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A RELATION BETWEEN STANDARD CONJECTURES AND THEIR ARITHMETIC ANALOGUES

Dedicated to the memory of Professor Nobuo Sasakura

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1. Introduction

Let X be a smooth projective variety of dimension n defined over an algebraically closed field k. We denote by $A^p(X)$ the vector space over $\boldsymbol{R}$ consisting of algebraic cycles of codimension p modulo homological equivalence. In [8] Grothendieck conjectured that $A^p(X)$ behaves like complex cohomology.

CONJECTURE 1.1 (Standard conjectures). *Let H be an ample line bundle on X and*

$$L_H : A^p(X) \to A^{p+1}(X)$$

the homomorphism intersecting with the first Chern class $c_1(H)$. Then the intersection pairing

$$\langle\,,\rangle = \deg(\cdot) : A^p(X) \times A^{n-p}(X) \to \boldsymbol{R}$$

is nondegenerate and for $p \leq n/2$, we have the following:

i) $L_H^{n-2p} : A^p(X) \to A^{n-p}(X)$ *is an isomorphism.*

ii) *For* $0 \neq x \in A^p(X)$ *such that* $L_H^{n+1-2p}(x) = 0$, $(-1)^p \deg(L_H^{n-2p}(x)x)$ *is positive.*

i) is called the hard Lefschetz conjecture and ii) is called the Hodge index conjecture. When the characteristic of k is zero, the Hodge index conjecture is true.

On the other hand, the notion of arithmetic cycles and the intersection theory for them were established by Arakelov [1] for surfaces and Gillet and Soulé [6] for higher dimensional varieties. Then it seems quite natural to ask whether analogues of standard conjectures hold in this situation.

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Let X be a regular scheme which is projective and flat over $\boldsymbol{Z}$. Such a scheme is called an arithmetic variety. For an arithmetic variety X the arithmetic Chow group $\widehat{CH}^p(X)$ is defined and the intersection product on $\widehat{CH}^p(X)_{\boldsymbol{Q}}$ is established in [6]. We denote by F_∞ the complex conjugation on the complex manifold $X(\boldsymbol{C})$ associated with the scheme $X \otimes_{\boldsymbol{Z}} \boldsymbol{C}$. A line bundle H on X together with an F_∞-invariant smooth hermitian metric $\| \|$ on the pull back $H_{\boldsymbol{C}}$ on $X(\boldsymbol{C})$ is called a hermitian line bundle. For a hermitian line bundle $(H, \| \|)$ on X, the arithmetic first Chern class $\hat{c}_1(H, \| \|) \in \widehat{CH}^1(X)$ can be defined. By intersecting with this class we obtain a homomorphism

$$L_{H,\| \|} : \widehat{CH}^p(X)_{\boldsymbol{R}} \to \widehat{CH}^{p+1}(X)_{\boldsymbol{R}}.$$

In [7] Gillet and Soulé proposed the following conjectures.

Conjecture 1.2 (Arithmetic analogues of standard conjectures). *Let n be the relative dimension of X over $\boldsymbol{Z}$ and H an ample line bundle on X. Then there exists an F_∞-invariant hermitian metric $\| \|$ on $H_{\boldsymbol{C}}$ satisfying the following for $2p \leq n+1$:*

i) *The intersection pairing*

$$\langle \, , \rangle = \widehat{\deg}(\, . \,) : \widehat{CH}^p(X)_{\boldsymbol{R}} \times \widehat{CH}^{n+1-p}(X)_{\boldsymbol{R}} \to \boldsymbol{R}$$

is nondegenerate.

ii) $L_{H,\| \|}^{n+1-2p} : \widehat{CH}^p(X)_{\boldsymbol{R}} \to \widehat{CH}^{n+1-p}(X)_{\boldsymbol{R}}$ *is an isomorphism.*

iii) *For $0 \neq x \in \widehat{CH}^p(X)_{\boldsymbol{R}}$ such that $L_{H,\| \|}^{n+2-2p}(x) = 0$, $(-1)^p \widehat{\deg}(L_{H,\| \|}^{n+1-2p}(x)x)$ is positive.*

When $n = 1$, iii) of Conjecture 1.2 was proved by Faltings [5] and by Hriljac [9] and this result was extended to higher dimensional varieties by Moriwaki [13]. He proved iii) of Conjecture 1.2 for any arithmetically ample hermitian line bundle $(H, \| \|)$ on an arithmetic variety X when $p = 1$. In [10] Künnemann reduced the conjectures to the similar conjectures for Arakelov Chow groups and proved them for projective spaces.

