H. UEDA
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MEROMORPHIC FUNCTIONS f AND g THAT SHARE TWO VALUES CM AND TWO OTHER VALUES IN THE SENSE OF $E_k(\beta, f) = E_k(\beta, g)$

HIDEHARU UEDA

1. Introduction

In this paper the term "*meromorphic function*" will mean a meromorphic function in C. We will use the standard notations of Nevanlinna theory: $T(r, f)$, $S(r, f)$, $m(r, \beta, f)$, $N(r, \beta, f)$, $\bar{N}(r, \beta, f)$, $N_1(r, \beta, f)$, $\bar{N}_1(r, \beta, f)$, $N_1(r, f)$, $\bar{N}_1(r, f)$, $\Theta(\beta, f)$ $(\beta \in C \cup \{\infty\})$, ... etc., and we assume that the reader is familiar with the basic results in Nevanlinna theory as found in [3].

For a nonconstant meromorphic function f, a number $\beta \in C \cup \{\infty\}$ and a positive integer or $+\infty$ k, we write $E_k(\beta, f) = \{z \in C : z$ *is a* β – *point of* f *with multiplicity less than or equal to* $k.\}$.

If two nonconstant meromorphic functions f and g satisfy $E_{+\infty}(\beta, f) = E_{+\infty}(\beta, g)$, then we say that *$f$ and g share β IM*. If f and g satisfy $E_k(\beta, f) = E_k(\beta, g)$ for all positive integers k, then we say that *f and g share β CM*.

The following Theorems A–C are due to Bhoosnurmath and Gopalakrishna [1]:

THEOREM A. *Let f and g be nonconstant meromorphic functions. Assume that there exist distinct 5 elements $a_1, \ldots, a_5$ in $C \cup \{\infty\}$ such that $E_k(a_j, f) = E_k(a_j, g)$ for $j = 1, \ldots, 5$, where k $(\geqq 3)$ is a positive integer or $+\infty$. Then $f \equiv g$.*

THEOREM B. *Let f and g be nonconstant meromorphic functions. Assume that there exist distinct 6 elements $a_1, \ldots, a_6$ in $C \cup \{\infty\}$ such that $E_2(a_j, f) = E_2(a_j, g)$ for $j = 1, \ldots, 6$. Then $f \equiv g$.*

THEOREM C. *Let f and g be nonconstant meromorphic functions. Assume that there exist distinct 7 elements $a_1, \ldots, a_7$ in $C \cup \{\infty\}$ such that $E_1(a_j, f) = E_1(a_j, g)$ for $j = 1, \ldots, 7$. Then $f \equiv g$.*

The case of $k = +\infty$ in Theorem A is a well-known result of Nevanlinna what is called *Five-Point Theorem* [5]. As we have pointed out in [6, p. 458], in

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the above three results, the assumption on the number of distinct elements $\{a_j\}$ satisfying $E_k(a_j, f) = E_k(a_j, g)$ cannot be improved.

In connection with Theorems B and C we showed in [7] the following Theorems D and E.

THEOREM D. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values* 0 *and* ∞ *CM, and further that they satisfy* $E_2(a_j, f) = E_2(a_j, g)$ *for* $j = 3, 4, 5$, *where* $a_3 = 1$, $a_4 = a$, $a_5 = b$. $(a, b \neq 0, \infty, 1;\ a \neq b)$ (i) *If* $\{a, b\} = \{\omega, \omega^2\}$, *where* $\omega(\neq 1)$ *is a third root of unity, then* $f^3 \equiv g^3$. (ii) *If* $\{a, b\} \neq \{\omega, \omega^2\}$, *then* $f \equiv g$.

THEOREM E. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values* 0 *and* ∞ *CM, and further that they satisfy* $E_1(a_j, f) = E_1(a_j, g)$ *for* $j = 3, \dots, 6$, *where* $a_3 = 1$, $a_4 = a$, $a_5 = b$, $a_6 = c$. $(a, b, c \neq 0, \infty, 1;\ a \neq b \neq c \neq a)$ (i) *If* $\{a, b, c\} = \{i, -1, -i\}$, *then* $f^4 \equiv g^4$. (ii) *If* $\{a, b, c\} = \{\alpha, -1, -\alpha\}$ $(\alpha \neq \pm i)$, *then* $f^2 \equiv g^2$. (iii) *If* $\{a, b, c\} \neq \{\alpha, -1, -\alpha\}$, *then* $f \equiv g$.

Gundersen [2] proved the following result which generalizes a well-known result of Nevanlinna what is called *Four-Point Theorem* [5].

THEOREM F. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values* 0 *and* ∞ *CM, and that they share two values* 1 *and* a $(\neq 0, \infty, 1)$ *IM.* (i) *If* $a = -1$, *then* $fg \equiv 1$, $f + g \equiv 0$ *or* $f \equiv g$. (ii) *If* $a = 1/2$, *then* $(f - (1/2))(g - (1/2)) \equiv 1/4$, $f + g \equiv 1$ *or* $f \equiv g$. (iii) *If* $a = 2$, *then* $(f - 1)(g - 1) \equiv 1$, $f + g \equiv 2$ *or* $f \equiv g$. (iv) *If* $a \neq -1, 1/2, 2$, *then* $f \equiv g$.

