P MILNES
KODAI MATH. J.
21 (1998), 306–317

MINIMAL H_3 ACTIONS AND SIMPLE QUOTIENTS OF A DISCRETE 6-DIMENSIONAL NILPOTENT GROUP

PAUL MILNES

Abstract

A simple C^*-algebra is introduced that is generated by a minimal effective action of the (discrete, nilpotent, non-abelian) Heisenberg group H_3 on the torus $\boldsymbol{T}^2$ It appears as a simple quotient of the group C^*-algebra $C^*(H_{6,4})$ of a 6-dimensional discrete nilpotent group $H_{6,4}$, and also as a C^*-crossed product generated by an action of $\boldsymbol{Z}^2$ on $\boldsymbol{T}^3$ The rest of the infinite dimensional simple quotients of $C^*(H_{6,4})$ are identified and displayed as C^*-crossed products generated by minimal effective actions, and also as matrix algebras over simple C^*-algebras from groups of lower dimension.

1. Introduction

In 3 dimensions there is a unique (up to isomorphism) connected, simply connected, nilpotent Lie group, which we call G_3 (following Nielsen [N]); G_3 ($= \boldsymbol{R}^3$ as a set) is the Heisenberg group with multiplication

$$(k,m,n)(k',m',n') = (k+k'+nm', m+m', n+n').$$

The faithful irreducible representations of the lattice subgroup H_3 ($= \boldsymbol{Z}^3$ as a set) of G_3 generate the irrational rotation algebras A_θ. In 4 dimensions there is also a unique such connected group G_4, and in 5 dimensions there are 6 such groups $G_{5,i}$, $1 \le i \le 6$. The main thrust in [MW1, MW2] was to find cocompact subgroups $H_4 \subset G_4$ and $H_{5,i} \subset G_{5,i}$, that would be analogous to $H_3 \subset G_3$, and then for these H's to identify the infinite dimensional simple quotients of $C^*(\mathrm{H})$, both the faithful ones (generated by a faithful representation of H) and the non-faithful ones, and also to give matrix representations over lower dimensional algebras for as many of the non-faithful quotients as possible. In the course of this work, it was observed that all flow presentations of simple C^*-algebras that arose used actions of abelian groups, namely, $\boldsymbol{Z}$ or $\boldsymbol{Z}^2$, or subgroups of them. It was also observed in the 5-dimensional setting that one of the groups, $H_{5,6}$,

---

1991 *Mathematics Subject Classification.* 22D25, 46L55.

Key words and phrases. minimal effective flow, simple C^*-algebra, discrete nilpotent group, semidirect product.

This research was supported in part by NSERC grant A7857
Received March 19, 1998.

differed from another, $H_{5,5}$, just by having one more (non-trivial) commutator. The same is true for a pair of 6-dimensional groups, $H_{6,7}$ and $H_{6,4}$. For these last 2 groups, the relevant commutator is at a 'higher level', i.e., its value is not in the centre of the group; this makes a greater change in the multiplication formula. Furthermore, the C^*-algebras $A_\theta^{6,4}$ generated by faithful irreducible representations of $H_{6,4}$ are simple quotients of $C^*(H_{6,4})$ and have flow presentations with H_3 acting on $\boldsymbol{T}^2$.

In §§2 and 3 of the present paper, these aspects of $H_{6,4}$ are given in some detail, and $A_\theta^{6,4}$ is shown to have another flow presentation, this time by an action of $\boldsymbol{Z}^2$ on $\boldsymbol{T}^3$. $A_\theta^{6,4}$ is also shown to have an automorphism of order 4, that is analogous to the Fourier automorphism of the irrational rotation algebra. In §4, the rest of the infinite dimensional simple quotients of $C^*(H_{6,4})$ are identified and displayed as C^*-crossed products generated by minimal effective actions, and also as matrix algebras over simple C^*-algebras from groups of lower dimension.

Preliminaries

To present the results and proofs of the paper, we need notation for semidirect products and C^*-crossed products; the discussion which follows is quite standard, appearing in [MW1], [Z-M] and many other places.

Suppose that N and K are discrete groups, the identity of each of them being denoted by e. Suppose that there is a homomorphism $s \mapsto \sigma_s$ from K into the automorphism group of N. Then $G = N \times K$ becomes a group, the *semidirect product* of N and K, with the multiplication formula

$$(t,s)(t',s') = (t\sigma_s(t'), ss').$$

We will usually write $s(t)$ instead of $\sigma_s(t)$.

Conversely, if N is a normal subgroup of G with quotient group $K = G/N$ suitably embedded as a subgroup in G, then G is canonically isomorphic to a semidirect product $N \times K$, whose automorphisms are determined by G, $\sigma_s(t) = sts^{-1}$ (product in G).

Now replace N by a C^*-algebra A with identity 1 and assume that we have a homomorphism $s \mapsto \sigma_s$ from K into the automorphism group of A. Then, for f and g in the Banach space $\ell^1(K,A)$, the convolution product $f * g$ and involution f^* are defined by

$$f * g(s') = \sum_{s \in K} f(s)\sigma_s(g(s^{-1}s')) \quad \text{and} \quad f^*(s) = \sigma_s(f(s^{-1})^*);$$

with these definitions, $\ell^1(K,A)$ becomes a Banach *-algebra. The *C^*-crossed product* $C^*(A,K)$ is defined to be the enveloping C^*-algebra of $\ell^1(K,A)$.