The aim of this paper is to resolve Conjecture 1.2 into other well-known conjectures including original standard conjectures. In [5, 9] for an arithmetic surface X the Hodge index theorem was proved by the positivity of Néron-Tate height pairing of the Jacobian of X and by some arguments on the intersection of cycles of X whose supports do not meet the generic fiber $X_{\boldsymbol{Q}}$. Our main result can be regarded as a generalisation of their methods to higher dimensional varieties. As a consequence, all results as mentioned above can be obtained, independent of the notion of the arithmetic ampleness.

In [12], Künnemann and Maillot prove the conjectures for an arithmetic variety with a cellular decomposition by the same method as ours. The author would like to thank Professor Klaus Künnemann for informing him of this and for giving him some comments for the first version of the paper.

2. Statement of the main theorem

We first recall some basic facts of Arakelov intersection theory. Throughout the paper, n is the relative dimension of an arithmetic variety X over $\boldsymbol{Z}$. For an arithmetic variety X, we put

$$Z^{p,p}(X) = \{\omega;\ \text{real closed } (p,p)\text{-form on } X(\boldsymbol{C}) \text{ with } F_\infty^*\omega = (-1)^p\omega\},$$

$$H^{p,p}(X) = \{c \in H^{p,p}(X(\boldsymbol{C}));\ c \text{ is real and } F_\infty^* c = (-1)^p c\}.$$

Then we have an exact sequence

$$CH^{p-1,p}(X)_{\boldsymbol{R}} \xrightarrow{\rho} H^{p-1,p-1}(X) \xrightarrow{a} \widehat{CH}^p(X)_{\boldsymbol{R}}$$

$$\xrightarrow{(\zeta,\omega)} CH^p(X)_{\boldsymbol{R}} \oplus Z^{p,p}(X) \xrightarrow{cl-\pi} H^{p,p}(X) \longrightarrow 0,$$

where the definitions of $CH^{p,p-1}, \rho, a, \zeta$ and ω are seen in [6, 3.3]. The map $cl : CH^p(X)_{\boldsymbol{R}} \to H^{p,p}(X)$ is the cycle class map and π is the canonical projection map. The map $\rho : CH^{p-1,p}(X)_{\boldsymbol{R}} \to H^{p-1,p-1}(X)$ coincides with the regulator map up to constant factor by [6, Theorem 3.5].

We fix a smooth F_∞-invariant Kähler metric h on $X(\boldsymbol{C})$. The pair $\overline{X} = (X, h)$ is called an Arakelov variety. By identifying an element of $H^{p,p}(X)$ with a harmonic (p,p)-form with respect to h, we can regard $H^{p,p}(X)$ as a subspace of $Z^{p,p}(X)$. We put $CH^p(\overline{X}) = \omega^{-1}(H^{p,p}(X))$ and call it Arakelov Chow group.

Assumption 1. *For $p \geq 0$, the vector space $H^{p,p}(X)$ is spanned by the images of ρ and cl, that is,*

$$H^{p,p}(X) = \operatorname{Im} \rho \oplus \operatorname{Im} cl.$$

This was first conjectured by Beilinson [2, Conjecture 3.7 (b)] and when $p = 0$ it is true by Dirichlet unit theorem. If Assumption 1 holds for X and for $p-1$, by the definition of the Arakelov Chow group we obtain the following exact sequence:

$$CH^{p-1}(X)_{\boldsymbol{R}} \xrightarrow{a\cdot cl} CH^p(\overline{X})_{\boldsymbol{R}} \xrightarrow{\zeta} CH^p(X)_{\boldsymbol{R}} \longrightarrow 0.$$

Let $(H, \|\ \|)$ be a hermitian line bundle on X. Suppose that H is ample and that $\|\ \|$ is a positive metric. Then the first Chern form of $(H, \|\ \|)$ determines an F_∞-invariant Kähler metric h on $X(\boldsymbol{C})$. Since the product with the Kähler form respects harmonicity of forms, for the Arakelov variety $\overline{X} = (X, h)$ we can define a homomorphism

$$L_{H,\|\ \|} : CH^p(\overline{X})_{\boldsymbol{R}} \to CH^{p+1}(\overline{X})_{\boldsymbol{R}}.$$

Hence we can consider analogues of standard conjectures for $CH^p(\overline{X})_{\boldsymbol{R}}$.