In this paper in relation to Theorems A and F we prove the following two results.

THEOREM 1. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values* 0 *and* ∞ *CM, and that they satisfy* $E_k(a_j, f) = E_k(a_j, g)$ *for* $j = 3, 4$, *where* $a_3 = 1$, $a_4 = a$ $(\neq 0, \infty, 1, -1)$ *and* k $(\geqq 12)$ *is a positive integer.* (i) *If* $a = 1/2$, *then* $(f - (1/2))(g - (1/2)) \equiv 1/4$, $f + g \equiv 1$ *or* $f \equiv g$. (ii) *If* $a = 2$, *then* $(f - 1)(g - 1) \equiv 1$, $f + g \equiv 2$ *or* $f \equiv g$. (iii) *If* $a \neq -1, 1/2, 2$, *then* $f \equiv g$.

THEOREM 2. *Let f and g be nonconstant meromorphic functions. Assume that f and g share two values* 0 *and* ∞ *CM, and that they satisfy* $E_k(a_j, f) = E_k(a_j, g)$ *for* $j = 3, 4$, *where* $a_3 = 1$, $a_4 = -1$ *and* k $(\geqq 7)$ *is a positive integer. Then* $fg \equiv 1$, $f + g \equiv 0$ *or* $f \equiv g$.

2. Notations and terminology

In this section, we introduce some notations and terminology which will be needed to prove Theorems 1 and 2.

⟨i⟩ Let f, g be distinct nonconstant meromorphic functions. For $r > 0$, put $T(r) = \max\{T(r,f), T(r,g)\}$. We write $\sigma(r) = S(r)$ for every function $\sigma : (0, +\infty) \to (-\infty, +\infty)$ satisfying $\sigma(r)/T(r) \to 0$ for $r \to +\infty$ possibly outside a set of finite Lebesgue measure.

⟨ii⟩ Let f, g be nonconstant meromorphic functions. We denote by $\overline{N}_c(r, \beta; f, g) \equiv \overline{N}_c(r, \beta)$ (resp. $\overline{N}_d(r, \beta; f, g) \equiv \overline{N}_d(r, \beta)$) the counting function of those common β-points of f and g with the same multiplicity (resp. with the different multiplicities), each point counted only once regardless of multiplicity, and we write $\overline{N}_i(r, \beta; f, g) \equiv \overline{N}_i(r, \beta) = \overline{N}_c(r, \beta) + \overline{N}_d(r, \beta)$.

We say that *f and g share β CM$''$* if $\overline{N}(r, \beta, f) - \overline{N}_c(r, \beta) = S(r, f)$ and $\overline{N}(r, \beta, g) - \overline{N}_c(r, \beta) = S(r, g)$ hold. Similarly, if $\overline{N}(r, \beta, f) - \overline{N}_i(r, \beta) = S(r, f)$ and $\overline{N}(r, \beta, g) - \overline{N}_i(r, \beta) = S(r, g)$ hold, then we say that *f and g share β IM$''$*. These notions CM$''$ and IM$''$ are slight generalizations of CM and IM, respectively.

⟨iii⟩ Let f and g be nonconstant meromorphic functions. For β, γ $(\in C \cup \{\infty\})$, $\beta \neq \gamma$ we put

$$m_{\beta,\gamma}(r) \equiv m_{\beta,\gamma}(r; f, g) = m(r, \beta, f) + m(r, \gamma, f) + m(r, \beta, g) + m(r, \gamma, g),$$

$$\begin{aligned}\overline{N}_{\beta,\gamma}(r) \equiv \overline{N}_{\beta,\gamma}(r; f, g) &= \overline{N}(r; f = \beta,\ g \neq \beta) + \overline{N}(r; f = \gamma,\ g \neq \gamma)\\ &\quad + \overline{N}(r; g = \beta,\ f \neq \beta) + \overline{N}(r; g = \gamma,\ f \neq \gamma),\\ \tilde{N}'_{\beta,\gamma}(r) &\equiv \tilde{N}'_{\beta,\gamma}(r; f, g) = \overline{N}_c(r, \beta) + \overline{N}_c(r, \gamma),\\ \tilde{N}''_{\beta,\gamma}(r) &\equiv \tilde{N}''_{\beta,\gamma}(r; f, g) = \overline{N}_d(r, \beta) + \overline{N}_d(r, \gamma),\end{aligned}$$

$$\tilde{N}_{\beta,\gamma}(r) \equiv \tilde{N}_{\beta,\gamma}(r; f, g) = \tilde{N}'_{\beta,\gamma}(r; f, g) + \tilde{N}''_{\beta,\gamma}(r; f, g) = \overline{N}_i(r, \beta; f, g) + \overline{N}_i(r, \gamma; f, g),$$

where for example, $\overline{N}(r; f = \beta,\ g \neq \beta)$ denotes the counting function of those β-points of f which are not β-points of g, each point counted only once.

3. Preparations for the proof of Theorems 1 and 2

We often need a slight generalization of Theorem F:

THEOREM F$'$. *Theorem F remains still valid if CM and IM are replaced by CM$''$ and IM$''$, respectively.*

In order to prove this fact we have only to use the argument (due to Mues) of the proof of Theorem 1 in [4] by replacing CM and IM by CM$''$ and IM$''$, respectively.