For $a \in A$ and $s \in K$, the δ-functions a_s and δ_s in $\ell^1(K,A) \subset C^*(A,K)$ are defined by $a_s(s) = a$, $a_s(s') = 0$ otherwise, and $\delta_s(s) = 1$ (the identity of A), $\delta_s(s') = 0$ for $s' \neq s$.

2. The nilpotent group $H_{6,4}$

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ, and let unitaries U, V and W and subsidiary operators X and Y satisfy the commutation relations

$$\text{(CR)}\qquad \begin{cases} [U,V] = X,\ [V,W] = Y,\ [U,Y] = \lambda = [X,W], \\ \text{and } [U,W] = 1 = [U,X] = [V,X] = [V,Y] = [W,Y] = [Y,X]. \end{cases}$$

Let $A_\theta^{6,4}$ denote the C^*-algebra generated by these unitaries, concrete representations of which are given in §2. Note that U, V and W generate $A_\theta^{6,4}$; the unitaries X and Y have been introduced only to control the notation.

A "discrete group construction" in [MW1] shows how to construct a group from unitaries like these; use (CR) to collect terms in the product

$$\lambda^g Y^h W^k X^j V^m U^n \lambda^{g'} Y^{h'} W^{k'} X^{j'} V^{m'} U^{n'},$$

then the exponents give the multiplication for a group $H_{6,4}$ $(= \boldsymbol{Z}^6$, as a set), namely,

$$\text{(m)}\qquad \begin{cases} (g,h,k,j,m,n)(g',h',k',j',m',n') = \\ (g+g'+jk'+nh', h+h'+mk', k+k', j+j'+nm', m+m', n+n') \end{cases}$$

with inverse

$$(g,h,k,j,m,n)^{-1} = (-g+jk+nh-kmn, -h+km, -k, -j+nm, -m, -n).$$

We think of $H_{6,4}$ as the lattice subgroup of Nielsen's $G_{6,4} = \boldsymbol{R}^6$ [N], although, in fact, Nielsen's group has a quite different multiplication. The simplest isomorphism we have been able to devise of our $H_{6,4}$ onto a cocompact subgroup of Nielsen's $G_{6,4}$ is

$$(g,h,k,j,m,n) \mapsto (g+j-nm, h, -j+nm, k, n, m).$$

An essential (and obvious) property of $H_{6,4}$ is that

$$\pi : (g,h,k,j,m,n) \mapsto \lambda^g Y^h W^k X^j V^m U^n$$

is a representation of $H_{6,4}$ that generates $A_\theta^{6,4}$.

Presentations of $H_{6,4}$ as a semidirect product $N \times K$

1. The form of $G_{6,4}$ in [N] is not useful for our purposes. Accordingly, we have given $H_{6,4}$ in the form above because it is already a semidirect product $\boldsymbol{Z}^3 \times H_3$ with normal subgroup $N_1 = \boldsymbol{Z}^3 = (\boldsymbol{Z},\boldsymbol{Z},\boldsymbol{Z},0,0,0) \subset H_{6,4}$ being acted on by $K_1 = H_3 = H_{6,4}/N_1 = (0,0,0,\boldsymbol{Z},\boldsymbol{Z},\boldsymbol{Z})$ via

$$(j,m,n) : (g',h',k') \mapsto (g'+jk'+nh', h'+mk', k);$$

the multiplication formula (m) for $H_{6,4} = N_1 \times K_1$ is the result. This group is isomorphic to the semidirect product part of one of the (central) extension presentations of another discrete 6-dimensional group $H_{6,7}$; the addition of one further non-trivial commutator $[X, U] = Y$ to (CR) yields the fairly large cocycle that completes this presentation of $H_{6,7}$.

2. One observes that $N_2 = \boldsymbol{Z}^3 = (\boldsymbol{Z}, 0, 0, \boldsymbol{Z}, 0, \boldsymbol{Z}) \subset H_{6,4}$ is also a normal subgroup with $K_2 = H_3 = H_{6,4}/N_2 = (0, \boldsymbol{Z}, \boldsymbol{Z}, 0, \boldsymbol{Z}, 0)$. One consequence of these 2 presentations of $H_{6,4}$ as $\boldsymbol{Z}^3 \times H_3$ is an automorphism of $H_{6,4}$

$$\varphi_1 : (g, h, k, j, m, n) \mapsto (g - nh - jk, j, n, -h + km, -m, -k),$$

which will be used below (Remark 2 in §3) to yield an interesting automorphism of $A_\theta^{6,4}$.

