})$, given by

$$R_{l_1} < \cdots < R_{l_s} < R_{i_1} < \cdots < R_{i_r} < R_{u_1} < \cdots < R_{u_t}.$$

Then we take the path Γ_k to be the minimal simple path, going from x_0 to x_k^ε. Now we will move the fiber of the projection pr_1 along this path Γ_k. Suppose that the path Γ_k intersects the line $L_{i,j}$ of the labyrinth $\mathscr{L}(\mathscr{A})$. Then we will obtain a braiding of the i^{th} string and j^{th} string. To express this braiding in terms of braid generators suppose that locally at this intersection, Γ_k goes from $R_{l_1} < \cdots < R_{l_s} < R_i < R_j < R_{u_1} < \cdots < R_{u_t}$ to $R_{l_1} < \cdots < R_{l_s} < R_j < R_i < R_{u_1} < \cdots < R_{u_t}$. Let call as above $\{s+1,\ s+2\}$ the local index of the intersection of Γ_k and $L_{i,j}$. Then the braid of the i^{th} string and the j^{th} string will be σ_{s+1} if Γ_k crosses $L_{i,j}$ at a point in the domain $L_{i,j}'^-$ and will be σ_{s+1}^{-1} if Γ_k crosses $L_{i,j}$ at a point in the domain $L_{i,j}'^+$. Recording successively all these braids when the fiber moves from x_0 to x_k^ε we obtain the braid $\theta(\Gamma_k)$. Let A_{I_k} is the full twist of the Lemma 4.1. Denoting the braid $\theta(\Gamma_k)$ by β_k we have then the formula for the braid monodromy of $\mathscr{A}$

$$\theta(\gamma_k) = \beta_k \,.\, A_{I_k} \,.\, \beta_k^{-1}.$$

Note that we can use some formulas on the conjugation of $B(n)$ on $P(n)$ (see e.g. [2], [3]) to express the braid $\theta(\gamma_k)$ purely in term of pure braid generators. We will not repeat it here.

Remark 4.3. The use of the lines $L_{i,j}'$'s gives us a concrete method to determine which string is upper and which string is lower in the braiding of i^{th} and j^{th} strings, i.e. to determine precisely all braids in the conjugation in $\theta(\gamma_k)$.

As noted in [11], the braid monodromy of $\mathscr{A}$ is closely related to the fundamental group of its complement $\pi_1(\boldsymbol{C}^l \backslash \bigcup_{H\in\mathscr{A}} H)$. The intersection of the fiber $\boldsymbol{C}_{x_0}$ of pr_1 over x_0 with hyperplanes of $\mathscr{A}$ consists of n points. Then $\boldsymbol{C}_{x_0} \backslash (\boldsymbol{C}_{x_0} \cap (\bigcup_{H\in\mathscr{A}} H))$ is a punctured complex line with n removed points. Let $g_1, \dots, g_n$ denote the generators of the free group $\pi_1(\boldsymbol{C}_{x_0} \backslash (\boldsymbol{C}_{x_0} \cap (\bigcup_{H\in\mathscr{A}} H)))$. The braid group $B[\boldsymbol{C}_{x_0}, \boldsymbol{C}_{x_0} \cap (\bigcup_{H\in\mathscr{A}} H)]$ can be naturally considered as a group of automorphisms of $\pi_1(\boldsymbol{C}^l \backslash \bigcup_{H\in\mathscr{A}} H)$. Let identify $g_1, \dots, g_n$ with their images in $\boldsymbol{C}^l \backslash \bigcup_{H\in\mathscr{A}} H$ by the homomorphism induced from the embedding $\boldsymbol{C}_{x_0} \backslash (\boldsymbol{C}_{x_0} \cap (\bigcup_{H\in\mathscr{A}} H)) \subset \boldsymbol{C}^l \backslash \bigcup_{H\in\mathscr{A}} H$. Remind that for each multiple point P_k, $1 \le k \le N$ we denote by $\mathscr{I}_k$ the set of indices of all lines of $\mathscr{A}$ going through P_k. Then we have the following corollary (cf. [10]).

COROLLARY 4.4. *The fundamental group of the complement to the arrangement $\mathscr{A}$, $\pi_1(\boldsymbol{C}^l \backslash \bigcup_{H\in\mathscr{A}} H)$, is generated by elements $g_1, \dots, g_n$, with the defining relations*

$$g_i = \beta_k \,.\, A_{L_k} \,.\, \beta_k^{-1} \,.\, g_i,$$

$i \in \mathscr{I}_k,\ k = 1, \dots, N.$

Remark 4.5. From the above presentation of $\pi_1(\boldsymbol{C}^l \backslash \bigcup_{H\in\mathscr{A}} H)$ we can simplify the defining relations to get the presentation given in [6]. If instead of the system of Γ_k; $k = 1, \ldots, N$ we follow the way of Arvola to choose a PL graph in the x-plane $\boldsymbol{C}^1$, by the method mentioned above using the labyrinth we can also obtain the Arvola's presentation for the fundamental group of the complement.

COROLLARY 4.6. *The labyrinth of an arbitrary arrangement $\mathscr{A}$ in $\boldsymbol{C}^2$ determines the homotopy type of its complement.*

Proof. According to Libgober [10], the complement of an arrangement $\mathscr{A}$ is homotopy to the standard 2-complex modelled on the braid monodromy presentation of its π_1. The braid monodromy presentation of π_1 of its complement given in the Corollary 4.4 is determined by its labyrinth. So, the corollary is proved.

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