In the rest of this section, we assume that f and g are *distinct* nonconstant meromorphic functions sharing $a_1 = 0$ and $a_2 = \infty$ CM and satisfying $E_k(a_j, f) = E_k(a_j, g)$ for $j = 3, 4$, where $a_3 = 1$, $a_4 = a$ $(\neq 0, \infty, 1)$ and k $(\geqq 2)$ is a positive integer. We write, for example, $N(r, 0, f) = N(r, 0, g) = N(r, 0)$, $N(r, \infty, f) = N(r, \infty, g) = N(r, \infty)$, $\overline{N}(r, 0, f) = \overline{N}(r, 0, g) = \overline{N}(r, 0)$, $\overline{N}(r, \infty, f) = \overline{N}(r, \infty, g) = \overline{N}(r, \infty)$, $N_1(r, 0, f) = N_1(r, 0, g) = N_1(r, 0)$, $N_1(r, \infty, f) = N_1(r, \infty, g) = N_1(r, \infty)$, $\overline{N}_1(r, 0, f) = \overline{N}_1(r, 0, g) = \overline{N}_1(r, 0)$, $\overline{N}_1(r, \infty, f) = \overline{N}_1(r, \infty, g) = \overline{N}_1(r, \infty)$.

LEMMA 1. $S(r) = S(r,f) = S(r,g)$.

Proof. Let $d \in C$ be different from a_j $(j = 1,2,3,4)$, and let $b_j = (a_j - d)^{-1}$ $(j=1,2,3,4)$. Then $b_1,\dots,b_4$ are all distinct and finite. If we put $F=(f-d)^{-1}$ and $G = (g-d)^{-1}$, then F and G share b_1 and b_2 CM and satisfy $E_k(b_j,F) = E_k(b_j,G)$ for $j = 3,4$. By the second fundamental theorem and the fact that $F \not\equiv G$

$$\begin{aligned} 2T(r,F) &\leqq \sum_{j=1}^{4} \bar{N}(r,b_j,F) + S(r,F) \\ &\leqq \sum_{j=1}^{2} \bar{N}(r,b_j,F) + \tilde{N}_{b_3,b_4}(r;F,G) \\ &\quad + \sum_{j=3}^{4} \bar{N}(r;F=b_j,\ G \neq b_j) + S(r,F) \\ &\leqq N(r,0,F-G) + \{2/(k+1)\}T(r,F) + S(r,F) \\ &\leqq T(r,F) + T(r,G) + \{2/(k+1)\}T(r,F) + S(r,F), \end{aligned}$$

i.e.,

$$T(r,F) \leqq \{(k+1)/(k-1)\}T(r,G) + S(r,F). \tag{3.1}$$

(3.1) is still valid when we exchange F and G, so that

$$T(r,G) \leqq \{(k+1)/(k-1)\}T(r,F) + S(r,G). \tag{3.2}$$

Taking $T(r,F) = T(r,f) + O(1)$ and $T(r,G) = T(r,g) + O(1)$ into account, we immediately deduce Lemma 1 from (3.1) and (3.2). ■

LEMMA 2. *Let $\tilde{n}(r;f'=g'=0,f\neq 0,g\neq 0)$ denote the number of distinct common zeros of f' and g' which are neither zeros of f nor g in $|z| \leqq r$. Put $\tilde{N}(r;f'=g'=0,f\neq 0,g\neq 0) = \int_0^r\{\tilde{n}(t;f'=g'=0,f\neq 0,g\neq 0)-\tilde{n}(0;f'=g'=0,f\neq 0,g\neq 0)\}/t\,dt + \tilde{n}(0;f'=g'=0,f\neq 0,g\neq 0)\log r$. If g/f is not a constant, then $\tilde{N}(r;f'=g'=0,\ f\neq 0,\ g\neq 0) = S(r)$.*

Proof. Since f and g share 0 and ∞ CM, there is an entire function α satisfying $g = e^{\alpha}f$, where α is nonconstant. Assume that there is a point z_0 such that $f'(z_0) = g'(z_0) = 0$, $f(z_0) \neq 0$ and $g(z_0) \neq 0$. The differentiation of $g = e^{\alpha}f$ gives $g' = e^{\alpha}(\alpha' f + f')$, and so we have $\alpha'(z_0) = 0$. Since α is entire, we deduce using the lemma of the logarithmic derivative that

$$\begin{aligned} \tilde{N}(r;f'=g'=0,f\neq 0,g\neq 0) &\leqq \bar{N}(r,0,\alpha') \leqq m(r,\alpha') + O(1) \\ &= m\{r,(e^{\alpha})'/e^{\alpha}\} + O(1) = S(r,e^{\alpha}) \\ &= S(r,g/f) \leqq S(r,f) + S(r,g) = S(r). \end{aligned}$$ ■