3. A quite different semidirect product presentation of $H_{6,4}$ as $N_3 \times K_3 = \boldsymbol{Z}^4 \times \boldsymbol{Z}^2$ shows that $H_{6,4}$ is related to a semidirect product presentation of $H_{6,7}$; it is also useful in identifying the 'other' (non-faithful) simple quotients of $C^*(H_{6,4})$ towards the end of the next section. For it the normal subgroup is $N_3 = \boldsymbol{Z}^4 = (\boldsymbol{Z}, \boldsymbol{Z}, 0, \boldsymbol{Z}, \boldsymbol{Z}, 0) \subset H_4$ with quotient $K_3 = H_{6,4}/N_3 = \boldsymbol{Z}^2 = (0, 0, \boldsymbol{Z}, 0, 0, \boldsymbol{Z})$ acting on $\boldsymbol{Z}^4$ via

$$(k, n) : (g', h', j', m') \mapsto (g' + kj' + nh' + knm', h' + km', j' + nm', m),$$

and yielding the multiplication for the semidirect product $N_3 \times K_3$,

$$(g, h, j, m, k, n)(g', h', j', m', k', n')$$
$$= (g + g' + kj' + nh' + knm', h + h' + km', j + j' + nm', m + m', k + k', n + n').$$

An isomorphism from $H_{6,4}$ to $N_3 \times K_3$ is given by

$$\varphi_2 : (g, h, k, j, m, n) \mapsto (g - jk, h - km, -j, -m, k, n).$$

(For $H_{6,7}$, the formula for the action of $\boldsymbol{Z}^2 = (0, 0, \boldsymbol{Z}, 0, 0, \boldsymbol{Z})$ on $\boldsymbol{Z}^4 = (\boldsymbol{Z}, \boldsymbol{Z}, 0, \boldsymbol{Z}, \boldsymbol{Z}, 0)$ has 4 more terms,

$$(k, n) : (g', h', j', m') \mapsto (g' + kj' + nh' + knm' + j'n(n-1)/2 + m'n(n-1)(n-2)/6,$$
$$h' + km' + nj' + m'n(n-1)/2, j' + nm', m').$$

See [JM], presentation 2 in §2.)

4. The normal subgroup $N_4 = \boldsymbol{Z} = (\boldsymbol{Z}, 0, 0, 0, 0, 0) \subset H_{6,7}$ with quotient $K_4 = H_{6,4}/N_4 = (0, \boldsymbol{Z}, \boldsymbol{Z}, \boldsymbol{Z}, \boldsymbol{Z}, \boldsymbol{Z}) \cong H_{5,2}$ permits the interpretation that $A_\theta^{6,4}$ is generated by a representation of canonical commutation relations over $(H_{5,2}, b)$ (or a b-representation of $H_{5,2}$); here b is the cocycle

$$[(h, k, j, m, n), (h', k', j', m', n')] = jk' + nh', K_4 \times K_4 \to N_4 = \boldsymbol{Z}.$$

See p. 325 of [MW2] of more details.

3. Simple quotients of $C^*(H_{6,4})$

Let $\lambda = e^{2\pi i\theta}$ for an irrational θ. Since $A_\theta^{6,4}$ is generated by a representation of $H_{6,4}$, it is a quotient of $C^*(H_{6,4})$. There are a number of methods to prove that quotients of group C^*-algebras are simple; see pp. 318–9 in [MW2]. Of these, we will use the minimal flow method involving Corollary 5.16 in [EH]. In Proposition 1 we shall show that the first 2 concrete representations of $\mathrm{A}_\theta^{6,4}$ given below yield C^*-algebras that are generated by irreducible representations; since primitive ideals of $C^*(\mathrm{H}_{6,4})$ are maximal [H], it follows that these concrete representations are simple.

The next theorem asserts that the C^*-algebra $\mathrm{A}_\theta^{6,4}$ is simple and universal for the equations (CR), and has a unique tracial state. The existence of the unique tracial state is easy to verify directly (and can also be proved by citing results from the literature).

In this theorem we use a flow $\mathscr{F} = (\mathrm{H}_3, \boldsymbol{T}^2)$ to generate a C^*-crossed product $C^*(\mathscr{C}(\boldsymbol{T}^2), \mathrm{H}_3)$. The required action of H_3 on $\boldsymbol{T}^2$ is given by

$$(j,m,n) : (w,v) \mapsto (\lambda^{-n}w, \lambda^{-j+nm}w^{-m}v),$$

which looks awkward. However, since H_3 is not abelian, inversion must be used to transfer the action of H_3 to $\mathscr{C}(\boldsymbol{T}^2)$, $sf = f \circ s^{-1}$, so

$$(\star) \qquad \begin{cases} (j,m,n)f(w,v) = f \circ (j,m,n)^{-1}(w,v) \\ = f((-j+nm,-m,-n)(w,v)) = f(\lambda^n w, \lambda^j w^m v). \end{cases}$$

If we let w and v also denote the generators $(w,v) \mapsto w$ and $(w,v) \mapsto v$ of $\mathscr{C}(\boldsymbol{T}^2)$, then

$$(\star) \qquad (j,m,n) : w \mapsto \lambda^n w, \quad v \mapsto \lambda^j w^m v$$

generates the action of H_3 on $\mathscr{C}(\boldsymbol{T}^2)$.

1. Theorem. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*

(a) *There is a unique (up to isomorphism) C^*-algebra $\mathrm{A}_\theta^{6,4}$ generated by unitaries U, V, W, X and Y satisfying* (CR); *$\mathrm{A}_\theta^{6,4}$ is simple and is universal for the equations* (CR). *Let H_3 act on $\mathscr{C}(\boldsymbol{T}^2)$ as indicated in the previous paragraph; then*

$$\mathrm{A}_\theta^{6,4} \cong C^*(\mathscr{C}(\boldsymbol{T}^2), \mathrm{H}_3).$$

(b) *Let π' be a representation of $\mathrm{H}_{6,4}$ such that $\pi = \pi'$ (as scalars) on the center $(\boldsymbol{Z},0,0,0,0,0)$ of $\mathrm{H}_{6,4}$, and let A be the C^*-algebra generated by π'. Then $\mathrm{A} \simeq \mathrm{A}_\theta^{6,4}$ via a unique isomorphism ω such that the following diagram commutes.*

$$\begin{tikzcd} \mathrm{H}_{6,4} \arrow[rr, "\pi"] \arrow[dr, "\pi'"'] & & \mathrm{A}_\theta^{6,4} \arrow[dl, "\omega"] \\ & \mathrm{A} & \end{tikzcd}$$

(c) *The C^*-algebra $\mathrm{A}_\theta^{6,4}$ has a unique tracial state.*

Proof. The proof can be much as for Theorem 1.1 in [MW2]; we give some details.