Conjecture 2.1. *For an ample line bundle H on X, there exists a positive F_∞-invariant hermitian metric $\|\ \|$ on $H_{\boldsymbol{C}}$ satisfying the following for $2p \leq n+1$:*

i) *The intersection pairing*

$$\langle\, , \rangle : CH^p(\overline{X})_{\boldsymbol{R}} \times CH^{n+1-p}(\overline{X})_{\boldsymbol{R}} \to \boldsymbol{R}$$

is nondegenerate.

ii) $L_{H,\|\,\|}^{n+1-2p} : CH^p(\overline{X})_{\boldsymbol{R}} \to CH^{n+1-p}(\overline{X})_{\boldsymbol{R}}$ *is an isomorphism.*

iii) *For* $0 \neq x \in CH^p(\overline{X})_{\boldsymbol{R}}$ *such that* $L_{H,\|\,\|}^{n+2-2p}(x) = 0$, $(-1)^p \widehat{\deg}(L_{H,\|\,\|}^{n+1-2p}(x)x)$ *is positive.*

THEOREM 2.1. *Analogues of standard conjectures for* $CH^p(\overline{X})_{\boldsymbol{R}}$ *imply ones for* $\widehat{CH}^p(\overline{X})_{\boldsymbol{R}}$.

Theorem 2.1 was proved by Künnemann in [10]. He also proved Conjecture 2.1 for projective spaces and $\mathcal{O}(1)$ with Fubini-Study metric. Recently the author proved Conjecture 2.1 for regular quadric hypersurfaces in $\boldsymbol{P}^n$ [15].

From now on, every arithmetic variety X is assumed to be irreducible. Then X is defined over the ring of integers $\mathcal{O}_K$ of an algebraic number field K such that the generic fiber X_K is geometrically irreducible. The cycle class map $cl : CH^p(X)_{\boldsymbol{R}} \to H^{p,p}(X)$ factors through the Chow group of the generic fiber and the restriction map $CH^p(X) \to CH^p(X_K)$ is surjective. Hence the image of cl coincides with the image of $CH^p(X_K)_{\boldsymbol{R}}$. We denote it by $A^p(X_K)$. Then the preceding exact sequence yields

$$0 \longrightarrow A^{p-1}(X_K) \xrightarrow{a} CH^p(\overline{X})_{\boldsymbol{R}} \xrightarrow{\zeta} CH^p(X)_{\boldsymbol{R}} \longrightarrow 0.$$

For an ample line bundle H on X, we can consider the standard conjectures for $A^p(X_K)$ although X_K is not defined over an algebraically closed field. For an algebraic closure $\overline{K}$ of K, standard conjectures for $(X_{\overline{K}}, H_{\overline{K}})$ imply ones for (X_K, H_K). In particular, the Hodge index theorem for $A^p(X_K)$ holds.

We define

$$CH^p(X)^0 = \mathrm{Ker}(CH^p(X) \xrightarrow{cl} A^p(X_K)).$$

For $x \in CH^p(X)^0_{\boldsymbol{R}}$, we denote by $\tilde{x} \in CH^p(\overline{X})_{\boldsymbol{R}}$ a lifting of x. Then we define a pairing

$$\langle\, , \rangle : CH^p(X)^0_{\boldsymbol{R}} \times CH^{n+1-p}(X)^0_{\boldsymbol{R}} \to \boldsymbol{R}$$

by $\langle x, y \rangle = \widehat{\deg}(\tilde{x}\tilde{y})$. This definition is independent of the choice of liftings. Since the homomorphism

$$L_H : CH^p(X)^0_{\boldsymbol{R}} \to CH^{p+1}(X)^0_{\boldsymbol{R}}$$

for a line bundle H can also be defined, we can consider analogues of standard conjectures for $CH^p(X)^0_{\boldsymbol{R}}$. We note that the standard conjectures for $CH^p(X)^0_{\boldsymbol{R}}$ have no concern with a metric on H.

Here we state our main theorem.