LEMMA 3. *Let $n_1'(r,f)$ denote the number of multiple points of f in $|z| \leqq r$ such that $f \neq 0, \infty, 1, a$, where a point of multiplicity m is counted $(m-1)$ times, and put $N_1'(r,f) = \int_0^r \{n_1'(t,f) - n_1'(0,f)\}/t\,dt + n_1'(0,f)\log r$. If $N_1'(r,g)$ is similarly defined, then*

$$\tilde{N}_{1,a}''(r;f,g) + k\overline{N}_{1,a}(r;f,g) + N_1'(r,f) + N_1'(r,g) \leqq 2\{\overline{N}(r,0) + \overline{N}(r,\infty)\} + S(r). \tag{3.3}$$

Proof. By the first and the second fundamental theorems

$$\begin{aligned}
(3.3)'\quad & m_{1,a}(r;f,g) + 2\tilde{N}_{1,a}'(r;f,g) + 3\tilde{N}_{1,a}''(r;f,g) + (k+1)\overline{N}_{1,a}(r;f,g) \\
&\leqq m_{1,a}(r;f,g) + N(r,1,f) + N(r,1,g) + N(r,a,f) + N(r,a,g) \\
&= 2\{T(r,f) + T(r,g)\} + O(1) \\
&\leqq \sum_{j=1}^{4}\{\overline{N}(r,a_j,f) + \overline{N}(r,a_j,g)\} - \{N_1'(r,f) + N_1'(r,g)\} + S(r) \\
&= 2\{\overline{N}(r,0) + \overline{N}(r,\infty) + \tilde{N}_{1,a}(r;f,g)\} + \overline{N}_{1,a}(r;f,g) \\
&\quad - \{N_1'(r,f) + N_1'(r,g)\} + S(r),
\end{aligned}$$

from which we immediately deduce (3.3). ■

Now, we introduce some auxiliary functions:

$$\phi_1 = \frac{f'g'(f-g)^2}{fg(f-1)(g-1)(f-a)(g-a)} \quad (\not\equiv 0), \tag{3.4}$$

$$\phi_2 = \frac{f'f}{(f-1)(f-a)} - \frac{g'g}{(g-1)(g-a)}, \tag{3.5}$$

$$\phi_3 = \frac{f'}{f(f-1)(f-a)} - \frac{g'}{g(g-1)(g-a)}, \tag{3.6}$$

$$\phi_4 = \left(\frac{f''}{f'} - 2\frac{f'}{f} - \frac{f'}{f-1} - \frac{f'}{f-a}\right) - \left(\frac{g''}{g'} - 2\frac{g'}{g} - \frac{g'}{g-1} - \frac{g'}{g-a}\right), \tag{3.7}$$

$$\phi_5 = \left(\frac{f''}{f'} + 2\frac{f'}{f} - \frac{f'}{f-1} - \frac{f'}{f-a}\right) - \left(\frac{g''}{g'} + 2\frac{g'}{g} - \frac{g'}{g-1} - \frac{g'}{g-a}\right), \tag{3.8}$$

$$\phi_6 = \phi_4^2 - (1+a)^2\phi_1, \tag{3.9}$$

$$\phi_7 = \phi_5^2 - (1+a)^2\phi_1, \tag{3.10}$$

$$\phi_8 = \left(\frac{f''}{f'} - 2\frac{f'}{f} - \frac{f'}{f-1} + \frac{af'}{f-a}\right) - \left(\frac{g''}{g'} - 2\frac{g'}{g} - \frac{g'}{g-1} + \frac{ag'}{g-a}\right), \tag{3.11}$$

and

$$\phi_9 = \left(\frac{f''}{f'} + (1-a)\frac{f'}{f} + \frac{af'}{f-1} - \frac{f'}{f-a}\right) - \left(\frac{g''}{g'} + (1-a)\frac{g'}{g} + \frac{ag'}{g-1} - \frac{g'}{g-a}\right). \tag{3.12}$$

We remark that for the case $a = -1$, $\phi_8 \equiv \phi_4$ and $\phi_9 \equiv \phi_5$ hold. With the aid of these auxiliary functions we obtain some basic estimates:

LEMMA 4. (i)

$$2\{N_1(r,0) + N_1(r,\infty)\} + \bar{N}_1'(r,f) + \bar{N}_1'(r,g) \leqq \bar{N}_{1,a}(r) + S(r). \tag{3.13}$$

(ii) *If neither* $\phi_2 \equiv 0$ *nor* $\phi_3 \equiv 0$, *then*

$$\bar{N}(r,0) + \bar{N}(r,\infty) \leqq 2\{\bar{N}_{1,a}(r) + \tilde{N}_{1,a}''(r)\} + S(r). \tag{3.14}$$

(iii) *If neither* $\phi_6 \equiv 0$ *nor* $\phi_7 \equiv 0$, *then*

$$\bar{N}(r,0) + \bar{N}(r,\infty) \begin{cases} \leqq 4\bar{N}_{1,a}(r) + 4\{\bar{N}_1'(r,f) + \bar{N}_1'(r,g)\} \\ \qquad + \bar{N}_1(r,0) + \bar{N}_1(r,\infty) + S(r) \quad (a \neq -1), \\ \leqq 2\bar{N}_{1,a}(r) + 2\{\bar{N}_1'(r,f) + \bar{N}_1'(r,g)\} \\ \qquad + \bar{N}_1(r,0) + \bar{N}_1(r,\infty) + S(r) \quad (a = -1), \end{cases} \tag{3.15}$$

where for example, $\bar{N}_1'(r,f)$ *denotes the counting function of multiple points of* $f(\neq 0, \infty, 1, a)$, *each point counted only once.*