One must note first that the flow $(\mathrm{H}_3, \boldsymbol{T}^2)$ above is minimal and effective; so $C^*(\mathscr{C}(\boldsymbol{T}^2), \mathrm{H}_3)$ is simple, by Corollary 5.16 of Effros and Hahn [EH].

Once the simplicity of $C^*(\mathscr{C}(\boldsymbol{T}^2), \mathrm{H}_3)$ is established, it is straightforward to prove the rest of (a) using the correspondence of

$$\delta_{(0,0,1)},\ \delta_{(0,1,0)},\ \delta_{(1,0,0)},\ v_{(0,0,0)},\ w_{(0,0,0)} \in \ell^1(\mathrm{H}_3, \mathscr{C}(\boldsymbol{T}^2)) \subset C^*(\mathscr{C}(\boldsymbol{T}^2), \mathrm{H}_3)$$

to U, V, X, W, Y, respectively; see [MW2; proof of Theorem 1.1], for example, also for (b) and (c). ■

Concrete representations of $\mathrm{A}_\theta^{6,4}$

As well as giving a pleasing connection with geometry and topology, the minimal flow presentations of these C^*-algebras (as in Theorem 1) also provide the most attractive concrete representations of the algebras. The first 2 concrete representations of $\mathrm{A}_\theta^{6,4}$ arise in this setting.

1. The first arises from the flow $(\mathrm{H}_3, \boldsymbol{T}^2)$ in the theorem. If we use the '$f \mapsto sf$' notation for the action of H_3 on $\mathscr{C}(\boldsymbol{T}^2)$ as at $(\star)$, then on $\mathscr{H}_1 = L^2(\boldsymbol{T}^2)$ the unitaries $U : f \mapsto (0,0,1)f$, $V : f \mapsto (0,1,0)f$ and $W : f \mapsto vf$ (along with their subsidiary operators $X : f \mapsto (1,0,0)f$ and $Y : f \mapsto wf$) satisfy (CR), so the C^*-subalgebra $\mathscr{A}_1 \subset \mathscr{L}(\mathscr{H}_1)$ they generate is isomorphic to $\mathrm{A}_\theta^{6,4}$.

2. The second is connected with the semidirect product $N_3 \times K_3 = \boldsymbol{Z}^4 \times \boldsymbol{Z}^2 \cong \mathrm{H}_{6,4}$ (and is similar to the flow representation of $\mathrm{A}_\theta^{6,7}$ in [JM; §3]). Let $\boldsymbol{Z}^2$ act on $\mathscr{H}_2 = L^2(\boldsymbol{T}^3)$ via

$$(\diamond) \qquad (k,n) : f \mapsto f \circ \alpha_1^k \circ \alpha_2^n,$$

where the commuting homeomorphisms

$$\alpha_1 : (x,w,v) \to (x, \lambda w, xv) \quad \text{and} \quad \alpha_2 : (x,w,v) \to (\lambda x, w, wv)$$

generate a minimal distal flow $\mathscr{F}_1 = (\boldsymbol{Z}^2, \boldsymbol{T}^3)$. Then the unitaries defined on $\mathscr{H}_2$ by

$$U_0 f = f \circ \alpha_2, \quad V_0 f = vf, \quad W_0 f = f \circ \alpha_1, \quad X_0 f = wf, \quad \text{and} \quad Y_0 f = xf,$$

where x also denotes the function $(x,w,v) \mapsto x$ in $\mathscr{C}(\boldsymbol{T}^3)$, satisfy

$$(\mathrm{CR})_0 \qquad [U_0, V_0] = X_0, \quad [W_0, V_0] = Y_0, \quad [U_0, Y_0] = \lambda = [W_0, X_0].$$

With the correspondence $U, V, X, W, Y \sim U_0, V_0^{-1}, X_0^{-1}, W_0, Y_0$, one sees that these are just the (non-trivial) commutators in (CR), so the correspondence generates an isomorphism of $\mathrm{A}_\theta^{6,4}$ onto the C^*-subalgebra $\mathscr{A}_2 \subset \mathscr{L}(\mathscr{H}_2)$ generated by $U_0, \ldots, Y_0$. $\mathrm{A}_\theta^{6,4}$ is also isomorphic to the C^*-crossed product $C^*(\mathscr{C}(\boldsymbol{T}^3), \boldsymbol{Z}^2)$ generated by $\mathscr{F}_1$.

Note that, in $(\diamond)$, the inversion in $f \mapsto f \circ s^{-1}$ (as at $(\star)$) has been omitted; this is possible because $\boldsymbol{Z}^2$ is abelian. We note as well that this representation differs from the concrete representation for $\mathrm{A}_\theta^{6,7}$ on $L^2(\boldsymbol{T}^3)$ [JM; §3] only in that, for $\mathrm{A}_\theta^{6,7}$, the formula for α_2 is $(x,w,v) \to (\lambda x, xw, wv)$.