Main Theorem. *Let X be an arithmetic variety defined over the ring of integers $\mathcal{O}_K$ of an algebraic number field K. Suppose that the generic fiber X_K is geometrically irreducible. Let H be an ample line bundle on X. Given a positive F_∞-invariant hermitian metric $\| \;\|$ on H_C, we define a metric $\| \;\|_\sigma$ by $| \;\|_\sigma = \exp(\sigma)\| \;\|$ for $\sigma \in \boldsymbol{R}$.*

i) *We assume that the pairing*

$$\langle\, , \rangle : A^{p-1}(X_K) \times A^{n+1-p}(X_K) \to \boldsymbol{R}$$

is nondegenerate. Then the pairing

$$\langle\, , \rangle : CH^p(\overline{X})_{\boldsymbol{R}} \times CH^{n+1-p}(\overline{X})_{\boldsymbol{R}} \to \boldsymbol{R}$$

is nondegenerat[illegible]ly if the pairings

$$\langle\, , \rangle : A^p(X_K) \times A^{n-p}(X_K) \to \boldsymbol{R}$$

and

$$\langle\, , \rangle : CH^p(X)^0_{\boldsymbol{R}} \times CH^{n+1-p}(X)^0_{\boldsymbol{R}} \to \boldsymbol{R}$$

are nondegenerate. In particular, the pairing for $CH^p(\overline{X})_{\boldsymbol{R}}$ is nondegenerate for any p if and only if the pairings for $A^p(X_K)$ and $CH^p(X)^0_{\boldsymbol{R}}$ are nondegenerate for any p.

ii) *We assume that the conclusion of* i) *holds for p. Then the hard Lefschetz theorem for $CH^p(\overline{X})_{\boldsymbol{R}}$ with $(H, \| \;\|_\sigma)$ for almost all σ is equivalent to the hard Lefschetz theorem for $CH^p(X)^0_{\boldsymbol{R}}$ with $c_1(H)$.*

iii) *The Hodge index theorem for $CH^p(\overline{X})_{\boldsymbol{R}}$ with $\hat{c}_1(H, \| \;\|_\sigma)$ for $0 \ll -\sigma$ is equivalent to the Hodge index theorem for $CH^p(X)^0_{\boldsymbol{R}}$ with $c_1(H)$.*

Remark. After arithmetic ampleness of hermitian line bundles was defined in [16], a finer version of Conjecture 1.2 was proposed [14]. This says that for arithmetically ample hermitian line bundle $(H, \| \;\|)$ on X, ii) and iii) of Conjecture 1.2 hold. For arbitrary hermitian line bundle $(H, \| \;\|)$ with a positive metric $\| \;\|$ and for $0 \ll -\sigma$, the hermitian line bundle $(H, \| \;\|_\sigma)$ becomes arithmetically ample. But it is difficult to compare the upper bound of σ such that $(H, \| \;\|_\sigma)$ is arithmetically ample with the upper bound of σ for which iii) of Main Theorem holds.

Corollary 2.2. *Let X be a Grassmannian or a projective smooth toric scheme or a generalized flag scheme, which is a quotient scheme of a split reductive group scheme by a Borel subgroup, defined over a ring of integers $\mathcal{O}_K$. Then for any ample line bundle H with a positive metric $\| \;\|$ and for $0 \ll -\sigma$, Conjecture* 1.2 *with $(H, \| \;\|_\sigma)$ holds.*

Proof. Since X has a stratification whose strata are all isomorphic to affine spaces over $\mathcal{O}_K$, Assumption 1 holds for X. Since $CH^p(X)^0_{\boldsymbol{R}}$ vanish for all p, all conditions about them are vacuous. For an embedding $\tau : K \to \boldsymbol{C}$, we denote by

X_τ the complex manifold associated with the scheme $X \otimes_\tau \boldsymbol{C}$. Then the cycle class map $cl : CH^p(X_K)_{\boldsymbol{C}} \to H^{p,p}(X_\tau)$ is bijective. So the standard conjectures for X_K are valid. Hence Main Theorem holds for X. □

3. Proof of the main theorem

By Theorem 2.1 we have only to prove Conjecture 2.1. We begin by the proof of i). We take splittings of surjections $\widehat{CH}^p(\overline{X})_{\boldsymbol{R}} \to A^p(X_K)$ and $\zeta^{-1}(CH^p(X)^0_{\boldsymbol{R}}) \to CH^p(X)^0_{\boldsymbol{R}}$. Then we have a decomposition of an Arakelov Chow group

$$\widehat{CH}^p(\overline{X})_{\boldsymbol{R}} \simeq A^p(X_K) \oplus CH^p(X)^0_{\boldsymbol{R}} \oplus A^{p-1}(X_K).$$