(iv) *If neither* $\phi_8 \equiv 0$ *nor* $\phi_9 \equiv 0$, *then*

$$\bar{N}(r,0) + \bar{N}(r,\infty) \leqq \tilde{N}_{1,a}''(r) + 2\bar{N}_{1,a}(r) + 2\{\bar{N}_1'(r,f) + \bar{N}_1'(r,g)\} + \bar{N}_1(r,0) + \bar{N}_1(r,\infty) + S(r) \quad (a \neq -1). \tag{3.16}$$

Proof. (i) From the fundamental estimate of the logarithmic derivative it follows that $m(r,\phi_1) = S(r)$ (cf. [4, p. 171]). The poles of ϕ_1 occur with multiplicity 1 due to the case [i] the 1- or a- points of f (resp. g) which are simple points of $g(\neq 1, a)$ (resp. $f(\neq 1, a)$), and with multiplicity 2 due to the case [ii] the common roots of $f = 1$ (resp. $f = a$) and $g = a$ (resp. $g = 1$). Hence we have $N(r,\infty,\phi_1) = \bar{N}_{1,a}(r) - \{\bar{N}_1'(r,f;g=1,a) + \bar{N}_1'(r,g;f=1,a)\}$, where for example, $\bar{N}_1'(r,f;g=1,a)$ denotes the counting function of those multiple points of $f(\neq 0,\infty,1,a)$ which are either 1- or a-points of g, each point counted only once. Since $\phi_1 \not\equiv 0$, we obtain from the first fundamental theorem

$$\begin{aligned} &2\{N_1(r,0) + N_1(r,\infty)\} + \bar{N}_1'(r,f;g \neq 1,a) + \bar{N}_1'(r,g;f \neq 1,a) \\ &\leqq N(r,0,\phi_1) \leqq T(r,\phi_1) + O(1) \\ &= \bar{N}_{1,a}(r) - \{\bar{N}_1'(r,f;g=1,a) + \bar{N}_1'(r,g;f=1,a)\} + S(r), \end{aligned} \tag{3.13'}$$

where for example, $\overline{N}_1'(r, f; g \neq 1, a)$ denotes the counting function of those multiple points of $f(\neq 0, \infty, 1, a)$ which are neither 1- nor a-points of g, each point counted only once. From (3.13)′ we immediately deduce (3.13). ■

(ii) From our assumption that $\phi_2 \not\equiv 0$ and $\phi_3 \not\equiv 0$, it follows that

$$\begin{aligned}\overline{N}(r,0) &\leqq N(r,0,\phi_2) \leqq T(r,\phi_2) + O(1) = m(r,\phi_2) + N(r,\infty,\phi_2) + O(1)\\ &\leqq \overline{N}_{1,a}(r) + \tilde{N}''_{1,a}(r) + S(r)\end{aligned}$$

and

$$\begin{aligned}\overline{N}(r,\infty) &\leqq N(r,0,\phi_3) \leqq T(r,\phi_3) + O(1) = m(r,\phi_3) + N(r,\infty,\phi_3) + O(1)\\ &\leqq \overline{N}_{1,a}(r) + \tilde{N}''_{1,a}(r) + S(r).\end{aligned}$$

Combining these inequalities we have (3.14). ■

(iii) Let z_0 be a common simple zero of f and g. Then we easily see that $\phi_6(z_0) = 0$. Hence our assumption $\phi_6 \not\equiv 0$ gives

$$\begin{aligned}\overline{N}(r,0) &\leqq N(r,0,\phi_6) + \overline{N}_1(r,0) \leqq T(r,\phi_6) + \overline{N}_1(r,0) + O(1)\\ &= m(r,\phi_6) + N(r,\infty,\phi_6) + \overline{N}_1(r,0) + O(1)\\ &= N(r,\infty,\phi_6) + \overline{N}_1(r,0) + S(r)\\ &\leqq 2\overline{N}_{1,a}(r) + 2\{\overline{N}_1'(r,f) + \overline{N}_1'(r,g)\} + \overline{N}_1(r,0) + S(r).\end{aligned}$$

(In particular, if $a = -1$, then we obtain

$$\begin{aligned}\overline{N}(r,0) &\leqq N(r,0,\phi_4) + \overline{N}_1(r,0) \leqq T(r,\phi_4) + \overline{N}_1(r,0) + O(1)\\ &= N(r,\infty,\phi_4) + \overline{N}_1(r,0) + S(r)\\ &\leqq \overline{N}_{1,a}(r) + \overline{N}_1'(r,f) + \overline{N}_1'(r,g) + \overline{N}_1(r,0) + S(r).)\end{aligned}$$