3. A third comes from a representation ρ of $\mathrm{H}_{6,4}$ on $\ell^2(\boldsymbol{Z}^5)$,

$$\rho(g,h,k,j,m,n) : \delta_{(h',j',k',m',n')} \mapsto \lambda^{g+kj'+nh'}\delta_s,$$

where

$$s = (h+h'+mk', k+k', j+j'+nm', m+m', n+n').$$

The unitaries $U = \rho(0,0,0,0,0,1)$, $V = \rho(0,0,0,0,1,0)$ and $W = \rho(0,0,1,0,0,0)$ satisfy (CR), so the C^*-algebra generated by ρ is isomorphic to $\mathrm{A}_\theta^{6,4}$.

Remarks. 1. The other infinite dimensional simple quotients of $C^*(\mathrm{H}_{6,4})$, A_1 and A_2 in the next section, each have concrete representations entirely analogous to the last 2 concrete representations of $\mathrm{A}_\theta^{6,4}$.

2. The automorphism φ_1 of $\mathrm{H}_{6,4}$ in the previous section produces an automorphism φ_1^* of $\mathrm{A}_\theta^{6,4}$ generated by

$$\varphi_1^* : Y^h W^k X^j V^m U^n \mapsto Y'^h W'^k X'^j V'^m U'^n$$

with $U' = W^{-1}$, $V' = V^{-1}$, $X' = Y^{-1}$, $W' = U$ and $Y' = X$. One verifies readily that U', V', X', W' and Y' satisfy (CR). φ_1^* is of order 4, like the Fourier automorphism $U \mapsto V$, $V \mapsto U^{-1}$, of A_θ^3; thus its square

$$\varphi_1^{*2} : Y^h W^k X^j V^m U^n \mapsto Y^{-h} W^{-k} X^{-j} V^m U^{-n}$$

is analogous to the flip automorphism $U \mapsto U^{-1}$, $V \mapsto V^{-1}$, of A_θ^3. See Walters [W] for more on these automorphisms of A_θ^3.

1. Proposition. *The concrete representations in* 1 *and* 2 *above of* $\mathrm{A}_\theta^{6,4}$ *are irreducible, from the same primitive ideal kernel in* $C^*(\mathrm{H}_{6,4})$, *and are not unitarily equivalent.*

Proof. We show that representation 1 is irreducible, and start by noting that, in our notation, $\{w^m v^n : m,n \in \boldsymbol{Z}\}$ is the usual basis for $\mathscr{H}_1 = L^2(\boldsymbol{T}^2)$. Then we have

$$U : w^m v^n \mapsto \lambda^m w^m v^n, \quad V : w^m v^n \mapsto w^{m+n} v^n \quad \text{and} \quad W : w^m v^n \mapsto \lambda^m w^m v^{n+1}.$$

Suppose that $T \in B(\mathscr{H}_1)$ commutes with U, V and W; we must show that T is a multiple of the identity. Let matrix coefficients for T be given by

$$T w^m v^m = \sum_{m',n' \in \boldsymbol{Z}} a_{m',n'}^{m,n} w^{m'} v^{n'},$$

with $\sum_{m',n' \in \boldsymbol{Z}} |a_{m',n'}^{m,n}|^2 < \infty$ and, in fact, uniformly bounded in m, n.

Now $TW = WT$, $TV = VT$ and $TU = UT$ imply that

$$(1)\ a_{m',n'}^{m,n} = a_{m',n'+1}^{m,n+1}, \quad (2)\ \lambda^m a_{m',n'}^{m,n} = \lambda^{m'} a_{m',n'}^{m,n} \quad \text{and} \quad (3)\ a_{m',n'}^{m,n} = a_{m'+n',n'}^{m+n,n},$$

respectively. Then (3), with $n = 0$, implies that (4) $a_{m',n'}^{m,0} = 0$ if $n' \neq 0$, because of the convergence condition. Also (1) and (4) imply that $a^{m,n}_{m',n'} = 0$ if $n \neq n'$, (2) implies that $a^{m,n}_{m',n'} = 0$ if $m \neq m'$, and then (1) and (3) imply that $a^{m,n}_{m,n}$ is constant for all m, n, i.e., T is a multiple of the identify.

The proof that representation 2 is irreducible is similar, and is omitted. Since the unitaries U_0, V_0^{-1}, X_0^{-1}, W_0, Y_0 satisfy (CR),

$$\pi' : (h,k,j,m,n) \mapsto Y_0^h W_0^k X_0^{-j} V_0^{-m} U_0^n$$

gives the representation of $H_{6,4}$, and hence of $\ell^1(H_{6,4})$ and of $C^*(H_{6,4})$, that generates $\mathscr{A}_2$. But it follows from Theorem 1 that the map

$$\sum a_{h,k,j,m,n} Y^h W^k X^j V^m U^n \mapsto \sum a_{h,k,j,m,n} Y_0^h W_0^k X_0^{-j} V_0^{-m} U_0^n, \quad \mathscr{A}_1 \to \mathscr{A}_2,$$

is an isometry, so π' and the representation $\pi : (h,k,j,m,n) \mapsto Y^h W^k X^j V^m U^n$ generating $\mathscr{A}_1$ have the same kernel $I \subset C^*(H_{6,4})$.