If $x \in A^{p-1}(X_K)$, we have $\widehat{\deg}(a(x)y) = \deg(x\omega(y))$. Hence if $\omega(y) = 0$, then $\langle a(x), y \rangle = 0$. From this computation for the above decompositions of $\widehat{CH}^p(\overline{X})_{\boldsymbol{R}}$ and $\widehat{CH}^{n+1-p}(\overline{X})_{\boldsymbol{R}}$ the pairing is described by the following matrix:

$$\begin{pmatrix} * & * & A_p \\ & B_p & 0 \\ A_{p-1} & 0 & 0 \end{pmatrix},$$

where A_p represents the pairing

$$\langle\, , \rangle : A^p(X_K) \times A^{n-p}(X_K) \to \boldsymbol{R}$$

and B_p represents the pairing

$$\langle\, , \rangle : CH^p(X_K)^0_{\boldsymbol{R}} \times CH^{n+1-p}(X)^0_{\boldsymbol{R}} \to \boldsymbol{R}.$$

Since the Hodge index theorem for $A^i(X_K)$ is true for any i, it holds that $\dim_{\boldsymbol{R}} A^p(X_K) \leq \dim_{\boldsymbol{R}} A^{n-p}(X_K)$. Hence under the condition of the non-degeneracy of A_{p-1}, the above matrix is nondegenerate if and only if B_p and A_p are nondegenerate.

We turn to the proof of ii). Let $\operatorname{Prim} A^p(X_K)$ be the $\boldsymbol{Z}$-module consisting of primitive cycles, namely,

$$\operatorname{Prim} A^p(X_K) = \{x \in A^p(X_K); L_H^{n+1-2p}(x) = 0\}.$$

We take a splitting of the surjection $\zeta^{-1}(CH^p(X)^0_{\boldsymbol{R}}) \to CH^p(X)^0_{\boldsymbol{R}}$. Since the pairing B_p is nondegenerate, we can choose a splitting of the surjection $\widehat{CH}^p(\overline{X})_{\boldsymbol{R}} \to A^p(X_K)$ whose image is orthogonal to the image of the splitting of $CH^p(X)^0_{\boldsymbol{R}}$ by $L^{n+1-2p}_{H,\|\,\|}$. Since $A^p(X_K) \simeq L_H A^{p-1}(X_K) \oplus \operatorname{Prim} A^p(X_K)$, we have a decomposition

$$\widehat{CH}^p(\overline{X})_{\boldsymbol{R}} \simeq L_H A^{p-1}(X_K) \oplus \operatorname{Prim} A^p(X_K) \oplus CH^p(X)^0_{\boldsymbol{R}} \oplus A^{p-1}(X_K).$$

Since i) of Main Theorem is assumed, we have only to prove that the pairing

$$\langle\, , L^{n+1-2p}_{H,\|\,\|_\sigma} \rangle : \widehat{CH}^p(\overline{X})_{\boldsymbol{R}} \times \widehat{CH}^p(\overline{X})_{\boldsymbol{R}} \to \boldsymbol{R}$$

is nondegenerate for almost all σ. For the above decomposition, the pairing is described by the matrix

$$\begin{pmatrix} * & * & 0 & C_{p-1} \\ & D_{\sigma,p} & 0 & 0 \\ 0 & 0 & E_p & 0 \\ C_{p-1} & 0 & 0 & 0 \end{pmatrix},$$

where C_{p-1} represents the pairing

$$\langle\,,L_H^{n+2-2p}\rangle : A^{p-1}(X_K) \times A^{p-1}(X_K) \to \boldsymbol{R}$$

and E_p represents the pairing

$$\langle\,,L_H^{n+1-2p}\rangle : CH^p(X)_{\boldsymbol{R}}^0 \times CH^p(X)_{\boldsymbol{R}}^0 \to \boldsymbol{R}.$$

For $x, y \in \operatorname{Prim} A^p(X_{\boldsymbol{R}})$ and their liftings $\tilde{x}, \tilde{y}$, we have

$$\begin{aligned}\langle\tilde{x}, L_{H,\|\,\|_\sigma}^{n+1-2p}\tilde{y}\rangle &= (\hat{c}_1(H,\|\,\|) - 2a(\sigma))^{n+1-2p}\tilde{x}\tilde{y} \\ &= \hat{c}_1(H,\|\,\|)^{n+1-2p}\tilde{x}\tilde{y} - 2\sigma(n+1-2p)a(c_1(H,\|\,\|)^{n-2p}xy).\end{aligned}$$