Next, let z_∞ be a common simple pole of f and g. Then we have $\phi_7(z_\infty) = 0$. Using the assumption that $\phi_7 \not\equiv 0$, we obtain

$$\begin{aligned}\overline{N}(r,\infty) &\leqq N(r,0,\phi_7) + \overline{N}_1(r,\infty) \leqq T(r,\phi_7) + \overline{N}_1(r,\infty) + O(1)\\ &= m(r,\phi_7) + N(r,\infty,\phi_7) + \overline{N}_1(r,\infty) + O(1)\\ &= N(r,\infty,\phi_7) + \overline{N}_1(r,\infty) + S(r)\\ &\leqq 2\overline{N}_{1,a}(r) + 2\{\overline{N}_1'(r,f) + \overline{N}_1'(r,g)\} + \overline{N}_1(r,\infty) + S(r).\end{aligned}$$

(In particular, if $a = -1$, then we get

$$\begin{aligned}\bar{N}(r,\infty) &\leqq N(r,0,\phi_5)+\bar{N}_1(r,\infty) \leqq T(r,\phi_5)+\bar{N}_1(r,\infty)+O(1)\\ &= N(r,\infty,\phi_5)+\bar{N}_1(r,\infty)+S(r)\\ &\leqq \bar{N}_{1,a}(r)+\{\bar{N}_1'(r,f)+\bar{N}_1'(r,g)\}+\bar{N}_1(r,\infty)+S(r).)\end{aligned}$$

The combination of the above two estimates yields (3.15). ■

(iv) If z_0 (resp. z_∞) is a common simple zero (resp. pole) of f and g, then $\phi_8(z_0)=0$ (resp. $\phi_9(z_\infty)=0$). Since we assume that $\phi_8 \not\equiv 0$ and $\phi_9 \not\equiv 0$, we easily see that

$$\begin{aligned}\bar{N}(r,0)+\bar{N}(r,\infty) &\leqq N(r,0,\phi_8)+N(r,0,\phi_9)+\bar{N}_1(r,0)+\bar{N}_1(r,\infty)\\ &\leqq N(r,\infty,\phi_8)+N(r,\infty,\phi_9)+\bar{N}_1(r,0)+\bar{N}_1(r,\infty)+S(r)\\ &\leqq \tilde{N}_{1,a}''(r)+2\bar{N}_{1,a}(r)+2\{\bar{N}_1'(r,f)+\bar{N}_1'(r,g)\}\\ &\quad +\bar{N}_1(r,0)+\bar{N}_1(r,\infty)+S(r).\end{aligned}$$ ■

4. Proof of Theorems 1 and 2

In what follows we assume that f and g are *distinct* and satisfy the assumptions of Theorem 1 or 2, and so there is an entire function α satisfying $g=e^\alpha f$ $(e^\alpha \not\equiv 1)$.

CASE 1. We first consider the case that e^α is a constant C $(\neq 0,1)$. From the assumptions $E_k(1,f)=E_k(1,g)$ and $E_k(a,f)=E_k(a,g)$ it follows that $\Theta(1,g)$, $\Theta(a,g) \geqq k/(k+1)$. If $C \neq a$, we also obtain $\Theta(C,g) \geqq k/(k+1)$, and so $\Theta(1,g)+\Theta(a,g)+\Theta(C,g) \geqq 3k/(k+1)>2$, a contradiction. This shows $C=a$. Further if $a^2 \neq 1$, we also obtain $\Theta(a^2,g) \geqq k/(k+1)$, and so $\Theta(1,g)+\Theta(a,g)+\Theta(a^2,g) \geqq 3k/(k+1)>2$, a contradiction. This shows $a^2=1$, i.e., $a=-1$ and $f+g\equiv 0$. In this case we remark that $N(r,1,f)=N(r,-1,g)$ *and* $N(r,-1,f)=N(r,1,g)$ *are not necessarily* $S(r)$!

CASE 2. We next consider the case that e^α is nonconstant. We divide our argument into several subcases:

2.1. The case $\phi_2 \equiv 0$

$\phi_2 \equiv 0$ implies that any 1- and a-point of f (resp. g) is a 1- or an a-point of g (resp. f). By making use of Lemma 2, we deduce from the assumptions $E_k(a_j,f)=E_k(a_j,g)$ for $j=3,4$ with $a_3=1$, $a_4=a$ that $\bar{N}(r;f=1,\ g=a)+\bar{N}(r;f=a,\ g=1)=S(r)$, (where $\bar{N}(r;f=1,g=a)$ denotes the counting function of common roots of $f=1$ and $g=a$, each counted only once,) and so by Lemma 1 f and g share two values 1 and a IM$''$. Hence by Theorem F$'$ f and g are connected with one of the relations stated in Theorem F. Further, straightforward computations show that only two relations $(f-(1/2))(g-(1/2)) \equiv 1/4$ (with $a=1/2$) and $(f-1)(g-1) \equiv 1$ (with $a=2$) are suitable for $\phi_2 \equiv 0$.

2.2. The case $\phi_3 \equiv 0$

The same reasoning as in the case 2.1 shows that only two relations $f+g \equiv 2$ (with $a=2$) and $f+g \equiv 1$ (with $a=1/2$) are suitable for $\phi_3 \equiv 0$.