To see that π and π' are not unitarily equivalent, suppose that T is a linear isometry of $\mathscr{H}_1$ onto $\mathscr{H}_2$, and that T intertwines U, V, W and U_0, V_0^{-1}, W_0 and is given by

$$Tw^m v^n = \sum_{a,b,c \in \mathbf{Z}} d^{m,n}_{a,b,c} x^a w^b v^c$$

Then the first equation alone gives the contradiction $T(\mathscr{H}_1) \subsetneqq \mathscr{H}_2$. For

$$U_0(x^a w^b v^c) = \lambda^a x^a w^{b+c} v^c,$$

and $TU = U_0 T$ gives $\lambda^m d^{m,n}_{a,b,c} = \lambda^a d^{m,n}_{a,b-c,c}$, so $d^{m,n}_{a,b,c} = 0$ if $c \neq 0$, since

$$\sum_{a,b,c} |d^{m,n}_{a,b,c}|^2 < \infty. \qquad \blacksquare$$

The following corollary is a very special case of a classical result of Thoma [T].

1. COROLLARY. $H_{6,4}$ *is not a type* I *group.*

4. Other simple quotients of $A_\theta^{6,4}$

When $\lambda = e^{2\pi i\theta}$ for an irrational θ, $A_\theta^{6,4}$ is a simple quotient of $C^*(H_{6,4})$ and the representation

$$\pi : (g,h,k,j,m,n) \to \lambda^g Y^h W^k X^j V^m U^n, \quad H_{6,4} \to A_\theta^{6,4},$$

is faithful. But there are other infinite dimensional simple quotients of $C^*(H_{6,4})$; for them π is not faithful.

When π is not faithful, the action of H_3 in the first concrete representation of $A_\theta^{6,4}$ becomes intractable; accordingly, we work with the formulation of the second concrete representation of $A_\theta^{6,4}$, the commutator equations

$$(\mathrm{CR})_0 \qquad [U_0, V_0] = X_0, \quad [W_0, V_0] = Y_0, \quad [U_0, Y_0] = \lambda = [W_0, X_0].$$

So, now suppose that λ is a primitive q-th root of unity and that A is a simple quotient of $C^*(\mathrm{H}_{6,4})$ that is irreducibly represented and generated by unitaries U_0, V_0, W_0, X_0 and Y_0 satisfying $(\mathrm{CR})_0$. Then X_0^q commutes with each of U_0, V_0 and W_0; hence, by irreducibility, it is a multiple of the identity, $X_0^q = \mu'$. Similarly, $Y_0^q = \eta'$.

1. Suppose that neither μ' nor η' is a root of unity, and in fact, that μ' and η' are linearly independent, i.e.,

$$\mu'^r \eta'^s = 1 \quad \text{for } r, s \in \boldsymbol{Z} \text{ implies } r = s = 0.$$

In $(\mathrm{CR})_0$ substitute $X_0 = \mu X_1$ and $Y_0 = \eta Y_1$ with $\mu^q = \mu'$ and $\eta^q = \eta'$ and get

$$(\mathrm{CR})_1 \qquad \begin{cases} [U_0, V_0] = \mu X_1, \quad [W_0, V_0] = \eta Y_1, \quad [U_0, Y_1] = \lambda = [W_0, X_1], \\ \text{and } X_1^q = Y_1^q = 1. \end{cases}$$

If $\boldsymbol{Z}_q$ denotes the subgroup of $\boldsymbol{T}$ with q elements, the C^*-algebra A_1 generated by the unitaries can be presented as $C^*(\mathscr{C}(\boldsymbol{Z}_q^2 \times \boldsymbol{T}), \boldsymbol{Z}^2)$, and also concretely represented on $L^2(\boldsymbol{Z}_q^2 \times \boldsymbol{T})$, using a minimal distal flow $\mathscr{F}_2 = (\boldsymbol{Z}^2, \boldsymbol{Z}_q^2 \times \boldsymbol{T})$. The formulae are almost the same as for the second concrete representation of $A_\theta^{6,4}$ using $(\boldsymbol{Z}^2, \boldsymbol{T}^3)$, the only difference being that the homeomorphisms generating $\mathscr{F}_2$ are

$$\alpha_1 : (x, w, v) \to (x, \lambda w, \eta x v) \quad \text{and} \quad \alpha_2 : (x, w, v) \to (\lambda x, w, \mu w v).$$

2. Suppose that μ is not a root of unity, but η is a primitive q'-th root of unity; put $q'' = \operatorname{lcm}\{q, q'\}$. Then $W^{q''}$ is also a multiple of the identity, and we can assume that $W^{q''} = 1$, since W is a generator. The equations become

$$(\mathrm{CR})_2 \qquad \begin{cases} [U, V] = \mu X_1, \quad [W, V] = \eta Y_1, \quad [U, Y_1] = \lambda = [W, X_1], \\ \text{and } X_1^q = Y_1^q = W^{q''} = 1. \end{cases}$$

The generated C^*-algebra A_2 can be represented as $C^*(C(\boldsymbol{Z}_q^2 \times \boldsymbol{T}), \boldsymbol{Z}^2)$, and concretely on $L^2(\boldsymbol{Z}_q^2 \times \boldsymbol{T})$, with the same formulae as for A_1.