Hence $D_{\sigma,p} = D_{0,p} - 2\sigma(n+1-2p)C_p'$, where C_p' represents the pairing

$$\langle\,,L_H^{n-2p}\rangle : \operatorname{Prim} A^p(X_K) \times \operatorname{Prim} A^p(X_K) \to \boldsymbol{R}.$$

Since the standard conjectures for X_K are assumed, C_{p-1} and C_p' are nondegenerate. Hence $D_{\sigma,p}$ is nondegenerate for all but finitely many σ. Therefore the above matrix is nondegenerate for almost all σ if and only if E_p is nondegenerate.

Now we will prove iii). Let x be a primitive element in $CH^p(\overline{X})_{\boldsymbol{R}}$ with respect to $L_{H,\|\,\|}$, that is, $L_{H,\|\,\|}^{n+2-2p}(x) = 0$ holds. Then for a harmonic $(p-1,p-1)$-form ω we have

$$\begin{aligned}L_{H,\|\,\|_\sigma}^{n+2-2p}(x + a(\omega)) &= (\hat{c}_1(H,\|\,\|) - 2a(\sigma))^{n+2-2p}(x + a(\omega)) \\ &= (\hat{c}_1(H,\|\,\|)^{n+2-2p} - 2\sigma(n+2-2p)a(c_1(H,\|\,\|)^{n+1-2p}))(x+a(\omega)) \\ &= a(c_1(H,\|\,\|)^{n+2-2p}\omega - 2\sigma(n+2-2p)c_1(H,\|\,\|)^{n+1-2p}\omega(x)).\end{aligned}$$

Hence $x + a(\omega) \in CH^p(\overline{X})_{\boldsymbol{R}}$ is primitive with respect to $L_{H,\|\,\|_\sigma}$ if and only if

$$c_1(H,\|\,\|)^{n+2-2p}\omega = 2\sigma(n+2-2p)c_1(H,\|\,\|)^{n+1-2p}\omega(x).$$

Since $c_1(H,\|\,\|)^{n+2-2p}\omega(x) = 0$, we have

$$\omega(x) = \begin{cases} \omega_0(x) + c_1(H,\|\,\|)\omega_1(x) & \text{if } 2p < n+1 \\ c_1(H,\|\,\|)\omega_1(x) & \text{if } 2p = n+1, \end{cases}$$

where each $\omega_i(x)$ is a primitive harmonic form. If $2p = n+1$, then we set

$\omega_0(x) = 0$. By the above equality we have

$$\omega = 2\sigma(n+2-2p)\omega_1(x).$$

We first assume $\omega(x) = 0$. If we denote $x' = \zeta(x)$, then $x' \in CH^p(X)^0_{\boldsymbol{R}}$. Since x is primitive with respect to $(H, \| \|_\sigma)$, x' is also primitive with respect to H. Moreover we have

$$(-1)^p \widehat{\deg}(L_{H,\| \|_\sigma}^{n+1-2p}(x)x) = (-1)^p \widehat{\deg}(L_H^{n+1-2p}(x')x')$$

and by the Hodge index theorem for $CH^p(X)^0_{\boldsymbol{R}}$ it is positive.

We next consider the case of $\omega(x) \neq 0$. Then any primitive cycle y in $CH^p(\bar{X})_{\boldsymbol{R}}$ with respect to $L_{H,\| \|_\sigma}$ is written by $x + a(\omega)$ where x is a primitive cycle with respect to $L_{H,\| \|}$ and $\omega = 2\sigma(n+2-2p)\omega_1(x)$. Then we have

$$\begin{aligned} L_{H,\| \|_\sigma}^{n+1-2p}(y)y &= (\hat{c}_1(H, \| \|) - 2a(\sigma))^{n+1-2p}(x + a(\omega))^2 \\ &= (\hat{c}_1(H, \| \|)^{n+1-2p} - 2\sigma(n+1-2p)a(c_1(H, \| \|)^{n-2p}))(x^2 + 2a(\omega(x)\omega)). \end{aligned}$$