2.3. The case $\phi_6 \equiv 0$

First we consider the case $a \neq -1$. By (3.9)

$$\phi_4^2 \equiv (1+a)^2\phi_1. \tag{4.1}$$

The poles of the right hand side of (4.1) occur with multiplicity 1 due to the case [i] the 1- or a-points of f (resp. g) which are simple points of $g(\neq 1, a)$ (resp. $f(\neq 1, a)$), and with multiplicity 2 due to the case [ii] the common roots of $f=1$ (resp. $f=a$) and $g=a$ (resp. $g=1$).

On the other hand, the poles of the left hand side of (4.1) occur with multiplicity 2 due to the following two cases:

[iii] The 1- or a-points of f (resp. g) which are neither 1- nor a-points of g (resp. f),

[iv] the zeros of f' such that $f \neq 0, 1, a$ or the zeros of g' such that $g \neq 0, 1, a$, where the multiplicities of the zeros of f' and g' are different.

Hence we see that there are no points satisfying the above [i], [ii], [iii] or [iv], so that f and g share 1 and a IM. Therefore by Theorem F, f and g are connected with one of the relations stated in Theorem F. Further straightforward computations show that only two relations $(f-(1/2))(g-(1/2)) \equiv 1/4$ (with $a=1/2$) and $(f-1)(g-1) \equiv 1$ (with $a=2$) are suitable for $\phi_6 \equiv 0$.

We next consider the case $a=-1$. In this case $\phi_6 \equiv 0$ implies $\phi_4 \equiv 0$. $\phi_4 \equiv 0$ implies that any 1- and a-point of f (resp. g) is a 1- or an a-point of g (resp. f). The same argument as in the case 2.1 yields that f and g are connected with the relation with $a=-1$ stated in Theorem F, i.e., $fg \equiv 1$. But, a direct computation shows that this is not suitable for $\phi_4 \equiv 0$.

2.4. The case $\phi_7 \equiv 0$

The same reasoning as in the case 2.3 shows that only two relations $f+g \equiv 2$ (with $a=2$) and $f+g \equiv 1$ (with $a=1/2$) are suitable for $\phi_7 \equiv 0$.

2.5. The case $\phi_8 \equiv 0$

If $a=-1$, then $\phi_8 \equiv \phi_4$. Since we have already handled the case $\phi_4 \equiv 0$ with $a=-1$ in 2.3, we may consider the case $a \neq -1$. First we easily see that f and g share 1 IM by considering the residue of ϕ_8 at any 1-point of f or g, where we used the assumption $a \neq -1$. Next, we prove that f and g share a IM$''$, i.e., $\overline{N}(r; f=a, g\neq a) + \overline{N}(r; g=a, f\neq a) = S(r)$. To show this, we suppose that $\overline{N}(r; f=a, g\neq a) + \overline{N}(r; g=a, f\neq a) \neq S(r)$, and will seek a contradiction. Under this assumption, we have $-1 < a < 0$. In fact, (without loss of generality) we may assume that $\overline{N}(r; f = a,\ g \neq a) \neq S(r)$. From Lemma 2 we see that there exists a point z_a satisfying $f(z_a) = a$ with multiplicity p $(\geqq k+1)$ and $g(z_a) = b(\neq a, 1, 0, \infty)$ with multiplicity 1. By the computation of the residue of ϕ_8 at z_a we have $p-1+ap = 0$, i.e., $(a+1)p=1$, which gives $-1 < a < 0$. Further the same reasoning shows that if $\overline{N}(r; f = a,\ g \neq a) \neq S(r)$, then any a-point of f which is not an a-point of g has multiplicity $\geqq (a+1)^{-1} \equiv p_0$ $(\geqq k+1 \geqq 13)$. In the same way, if $\overline{N}(r; g = a,\ f \neq a) \neq S(r)$, then any a-point of g which is not an a-point of f has multiplicity $\geqq (a+1)^{-1} \equiv p_0$ $(\geqq k+1 \geqq 13)$. Hence (by taking the fact that f and g share 1 IM into account) in the same way as in $(3.3)'$ in Lemma 3 we have

$$\begin{aligned} m_{1,a}(r; f, g) &+ 2\tilde{N}'_{1,a}(r; f, g) + 3\tilde{N}''_{1,a}(r; f, g) \\ &+ p_0\{\overline{N}(r; f = a,\ g \neq a) + \overline{N}(r; g = a,\ f \neq a)\} \\ &\leqq 2\{\overline{N}(r, 0) + \overline{N}(r, \infty) + \tilde{N}_{1,a}(r; f, g)\} + \overline{N}(r; f = a,\ g \neq a) \\ &+ \overline{N}(r; g = a,\ f \neq a) - \{N'_1(r, f) + N'_1(r, g)\} + S(r), \end{aligned}$$

and so

$$\begin{aligned} (4.2) \qquad p_0\{\overline{N}(r; f = a,\ g \neq a) + \overline{N}(r; g = a,\ f \neq a)\} &\leqq 2\{\overline{N}(r, 0) + \overline{N}(r, \infty)\} \\ &+ \{\overline{N}(r; f = a,\ g \neq a) + \overline{N}(r; g = a,\ f \neq a)\} + S(r). \end{aligned}$$