3. Suppose that μ and η are not roots of unity, but are linearly dependent, $\mu^{r'} \eta^r = 1$ for some $r, r' \in \boldsymbol{Z}$. This case can be reduced to the previous one by a change of variables, much as for case 2 of $C^*(\mathrm{H}_{5,2})$ [MW2; p. 322]. Specifically, we have $\mu^{t'r''} = \eta^{tr''}$ with $(t, t') = 1$, so that $ts + t's' = 1$ for some $s, s' \in \boldsymbol{Z}$. The substitution

$$\begin{pmatrix} W' \\ U' \end{pmatrix} = \begin{pmatrix} t, & -t' \\ s', & s \end{pmatrix} \begin{pmatrix} W \\ U \end{pmatrix} = \begin{pmatrix} W^t U^{-t'} \\ W^{s'} U^s \end{pmatrix}$$

in $(\mathrm{CR})_1$ gives unitaries U', V, W', X' and Y' that generate the same algebra A_3

as U, V, W, X and Y (since $W = W'^s U''^{t'}$ and $U = W'^{-s'} U''^{t}$), and satisfy

$$\begin{cases} [U', V] = \mu'' X_1', \quad [W', V] = \eta'' Y_1', \quad [U', Y_1'] = \lambda = [W', X_1'], \\ \text{and } X_1'^q = Y_1'^q = 1, \end{cases}$$

i.e., $(\mathrm{CR})_1$ with some variables changed; the point of the substitution is that $\eta'' = \eta^t \mu^{-t'}$ is a root of unity (of order dividing r''), but $\mu'' = \eta^{s'} \mu^s$ is not. The method of 2 can now be applied.

4. When $\mu^{p'} = 1 = \eta^{q'}$, we can assume $W^{q''} = 1 = U^{p''} = V^{p''}$, where $p'' = \operatorname{lcm}\{q, p'\}$, and so the generated C^*-algebra A_4 is finite dimensional.

The preceding comments are summarized in the next theorem.

2. THEOREM. *A C^*-algebra A is isomorphic to a simple infinite dimensional quotient of $C^*(H_{6,4})$ if and only if A is isomorphic to $A_\theta^{6,4}$ for some irrational θ, or to a $C^*(\mathscr{C}(\boldsymbol{Z}_q^2 \times \boldsymbol{T}), \boldsymbol{Z}^2)$, as in case 1 or 2 above.*

A result that has been used implicitly above, and should be stated explicitly, is the analogue of Theorem 1 that holds for $A_i \cong C^*(\mathscr{C}(\boldsymbol{Z}_q^2 \times \boldsymbol{T}), \boldsymbol{Z}^2)$, $i = 1, 2$.

3. THEOREM. *For $i = 1, 2$, $A_i \cong C^*(\mathscr{C}(\boldsymbol{Z}_q^2 \times \boldsymbol{T}), \boldsymbol{Z}^2)$ is the unique (up to isomorphism) C^*-algebra generated by unitaries U, V, W, X and Y satisfying $(\mathrm{CR})_i$; A_i is simple and is universal for the equations $(\mathrm{CR})_i$.*

As for Theorem 1, the result is a consequence of the minimality and effectiveness of the flow involved.

Matrix representations of the non-faithful quotients

The algebras in 1 and 2 above have representations as matrix algebras with entries in simple C^*-algebras from groups of lower dimension. For the first one we need $A_{\varphi,\psi}^{5,2}$, the simple C^*-algebra generated by a faithful irreducible representation of $H_{5,2}$, in particular, by unitaries U_1, V_1 and W_1 satisfying

$$(\dagger) \qquad [U_1, V_1] = \mu_1, \quad [W_1, V_1] = \eta_1 \quad \text{and} \quad [U_1, W_1] = 1,$$

where $\mu_1 = e^{-2\pi i\varphi}$ and $\eta_1 = e^{-2\pi i\psi}$ are linearly independent; see [MW2; Theorem 2.1] (with the advice that the equations have been adjusted a little for the present context).

4. THEOREM. *The algebra A_1 in 1 above is isomorphic to $M_{q^2}(A_{\varphi,\psi}^{5,2})$ for*

$$e^{-2\pi i\varphi} = \mu^q \quad \text{and} \quad e^{-2\pi i\psi} = \eta^q.$$

Proof. Here we need $\mu_1 = \mu^q$ and $\eta_1 = \eta^q$, and unitaries U_1, V_1 and W_1 satisfying $(\dagger)$ and generating $A_{\theta,\varphi}^{5,2}$. Then define unitaries U', V', W', X' and Y' in $M_{q^2}(A_{\varphi,\psi}^{5,2})$ as follows (all unspecified entries being 0).