By substituting the above equality for ω and $\omega(x) = \omega_0(x) + c_1(H, \| \|)\omega_1(x)$ we have

$$\begin{aligned} \widehat{\deg}(L_{H,\| \|_\sigma}^{n+1-2p}(y)y) = \widehat{\deg}(\hat{c}_1(H, \| \|)^{n+1-2p}x^2) - \sigma(n+1-2p)\deg(c_1(H, \| \|)^{n-2p}\omega_0(x)^2) \\ + \sigma(n+3-2p)\deg(c_1(H, \| \|)^{n+2-2p}\omega_1(x)^2). \end{aligned}$$

The Hodge index theorem for the complex cohomology implies

$$(-1)^p \deg(c_1(H, \| \|)^{n-2p}\omega_0(x)^2) > 0$$

and

$$(-1)^{p+1} \deg(c_1(H, \| \|)^{n+2-2p}\omega_1(x)^2) > 0$$

if $\omega_i(x) \neq 0$. When $2p < n+1$, $\omega(x) \neq 0$ implies $\omega_0(x) \neq 0$ or $\omega_1(x) \neq 0$. Hence for $0 \ll -\sigma$, $(-1)^p \widehat{\deg}(L_{H,\| \|_\sigma}^{n+1-2p}(x)x)$ is positive. When $2p = n+1$, $\omega_0(x) = 0$ and $\omega_1(x) \neq 0$. Hence in this case the same inequality is obtained. □

4. A final remark

Main Theorem shows that analogues of standard conjectures for $\widehat{CH}^p(X)_{\boldsymbol{R}}$ are equivalent to ones for $CH^p(X)^0_{\boldsymbol{R}}$ and original standard conjectures. In this section we will consider a relation between the standard conjectures for $CH^p(X)^0_{\boldsymbol{R}}$ and ones for height pairing, which are first conjectured by Beilinson [3].

According to [11] we introduce the height pairing of X_K. We define

$$CH^p(X_K)^0 = \mathrm{Ker}(cl : CH^p(X_K) \to A^p(X_K))$$

and

$$CH^p_{\mathrm{fin}}(X) = \mathrm{Ker}(CH^p(X) \to CH^p(X_K)).$$

Then we have a short exact sequence

$$0 \to CH^p_{\mathrm{fin}}(X)_{\boldsymbol{R}} \to CH^p(X)^0_{\boldsymbol{R}} \to CH^p(X_K)^0_{\boldsymbol{R}} \to 0.$$

For the pairing of $CH^p(X)^0_{\boldsymbol{R}}$ as above, we assume the following: For arbitrary element $x \in CH^p(X_K)^0_{\boldsymbol{R}}$, there exists a lifting $\tilde{x} \in CH^p(X)^0_{\boldsymbol{R}}$ which is orthogonal to $CH^{n+1-p}_{\mathrm{fin}}(X)^0_{\boldsymbol{R}}$. Under this assumption for p and $n+1-p$ the height pairing of $CH^p(X_K)^0_{\boldsymbol{R}}$ can be defined. (See [3, 4, 11].)

We take splittings of surjections $CH^p(X)^0_{\boldsymbol{R}} \to CH^p(X_K)^0_{\boldsymbol{R}}$ and $CH^{n+1-p}(X)^0_{\boldsymbol{R}} \to CH^{n+1-p}(X_K)^0_{\boldsymbol{R}}$ whose images are orthogonal to $CH^{n+1-p}_{\mathrm{fin}}(X)_{\boldsymbol{R}}$ and $CH^p_{\mathrm{fin}}(X)_{\boldsymbol{R}}$ respectively. Then we can have decompositions

$$CH^p(X)^0_{\boldsymbol{R}} \simeq CH^p_{\mathrm{fin}}(X)_{\boldsymbol{R}} \oplus CH^p(X_K)^0_{\boldsymbol{R}}$$

and

$$CH^{n+1-p}(X)^0_{\boldsymbol{R}} \simeq CH^{n+1-p}_{\mathrm{fin}}(X)_{\boldsymbol{R}} \oplus CH^{n+1-p}(X_K)^0_{\boldsymbol{R}}.$$

In the same way as the proof of Main Theorem we can show that the standard conjectures for $CH^p(X)^0_{\boldsymbol{R}}$ are equivalent to these for $CH^p(X_K)^0_{\boldsymbol{R}}$ and for $CH^p_{\mathrm{fin}}(X)_{\boldsymbol{R}}$.

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