If z_β satisfies $f'(z_\beta)=0$, $f(z_\beta) \neq 0, 1, a$, (resp. $g'(z_\beta)=0$, $g(z_\beta) \neq 0, 1, a$) then $\phi_8 \equiv 0$ implies that $g'(z_\beta) = 0$, $g(z_\beta) \neq 0, 1$ (resp. $f'(z_\beta) = 0$, $f(z_\beta) \neq 0, 1$). Hence by Lemma 2

$$(4.3) \qquad \overline{N}'_1(r, f) + \overline{N}'_1(r, g) \leqq 2\tilde{N}(r; f' = g' = 0,\ f \neq 0,\ g \neq 0) = S(r).$$

In view of (3.13) we have

$$(4.4) \quad 2\{N_1(r, 0) + N_1(r, \infty)\} \leqq \overline{N}(r; f = a,\ g \neq a) + \overline{N}(r; g = a,\ f \neq a)\} + S(r).$$

Since we have already considered the case $\phi_6 \equiv 0$ in 2.3 and $\phi_7 \equiv 0$ in 2.4, we may now consider the case $\phi_6 \not\equiv 0$ and $\phi_7 \not\equiv 0$. Substituting (4.3) and (4.4) into (3.15) with $a \neq -1$, we obtain

$$(4.5) \quad 2\{\overline{N}(r, 0) + \overline{N}(r, \infty)\} \leqq 9\{\overline{N}(r; f = a,\ g \neq a) + \overline{N}(r; g = a,\ f \neq a)\} + S(r).$$

The combination of (4.2) and (4.5) gives $p_0 \leqq 10$, which is a contradiction. This proves that f and g share a IM$''$. Thus we deduce from Theorem F$'$ that f and g are connected with one of the relations with $a \neq -1$ stated in Theorem F. But straightforward computations show that none of the relations stated in Theorem F are suitable for $\phi_8 \equiv 0$, $\phi_6 \not\equiv 0$ and $\phi_7 \not\equiv 0$.

2.6. The case $\phi_9 \equiv 0$ (, $\phi_6 \not\equiv 0$, $\phi_7 \not\equiv 0$)

The same reasoning as in the case 2.5 shows that there is not a pair of f and g satisfying $\phi_9 \equiv 0$, $\phi_6 \not\equiv 0$ and $\phi_7 \not\equiv 0$.

2.7. The case $\phi_2 \not\equiv 0$, $\phi_3 \not\equiv 0$, $\phi_6 \not\equiv 0$, $\phi_7 \not\equiv 0$, $\phi_8 \not\equiv 0$, $\phi_9 \not\equiv 0$

First we consider the case $a \neq -1$. Combining (3.3), (3.15) and (3.13), we have

$$\tilde{N}''_{1,a}(r) + (k-15)\bar{N}_{1,a}(r) \leqq S(r). \tag{4.6}$$

On the other hand, using (3.3), (3.16) and (3.13) we have

$$(k-7)\bar{N}_{1,a}(r) \leqq \tilde{N}''_{1,a}(r) + S(r). \tag{4.7}$$

Substituting (4.7) into (4.6), it follows that $(k-11)\bar{N}_{1,a}(r) \leqq S(r)$. Since $k \geqq 12$, this implies that $\bar{N}_{1,a}(r) = S(r)$, and so $\tilde{N}''_{1,a}(r) = S(r)$ by (4.6).

Now assume that $a = -1$. Combining (3.3) and (3.14), we have

$$(k-4)\bar{N}_{1,a}(r) \leqq 3\tilde{N}''_{1,a}(r) + S(r). \tag{4.8}$$

On the other hand, we use (3.3), (3.15) and (3.13) to obtain

$$\tilde{N}''_{1,a}(r) + (k-7)\bar{N}_{1,a}(r) \leqq S(r). \tag{4.9}$$

Taking the fact $k \geqq 7$ into account, we deduce from (4.8) and (4.9) that $\tilde{N}''_{1,a}(r) = S(r)$ and $\bar{N}_{1,a}(r) = S(r)$.

Hence, $\bar{N}_{1,a}(r) = S(r)$ and $\tilde{N}''_{1,a}(r) = S(r)$ hold in both cases. From (3.13) and (3.14) we obtain $N(r,0) + N(r,\infty) = S(r)$, and so by Lemma 1 and the second fundamental theorem $\bar{N}(r,1,f)$, $\bar{N}(r,a,f) = T(r,f) + S(r)$ and $\bar{N}(r,1,g)$, $\bar{N}(r,a,g) = T(r,g) + S(r)$. On the other hand, $\bar{N}_{1,a}(r) = S(r)$ implies that f and g share two values 1 and a IM$''$, and so we deduce from Theorem F$'$ that f and g are connected with one of the relations in Theorem F. Therefore we obtain $fg \equiv 1$ with $a = -1$ in this case.

This completes the proof of Theorems 1 and 2. ■

Remark 1. The author does not know whether Theorem 1 holds for positive integers k $(3 \leqq k \leqq 11)$ or not.

Remark 2. The author does not know whether Theorem 2 holds for positive integers k $(3 \leqq k \leqq 6)$ or not.

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DEPARTMENT OF MATHEMATICS
DAIDO INSTITUTE OF TECHNOLOGY
DAIDO, MINAMI, NAGOYA 457-8530
JAPAN