If $U_2 \in M_q(\mathrm{A}^{5,2}_{\varphi,\psi})$ has U_1's on the diagonal and $I \in M_q(\mathrm{A}^{5,2}_{\varphi,\psi})$ is the identity matrix, then

$U' \in M_q(M_q(\mathrm{A}^{5,2}_{\varphi,\psi})) = M_{q^2}(\mathrm{A}^{5,2}_{\varphi,\psi})$ has U_2 in the upper righthand corner and I's on the subdiagonal, i.e.,

$$U' = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & U_2 \\ I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & I & 0 \end{pmatrix}.$$

Let $V_2 \in M_q(\mathrm{A}^{5,2}_{\varphi,\psi})$ have $V_1, \bar{\eta}V_1, \bar{\eta}^2 V_1, \ldots, \bar{\eta}^{q-1} V_1$ on the diagonal, and let $X_2 \in M_q(\mathrm{A}^{5,2}_{\varphi,\psi})$ have $1, \bar{\lambda}, \bar{\lambda}^2, \ldots, \bar{\lambda}^{q-1}$ on the diagonal. Then

$V' \in M_q(M_q(\mathrm{A}^{5,2}_{\varphi,\psi}))$ has $V_2, (\mu X_2)^{-1} V_2, (\mu X_2)^{-2} V_2, \ldots, (\mu X_2)^{-(q-1)} V_2$ on the diagonal, i.e.,

$$V' = \begin{pmatrix} V_2 & 0 & 0 & \cdots & 0 \\ 0 & (\mu X_2)^{-1} V_2 & 0 & \cdots & 0 \\ 0 & 0 & (\mu X_2)^{-2} V_2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (\mu X_2)^{-(q-1)} V_2 \end{pmatrix}.$$

Let $W_2 \in M_q(\mathrm{A}^{5,2}_{\varphi,\psi})$ have W_1 in the upper righthand corner and 1's on the subdiagonal, i.e.,

$$W_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & W_1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Then

$W' \in M_q(M_q(\mathrm{A}^{5,2}_{\varphi,\psi}))$ has W_2's on the diagonal.

$X' \in M_q(M_q(\mathrm{A}^{5,2}_{\varphi,\psi}))$ has X_2's on the diagonal.

$Y' \in M_q(M_q(\mathrm{A}^{5,2}_{\varphi,\psi}))$ has $I, \bar{\lambda}I, \bar{\lambda}^2 I, \ldots, \bar{\lambda}^{q-1} I$ on the diagonal.

Then the unitaries U', V', W', X' and Y' satisfy $(\mathrm{CR})_1$, e.g., $[W_2, V_2] = \eta$ and $[W_2, X_2^{-j}] = \bar{\lambda}^j$, so $[W', V'] = \eta Y'$. They also generate $M_{q^2}(\mathrm{A}^{5,2}_{\varphi,\psi}) = M_q(M_q(\mathrm{A}^{5,2}_{\varphi,\psi}))$, so $\mathrm{A}_1 \cong M_{q^2}(\mathrm{A}^{5,2}_{\varphi,\psi})$. ∎

5. THEOREM. *The algebra* A_2 *in* 2 *above is isomorphic to* $M_Q(\mathrm{A}^3_\gamma)$ *for suitable* Q *and* γ.

Proof. The difference here is in the simple C^*-algebra A generated by unitaries U_1, V_1 and W_1 satisfying (†), when η is a primitive q'-th root of unity. Now $\eta_1 = \eta^q$ is a primitive r-th root of unity, say, for some r dividing q'; note that $\mathrm{lcm}\{q, r\} = \mathrm{lcm}\{q, q'\} = q''$. Then $\mathrm{A} \cong M_r(\mathrm{A}^3_\gamma)$ for $e^{-2\pi i\gamma} = \mu_1^r = \mu^{qr}$ [MW1; Theorem 2.3], and we can regard U_1, V_1 and W_1 as members of $M_r(\mathrm{A}^3_\gamma)$ generating $M_r(\mathrm{A}^3_\gamma)$. We use them to construct matrices $U', \ldots, Y'$ in $M_{q^2}(\mathrm{A}) = M_{q^2 r}(\mathrm{A}^3_\gamma)$ as in Theorem 4; then, with $Q = q^2 r$, these matrices satisfy $(\mathrm{CR})_2$ and generate $M_Q(\mathrm{A}^3_\gamma)$. ∎

REFERENCES

[EH] E.G. EFFROS AND F HAHN, Locally Compact Transformation Groups and C^*-algebras, Mem. Amer. Math. Soc., **75**, Providence, RI, 1967

[H] R.E. HOWE, On representations of discrete, finitely generated, torsion-free, nilpotent groups, Pacific J. Math., **73** (1977), 281–305.

[JM] H.D. JUNGHENN AND P MILNES, Analysis on a discrete 6-dimensional nilpotent group, preprint.

[MW1] P MILNES AND S. WALTERS, Simple quotients of the group C^*-algebra of a discrete 4-dimensional nilpotent group, Houston J. Math., **19** (1993), 615–636.

[MW2] P MILNES AND S. WALTERS, Simple infinite dimensional quotients of $C^*(G)$ for discrete 5-dimensional nilpotent groups G, Illinois J. Math., **41** (1997), 315–340.

[N] O. NIELSEN, Unitary Representations and Coadjoint Orbits of Low Dimensional Nilpotent Lie Groups, Queen's Papers in Pure and Applied Mathematics, **63**, Queen's University, 1983.

[T] E. THOMA, Über unitäre Darstellungen abzählbarer, diskreter Gruppen, Math. Ann., **153** (1964), 111–138.

[W] S. WALTERS, Chern characters of Fourier modules, submitted.

[Z-M] G. ZELLER-MEIER, Produits croisés d'une C^*-algèbre par un groupe d'automorphismes, J. Math. Pures Appl., **47** (1968), 101–239.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WESTERN ONTARIO
MIDDLESEX COLLEGE
LONDON, CANADA
N6A 5B7